First name:______________________________

Last name:______________________________

Student ID:______________________________

Section:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 110 minutes.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the set of functions $\mathcal{V} = \text{span}\{\cos t, \sin t, \cos 2t, \sin 2t\}$, which is known to be a vector space over the field of real numbers $\mathbb{R}$.

(a) Find a basis for $\mathcal{V}$.

(b) Let $T(f) := \dfrac{d^2 f}{dt^2} + f$ where $f \in \mathcal{V}$. Show that operator T is linear.

(c) Find the null space of T.

(d) Find the range space of T.

(e) Note that $T : \mathcal{V} \to \mathcal{V}$. Considering the same basis both for the domain and the range (codomain) obtain the matrix representation of T.

Answer:

Q2.

Let four vectors $v_1,\, v_2,\, v_3,\, v_4$ in $\mathbb{R}^3$ be such that any three of them make a linearly independent set. Define $\mathcal{S}_1 := \operatorname{span}\{v_1,\, v_2\}$ and $\mathcal{S}_2 := \operatorname{span}\{v_3,\, v_4\}$. Show that $\dim(\mathcal{S}_1 \cap \mathcal{S}_2) = 1$.

Answer:

Q3.

Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation, where $\mathcal{V}$ is the set of all 2×2 real matrices and $\mathcal{W}$ is the set of all polynomials with degree no greater than two, defined as

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) := a_{11} + a_{12}t + (a_{21} + a_{22})t^2 \,.$$

Let

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

and

$$\mathcal{C} = (1,\, t,\, t^2)$$

be bases for $\mathcal{V}$ and $\mathcal{W}$, respectively.

(a) Find the matrix representation of T with respect to the given bases.

(b) Replace basis $\mathcal{C}$ with $\widehat{\mathcal{C}} = (1,\, 1+t,\, 1+t+t^2)$ and find the new matrix representation of T.

Answer:

Q4.

Give the definition of a field. For the below candidates determine the additive and multiplicative identities ($0_{\mathcal{F}}$ and $1_{\mathcal{F}}$) and additive and multiplicative inverses ($-a$ and a^{-1} for an arbitrary element a) *if $\mathcal{F}$ is a field.* (No need to show that $\mathcal{F}$ satisfies all the axioms of a field.)

(a) $\mathcal{F}$ is the set of rational functions. (A rational function f is such that it can be written as the ratio of two polynomials $f(s) = p(s)/q(s)$ where q is different than the zero polynomial.) Addition and multiplication are defined in the standard sense, i.e., $[f + g](s) = f(s) + g(s)$ and $[fg](s) = f(s)g(s)$.

(b) $\mathcal{F}$ is the extended real line $\mathbb{R} \cup \{-\infty\}$. Addition and multiplication are defined as

$$\begin{aligned} a \oplus b &:= \ln(e^a + e^b) \\ a \odot b &:= a + b\,. \end{aligned}$$

Moreover, the element $-\infty$ is assumed to satisfy the following.

$$\begin{aligned} -\infty + a &= -\infty \quad \forall a \in \mathcal{F} \\ e^{-\infty} &= 0 \\ \ln(0) &= -\infty\,. \end{aligned}$$

Answer: