First name:____________________________

Last name:____________________________

Student ID:____________________________

Section:____________________________

Signature:____________________________

Read before you start:

- There are six questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 150 minutes.

Q1	Q2	Q3	Q4	Q5	Q6	Total

Q1.

Let $\mathcal{P}_3$ be the set of all polynomials $f(t)$ with degree$(f) \leq 3$. Consider the mapping $\ell : \mathcal{P}_3 \to \mathcal{P}_3$ defined as

$$\ell(f) := \frac{d^2 f}{dt^2} + 3\frac{df}{dt}.$$

(a) Show that ℓ is a linear mapping.

(b) Find a basis for $\mathcal{P}_3$.

(c) Find bases for the null and range spaces of ℓ.

(d) Find the matrix representation of ℓ with respect to the basis chosen in part (b).

Answer:

Q2.

Let $Q \in \mathbb{R}^{n\times n}$ be a real symmetric positive definite matrix. We define $\langle x, y\rangle_Q := y^*Qx$ for $x, y \in \mathbb{C}^n$.

(a) Show that $\langle \cdot\,, \cdot\rangle_Q$ is an inner product.

(b) Given

$$Q = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

determine whether set $\mathcal{S}$ is orthogonal with respect to our inner product $\langle \cdot\,, \cdot\rangle_Q$. If not, apply Gram-Schmidt to orthogonalize it (with respect to $\langle \cdot\,, \cdot\rangle_Q$).

Answer:

Q3.

(a) Obtain the Jordan form and the transformation matrix for the below matrix.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

(b) Let matrix A have the characteristic polynomial $d(s) = (s-2)^{10}$ and the minimal polynomial $m(s) = (s-2)^3$. Moreover, it is known that $\operatorname{rank}(A-2I) = 6$ and $\operatorname{rank}(A-2I)^2 = 3$. Find possible Jordan form(s) of A.

Answer:

Q4.

Let

$$A = \begin{bmatrix} 5 & 1 \\ -4 & 1 \end{bmatrix}$$

(a) Compute the eigenvalues and eigenvectors of A.

(b) Compute $\sin(\pi A)$ and $\cos(\pi A)$.

(c) Compute the eigenvalues and eigenvectors of $\sin(\pi A)$.

Answer:

Q5.

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Compute the minimum norm $\bar{x} \in \mathbb{R}^3$ that minimizes $\|Ax - b\|$.

Answer:

Q6.

Consider the linear space of $n \times n$ real matrices $\mathbb{R}^{n\times n}$ over the field of real numbers $\mathbb{R}$. Let $\mathcal{S}$ denote the set of all $n \times n$ real symmetric matrices.

(a) Show that $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times n}$.

(b) Let function $f : \mathcal{S} \to \mathbb{R}$ be defined as $f(A) := \max_i |\lambda_i(A)|$, where $\lambda_i(A)$ denote the ith eigenvalue of matrix $A \in \mathcal{S}$. Show that f is a norm on $\mathcal{S}$.

Answer: