EE501 HW2

Q1. Consider linear space $\mathcal{V} = \{f \big| f : [0,\, 1] \to \mathbb{R}\,, f \text{ differentiable}\}$.

(a) Show that $\|f\| := \max_{t\in[0,\,1]} |f(t)| + \max_{t\in[0,\,1]} |\dot{f}(t)|$ is a norm on $\mathcal{V}$.

(b) Is $\|f\| := \max_{t\in[0,\,1]} |\dot{f}(t)|$ a norm on $\mathcal{V}$?

Q2. Show that for all $x \in \mathbb{R}^n$ the following hold.

(a) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$.

(b) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

(c) $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$.

Q3. Let $\mathcal{V}$ be a normed space. Show that $\|x\| - \|y\| \leq \|x + y\|$ for all $x,\, y \in \mathcal{V}$.

Q4. Consider the linear space of $n \times n$ matrices with complex entries. Show that $\langle A,\, B\rangle := \mathrm{tr}(B^*A)$ is an inner product, where A^* denotes the conjugate transpose of A and $\mathrm{tr}(A)$ is the trace of A.

Q5. Let $\mathcal{V}$ be a finite-dimensional inner product space. Suppose a linear mapping $P : \mathcal{V} \to \mathcal{V}$ satisfies the following.

- $P^2 = P$.
- $\langle x,\, y\rangle = 0$ for all $x \in \mathcal{N}(P)$ and $y \in \mathcal{R}(P)$.
- There exists some $x \in \mathcal{V}$ such that $Px \neq 0$.

Prove the following.

(a) $\mathcal{N}(P) \cap \mathcal{R}(P) = \{0\}$.

(b) $\max_{\|x\|\neq 0} \dfrac{\|Px\|}{\|x\|} = 1$.

Q6 [1]. Let

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$ and

$$\begin{aligned} f_1 &= e_1 + e_2 + e_3 \\ f_2 &= e_2 + e_3 \\ f_3 &= e_3 \end{aligned}$$

(a) Apply Gram-Schmidt process to basis $(f_1,\, f_2,\, f_3)$.

(b) Apply Gram-Schmidt process to basis $(f_3,\, f_2,\, f_1)$.

Q7 [1]. Consider the inner product space $\{f \,|\, f : [-\pi, \pi] \to \mathbb{R},\ f \text{ continuous}\}$ with

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)g(t)dt\,.$$

Verify that for any positive integer n, the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \frac{\sin(2t)}{\sqrt{\pi}}, \ldots, \frac{\sin(nt)}{\sqrt{\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\cos(2t)}{\sqrt{\pi}}, \ldots, \frac{\cos(nt)}{\sqrt{\pi}} \right\}$$

is orthonormal.

Q8 [1]. Prove or disprove the following claim.

Claim. There is an inner product on $\mathbb{R}^2$ whose induced norm is $\|x\| = |x_1| + |x_2|$.

Q9 [1]. Let $\mathcal{V}$ be a finite-dimensional inner product space. Given vectors $x, y \in \mathcal{V}$ show that the statements below are equivalent.

- $\langle x, y \rangle = 0$.
- $\|x\| \leq \|x + \alpha y\|$ for all $\alpha \in \mathbb{C}$.

Q10 [2]. Show that $\langle x, y \rangle := y^* A x$ is an inner product in $\mathbb{C}^2$ for

$$A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

Compute $\langle x, y \rangle$ for $x = [1 \ \ 2]^T$ and $y = [i \ \ 1+i]^T$.

Q11 [2]. Let $\mathcal{B}$ be a basis for a finite-dimensional inner product space. Prove the following.

(a) If $\langle x, z \rangle = 0$ for all $z \in \mathcal{B}$, then $x = 0$.

(b) If $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \mathcal{B}$, then $x = y$.

Q12 [2]. Let $T : \mathcal{V} \to \mathcal{V}$ be a linear transformation on an inner product space $\mathcal{V}$ with dimension n. Suppose $\|Tx\| = \|x\|$ for all x. Prove that $\dim(\mathcal{R}(T)) = n$.

Q13 [2]. Let $\mathcal{V}$ be an inner product space and $\{v_1, v_2, \ldots, v_m\}$ be an orthonormal subset of $\mathcal{V}$. Show that for all $x \in \mathcal{V}$ we can write

$$\|x\|^2 \geq \sum_{i=1}^{m} |\langle x, v_i \rangle|^2\,.$$

Q14 [2]. Consider inner product space $\mathcal{V} = \{f \,|\, f : [0, 1] \to \mathbb{R},\ f \text{ continuous}\}$ with inner product $\langle f, g \rangle := \int_0^1 f(t)g(t)dt$. Find an orthonormal basis for the subspace of $\mathcal{V}$ spanned by set $\{\sqrt{t}, t\}$.

References

[1] I. Lankham, B. Nachtergaele, and A. Schilling. *Lecture Notes for MAT67.* Fall 2007.

[2] Special thanks to J. Hernandez.