Q1. Let F be a field. For an arbitrary element $a \in F$ prove the following.

(a) The additive inverse $-a$ is unique.

(b) For $a \neq 0$ the multiplicative inverse a^{-1} is unique.

Q2. Let $F = \mathbb{R}_{>0}$ (the set of strictly positive real numbers). We define addition and multiplication, respectively, as

$$\begin{aligned} a \oplus b &:= ab \\ a \odot b &:= e^{\ln a \ln b} \end{aligned}$$

for $a, b \in F$.

(a) Find the additive identity 0_F and the multiplicative identity 1_F.

(b) Prove that F is a field.

(c) Given $a \in F$, find $-a$ and a^{-1}.

Q3. Let (V, F) be a linear space. Given $a \in F$ and $v \in V$ prove the following.

(a) $a0_V = 0_V$.

(b) $-0_V = 0_V$.

(c) $0_F v = 0_V$.

(d) $(-1_F)x = -x$.

(e) $av = 0_V$ implies either $a = 0_F$ or $v = 0_V$.

(f) $a0_V = 0_V$.

Q4. Show that the set of all $m \times n$ real matrices $\mathbb{R}^{m\times n}$, with usual matrix addition and multiplication of a matrix by a real scalar, is a linear space over the field of real numbers $\mathbb{R}$.

Q5. Let $P[0, 1]$ denote the set of all functions $f : [0, 1] \to \mathbb{R}$ that are piecewise continuous (i.e. discontinuous only at finitely many points with well-defined (finite) right and left limits where discontinuous). Show that $P[0, 1]$ is a linear space.

Q6. Let $S = \left\{ f \in C[0, 1] : f(\frac{1}{2}) = 0 \right\}$ where $C[0, 1]$ is the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Prove or disprove the claim that S is a subspace of $C[0, 1]$.

Q7. Given linear space V, let Y and Z be subspaces of V. Prove the following.

(a) The intersection $Y \cap Z$ is also a subspace of V.

(b) That $Y \cup Z$ is a subspace implies either $Y \subset Z$ or $Z \subset Y$.

Q8. Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined as

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) := \begin{bmatrix} a_{11} + a_{12} & a_{21} + a_{22} \\ a_{22} & a_{11} + a_{22} \end{bmatrix}$$

(a) Find the matrix representation of this transformation with respect to the standard basis:

$$B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

(b) Find a basis for the null space $\mathcal{N}(T)$.

(c) Find a basis for the range space $\mathcal{R}(T)$.

Q9. Let $B = (v_1, v_2, \ldots, v_n)$ be an ordered basis for a finite dimensional linear space V. Show that for each $x \in V$, coordinates $(a_1, a_2, \ldots, a_n)$ of x with respect to B (i.e., $x = a_1v_1 + a_2v_2 + \ldots + a_nv_n$) are unique.

Q10. Let V be the linear space of 2×2 real matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(a) Let $S_1 = \{A \in V : a_{11} + a_{21} = a_{22}\}$. Is S_1 a subspace of V? If so, find a basis for S_1.

(b) Repeat part (a) for $S_2 = \{A \in V : a_{12} = 1 + a_{11}\}$.

(c) Find a basis for V that contains

$$v_1 = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

as an element.

Q11. Consider the set of polynomials $W = \mathrm{Sp}\{p_1, p_2, p_3, p_4, p_5\}$ where $p_1(t) = 1 + t$, $p_2(t) = 2 + t^2$, $p_3(t) = t^4$, $p_4(t) = 1 - t + t^2$, and $p_5(t) = 1 - t + t^2 + 2t^4$. Find a basis for W.

Q12. Let $T : V \to W$ be a linear transformation, where V is the set of all 2×2 real matrices and W is the set of all polynomials with degree no greater than two, defined as

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) := a_{11} + a_{12}t + (a_{21} + a_{22})t^2 .$$

Let

$$B = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

and

$$C = (1, t, t^2)$$

be bases for V and W, respectively.

(a) Find the matrix representation of T with respect to the given bases.

(b) Replace basis C with $\widehat{C} = (1, 1+t, 1+t+t^2)$ and find the new matrix representation of T.

Q13. Let P be the set of all polynomials with real coefficients and with degree no greater than $n-1$. Let L be the set of all linear functions of the form $\ell : P \to \mathbb{R}$. *It is a fact that L is an n-dimensional linear space.*

(a) Given $t_1 \in \mathbb{R}$ let $\ell_1 : P \to \mathbb{R}$ be defined as $\ell_1(p) = p(t_1)$. Show that $\ell_1 \in L$.

(b) Take $n = 3$. Prove the following claim. Given three distinct real numbers t_1, t_2, t_3 we can always find three real numbers a_1, a_2, a_3 such that

$$\int_0^1 p(t)dt = a_1 p(t_1) + a_2 p(t_2) + a_3 p(t_3)$$

for all $p \in P$.