EE 501 (SUPPLEMENTARY NOTES)

COMPUTATION OF THE $R(A)$ AND $N(A)$

Our aim is to determine basis sets for $R(A)$ and $N(A)$, where A is an $m \times n$ real matrix. The analysis is similar in the complex case.

Elementary Column Operations (e.c.o.)

Let A be an $m \times n$ matrix. The three elementary column operations are:

i. Multiplication of a column by a nonzero real number.
ii. Adding the product of column j by a real number to column i, where $i \neq j$.
iii. Interchanging two columns.

Theorem: If B is obtained from A by a sequence of e.c.o., then A can be obtained from B by a sequence of e.c.o.

Proof: It is sufficient to show this for each of the three e.c.o. Let $A = [a_1 \; a_2 \; \ldots \; a_n]$, where $a_i \in R^m$ are columns of A.

i. Let c be a nonzero real number. Multiply a_i by c. Now, ca_i is the i-th column of B. A can be obtained from B by multiplying the i-th column by $c^{-1} = 1/c$.
ii. Multiply the j-th column of A by c and add it to column i. Now, $a_i + ca_j$ is the i-th column of B. To obtain A back, subtract the product of the j-th column of B (i.e. a_j) by c, from i-th column of B.
iii. If columns a_i and a_j are interchanged to obtain B, interchange the same two columns to obtain A back.

Theorem: Any of the three e.c.o. defined above corresponds to multiplying A from right by an $n \times n$ nonsingular matrix E where E is obtained by performing the corresponding e.c.o. on the unit matrix I_n.

Proof: Exercise.

Examples:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} ca_{11} & a_{12} \\ ca_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \text{ (First e.c.o.)}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (Third e.c.o.)}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12}+ca_{11} \\ a_{21} & a_{22}+ca_{21} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \text{ (Second e.c.o.)}$$

We therefore see that if B is obtained from A by a sequence of e.c.o., then $B = AE_1E_2 \ldots E_k = AK$ where $K = E_1E_2 \ldots E_k$ and $\det K \neq 0$. E_i is obtained by performing the i-th e.c.o. on I_n. Clearly, E_i^{-1} corresponds to an e.c.o. performed on I_n as well.

Definition: If B is obtained from A by a sequence of e.c.o., then B is said to be column equivalent to A.

Theorem: If B is column equivalent to A, then $R(B) = R(A)$.

Proof: Let $y \in R(A)$. Then, there exists an $x \in R^n$ such that $Ax = y$. Since $A = BK^{-1}$, $y = BK^{-1}x$. Let $\tilde{x} = K^{-1}x$. So, $y = B\tilde{x}$, which means that $y \in R(B)$. So $R(A) \subset R(B)$. Now, let $y' \in R(B)$. Then, there exists an $x' \in R^n$ such that $Bx' = y$. $B = AK$ and let $\tilde{x}' = Kx'$. Then, $y' \in R(A)$. So, $R(B) \subset R(A)$.

Remark: By using the theorem given above, we can simplify a matrix A by e.c.o. and obtain a matrix B which is column equivalent to A. Then we can find a basis for $R(B)$, which also turns out to be a basis for $R(A)$.

The Algorithm to find a basis for $R(A)$***:***

1. Set $i = 1$.
2. Find the smallest k such that
$$a_{ik} \neq 0$$
$$a_{lk} = 0 \quad \text{for all } l < i$$
If such a k does not exist, go to step 4.
3. For all $j > k$ such that
$$a_{ij} \neq 0$$
$$\text{and } a_{lj} = 0 \quad \text{for all } l < i$$
multiply column k by $-a_{ij}/a_{ik}$ and add to column j.
4. If $i = m$, stop. Otherwise, let $i \to i+1$ and go to step 2.

As a result of this algorithm, the matrix A is transformed into $\overline{A}$ by e.c.o. where each column of A is either zero or if k_i is the first non-zero entry (counting from the top) of column i, then $k_i \neq k_j$ when $i \neq j$.

The matrix $\overline{A}$ looks like

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ x & 0 & 0 & \dots & 0 \\ \vdots & x & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & x \\ \vdots & \vdots & 0 & \dots & \vdots \end{bmatrix}$$

By reordering columns of $\overline{A}$, we can have:

$1 \le k_1 < k_2 < \dots < k_r \le n$

where r is the number of non-zero columns of $\overline{A}$. Then $\{\overline{a}_1, \dots \overline{a}_r\}$ constitute a basis for $R(\overline{A}) = R(A)$. Prove this last statement as exercise.

Example:

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ -1 & 1 & -2 & -1 & -3 \\ 2 & -1 & 4 & 4 & 7 \\ 1 & -2 & 6 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & -1 & 4 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 \end{bmatrix} = \overline{A}$$

So, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for $R(\overline{A}) = R(A)$.

Elementary Row Operations (e.r.o.)

The three elementary row operations are:

i. Multiplication of a row by a <u>nonzero</u> real number.
ii. Adding the product of row j by a real number to row i, where $i \neq j$.
iii. Interchanging two rows.

Theorem: If B is obtained from A by a sequence of e.r.o., then A can be obtained from B by a sequence of e.r.o.
Proof: Exercise

Theorem: Let B be obtained from A by a sequence of e.r.o., then $B = KA$ where $K = E_k E_{k-1} \dots E_1$ is a non-singular $m \times m$ matrix and each E_i is obtained by performing the i-th e.r.o. on the $m \times m$ identity matrix I_m.
Proof: Exercise.

Definition: If B is obtained from A by a sequence of e.r.o., then B is said to be row equivalent to A.

Theorem: If B is row equivalent to A, then $N(B) = N(A)$.
Proof:
Let $x \in N(A)$. Then, $Ax = 0$, and $KAx = Bx = 0$. Hence $x \in N(B)$.
Next, let $x \in N(B)$. Then, $Bx = 0$, and $K^{-1}Bx = Ax = 0$. So, $x \in N(A)$.

The Algorithm to find a Basis for $N(A)$

I. First phase (e.r.o.)

1. Set $i = 1$.
2. Find the smallest k such that

$$a_{ki} \neq 0$$
$$a_{kj} = 0 \quad \text{for all } j < i$$

If such a k does not exist, go to step 4.

5. For all $j \neq k$ such that $a_{ji} \neq 0$, multiply row k by $-a_{ji}/a_{ki}$ and add to row j.
6. If $i = n$, stop. Otherwise, let $i \to i+1$ and go to step 2.

Note that this portion of the algorithm resembles the algorithm for finding a basis for $R(A)$. At this point, the rows are either all zeros or in the form

$0 \quad 0 \quad \ldots \quad 0 \quad x \quad x \quad \ldots \quad x$

II. Second phase (Reordering the rows)

1. Interchange the rows so that all the zero rows lie below the nonzero rows. Define integers $k_1, k_2, \ldots, k_r$ as follows: k_i is the location of the first non-zero entry of the i-th row.
2. The rows are ordered such that $k_1 < k_2 < \ldots < k_r$

The resulting matrix $\overline{A}$ (which is row equivalent to A) has the following properties:

i. Rows $r+1, r+2, \ldots, m$ have only zero elements.
ii. All elements to the left of (i, k_i) are zero for all $i = 1,2,\ldots,r$.
iii. All elements above (i, k_i) are zero for all $i = 1,2,\ldots,r$.

Define the index set $K = \{k_1, k_2, \ldots k_r\}$, where $1 \leq k_1 < k_2 < \ldots < k_r \leq n$.
Define also the index set $L = \{1,2,\ldots n\} - K$ as $L = \{l_1, l_2, \ldots l_{n-r}\}$
So, L is the set of integers from 1 to n which are not in K. Order l_i's such that $1 \leq l_1 < l_2 < \ldots < l_{n-r} \leq n$. In other words $K \cup L = \{1,2,\ldots n\}$.

III. The basis for $N(A)$

Solve the system of equations $\bar{A}x = 0$ $n-r$ times, each time by setting successively all components of x with indices in L equal to zero except one component which is set equal to one. Then, solve for the components of x with indices in K.

Let x_{l_i} be the solution obtained when the l_i-th component is set equal to 1 and other l_j's are set to 0, i.e.

$$x_{l_i} = \begin{bmatrix} u_{1,l_i} \\ u_{2,l_i} \\ \vdots \\ u_{n,l_i} \end{bmatrix}, \text{ where } u_{l_i,l_i} = 1 \text{ and } u_{l_j,l_i} = 0 \text{ for } i \neq j.$$

and u_{k_j,l_i}'s are obtained from $Ax_{l_i} = 0$. In this way, $n-r$ solutions $\{x_{l_1}, x_{l_2}, \ldots x_{l_{n-r}}\}$ are obtained. This set is a basis for $N(\bar{A}) = N(A)$.

Example:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 0 & 1 & -4 \\ -1 & 5 & -3 & -3 \\ 3 & 7 & -2 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & -2 \\ 0 & 4 & -2 & -4 \\ 0 & 10 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -2 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \bar{A}$$

$k_1 = 1$, $k_2 = 2$ and $l_1 = 3$, $l_2 = 4$

$$x_{l_1} = x_3 = \begin{bmatrix} u_{1,l_1} \\ u_{2,l_1} \\ 1 \\ 0 \end{bmatrix} \qquad Ax_{l_1} = 0 \qquad x_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$x_{l_2} = x_4 = \begin{bmatrix} u_{1,l_2} \\ u_{2,l_2} \\ 0 \\ 1 \end{bmatrix} \qquad Ax_{l_2} = 0 \qquad x_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } N(\bar{A}) = N(A)$$