MIDDLE EAST TECHNICAL UNIVERSITY

EE 501 Linear Systems Theory

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Field

Definition

A field is a set F, together with two mappings of $F \times F \to F$, called addition and multiplication, written as $(a, b) \to a + b$ and $(a, b) \to ab$ respectively with the following properties:

Addition:

- (A1) $a + b = b + a$ for all $a, b \in F$ (commutativity)
- (A2) $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$ (associativity)
- (A3) There is an element in F, denoted by 0_F, such that $a + 0_F = a$ $\forall a \in F$ (additive identity)
- (A4) For each $a \in F$ there is an element in F, denoted by $-a$, such that $a + (-a) = 0_F$ (additive inverse)

Field continued..

Multiplication:

- (M1) $ab = ba$ for all $a, b \in F$ (commutativity)
- (M2) $a(bc) = (ab)c$ for all $a, b, c \in F$ (associativity)
- (M3) There is an element in F, denoted by 1_F, such that $a1_F = a$ $\forall a \in F$ (multiplicative identity)
- (M4) For each $a \neq 0_F$ there is an element in F, denoted by a^{-1}, such that $aa^{-1} = 1_F$ (multiplicative inverse)

(D1) $a(b + c) = ab + ac \quad \forall a, b, c$ (distributive law).

Field continued..

Example

Set of real numbers $\mathbb{R}$ with standard addition and multiplication.

Example

Set of binary numbers with modulo 2 addition and multiplication.

$F = \{0, 1\}$

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Field continued..

Example

Let $F = \mathbb{R} \times \mathbb{R}$. Let us define $+$ and $\cdot$ as:
$x + y := (x_1 + y_1, x_2 + y_2)$,
$x \cdot y := (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$,
where $x = (x_1, x_2) \in F, y = (y_1, y_2) \in F$.

Note that this is nothing but complex number field $\mathbb{C}$. Then $0_F = (0, 0)$ and $1_F = (1, 0)$.

Exercise

Let $F = (0, \infty) = \mathbb{R}_+$ (positive real numbers) Given $x + y := xy$, $x \cdot y := e^{\ln(x)\ln(y)}$, show that F satisfies the axioms of field. Find 1_F and 0_F.

Question

Are polynomials a field? Are matrices a field?

Definition

A linear space V is a set, whose elements are called vectors associated with a field F, whose elements are called scalars. Vectors can be added and they can be multiplied by scalars. These operations satisfy the following properties:

Vector addition: $x + y, \quad + : V \times V \to V$

(A1) $x + y = y + x \quad \forall x, y \in V$ (commutativity)

(A2) $x + (y + z) = (x + y) + z \quad \forall x, y, z \in V$ (associativity)

(A3) $x + 0 = x \quad \forall x \in V$ (additive identity)

(A4) $x + (-x) = 0 \quad \forall x \in V$ (additive inverse)

Scalar multiplication: $ax, \quad \cdot : F \times V \to V$

(M1) $a(bx) = (ab)x$ for all $a, b \in F$, $x \in V$ (associativity)

(M2) $a(x + y) = ax + ay$ for all $a \in F$, $x, y \in V$ (distributive)

(M3) $(a + b)x = ax + bx$ for all $a, b \in F$, $x \in V$ (distributive)

(M4) $1x = x$ (unit rule)

Example

Show that $0x = 0$

Example

Set of all vectors $(a_1, a_2, \ldots, a_n)$ with $a_i \in F$. Addition, multiplication are defined componentwise. This space is denoted as F^n. Let $x, y \in F^n$
$x = (a_1, a_2, \ldots, a_n)$, $y = (b_1, b_2, \ldots, b_n)$
Addition: $x + y := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
Multiplication: $cx := (ca_1, ca_2, \ldots, ca_n)$
Most common examples are $\mathbb{R}^n$ and $\mathbb{C}^n$.

Linear Spaces

Example

Set of all real valued functions $t \to f(t)$ defined on the real line $F = \mathbb{R}$.

Example

Set of all polynomials with degree n with coefficients in F.

Example

Set of all polynomials with degree less than n with coefficients in F. Note that this linear space is a subset of the previous one for $F = \mathbb{R}$.

Definition

Let V be a linear space defined over field F, denoted by (V, F). A subset W of V is called a subspace if sums and scalar multiples of elements of W belong to W. That is,

(S1) $w_1 + w_2 \in W \quad \forall w_1, w_2 \in W$

(S2) $aw \in W \quad \forall w \in W$ and $\forall a \in F$

Remark

Subset has to be closed under addition and scalar multiplication. All other properties are inherited from the original linear space.

Example

linear space $V = \mathbb{R}^2$;
subspace $W = [a\ 0]^T : a \in \mathbb{R}$

Example

linear space $V = \mathbb{R}^2$;
subspace $W = [a\ 1]^T : a \in \mathbb{R}$

Example

linear space $V =$ set of all real valued functions $t \to f(t)$;
subspace $W_1 =$ set of all continuous functions,
subspace $W_2 =$ set of all functions periodic with π.

Remark

0 vector itself is a subspace and it is the smallest subspace.

Definition

The sum of two subsets Y and Z of a linear space X, denoted as $Y + Z$, is the set of all vectors of form $y + z$, $y \in Y$, $z \in Z$.

Example

Show that $Y + Z$ is a linear subspace of X if Y and Z are

Example

Prove that if Y and Z are subspaces of linear space X, so is their intersection $Y \cap Z$.

Example

If Y and Z are subspaces of linear space X, is their union $Y \cup Z$ a subspace?

Definition

Let (V, F) and (W, F) be two linear spaces defined over the same scalar field F. The product space of (V, F) and (W, F) is defined as

- $V \times W = \{(v, w) : v \in V, w \in W\}$
- $(v, w) + (x, y) := (v + x, w + y)$ (vector addition)
- $a(v, w) := (av, aw)$ (scalar multiplication)

Linear Spaces

Definition

A linear combination of n vectors $x_1, x_2, , \ldots, x_n$ of a linear space C is a vector of the form $a_1x_1, +a_2x_2+, \ldots, +a_nx_n,$ where a_i's are scalars in F.

Definition

The set of all linear combinations of $x_1, x_2, , \ldots, x_n$ is called the span of $\{x_1, x_2, , \ldots, x_n\}$; denoted by $sp\{x_1, x_2, , \ldots, x_n\}$.

Definition

Vectors $x_1, x_2, , \ldots, x_n$ in X are said to be linearly independent if $a_1x_1, +a_2x_2+, \ldots, +a_nx_n, = 0$ implies $a_i = 0, \forall i$. Otherwise, they are linearly dependent.

Example

i) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

ii) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

Example

Consider the linear space of polynomials with degree $n \leq 2$. Let subset S=$\{P_1, P_2, P_3\}$ be such that $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2, \quad \forall t$
Is this set linearly independent?

Example

S=$\{\cos(t), \sin(t), \cos(t - \pi/3)\}$

Definition

Let V be a linear space and (finite) set of vectors S=$\{x_1, \ldots, x_n\}$ be a subset of V. S is said to be a basis for V iff

- $Span(S) = V$
- S is a linearly independent set.

Definition

A (finite dimensional) linear space V has many bases. All these bases must have the same number of vectors. That number is called the dimension of V.

Example

i) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, ii) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

Definition

Ordered basis is a basis $(x_1, x_2, \ldots, x_n)$, where basis vectors are given in a specific ordering.

If $(x_1, x_2, \ldots, x_n)$ is an ordered basis of V and $y \in V$, then there is a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ such that $y = \sum_{i=1}^{n} a_i x_i$. Scalars $(a_1, a_2, \ldots, a_n)$ are called the components of y with respect to the ordered basis $(x_1, x_2, \ldots, x_n)$.

Example

With respect to some ordered basis $B_1 = (x_1, x_2)$ of $\mathbb{R}^2$, let the vectors y_1, y_2, y_3 be presented by $[y_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $[y_2]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[y_3]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. That is, $y_1 = 1x_1 + 1x_2$, $y_2 = 1x_1 + 0x_2$, $y_3 = 2x_1 + 3x_2$. Let our new basis be $B_2 = (y_1, y_2)$. Express y_3 w.r.t. this new basis.

Remark: For a given ordered basis, the representation of a vector is unique.

Normed linear spaces

Definition

Consider a linear space V over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. Let there be a function $x \to \|x\|$ that assigns to each $x \in V$, a nonnegative real number $\|x\| \in \mathbb{R} > 0$. Such function is called a **norm** if it satisfies the following properties.

(P1) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

(P2) $\|\alpha x\| = |\alpha| \, \|x\| \; \forall x \in V$ and $\alpha \in F$

(P3) $\|x\| = 0 \Leftrightarrow x = 0$

The expression "'$\|x\|$"' is read "'the norm of x"' and the function $\|.\|$ is said to be a norm on V.

The triplex $(V, F, \|.\|)$ is called a **normed space**.

Normed linear spaces

Norms can quantify **distance** between two points in our linear space.

The distance between $x_1, x_2 \in V$ is the norm of the vector $x_1 - x_2$ or $x_2 - x_1$: $\|(x_1 - x_2)\|$.

The norm of x, $\|x\|$ is the distance of x to the origin 0.

Now that we have a proper tool for measuring distance (norm), we can begin studying the “geometry” of the space (parallelism, orthogonality, area, volume, shape in general).

Example

Let $V = \mathbb{R}^2$, $F = \mathbb{R}$,

i) $\|x\|_1 := |\alpha_1| + |\alpha_2|$. Is $\|.\|_1$ a norm?

ii) $\|x\|_2 := (\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}}$. Is $\|.\|_2$ a norm?

iii) $\|x\|_\infty :=$ max$(|\alpha_1|, |\alpha_2|)$. Is $\|.\|_\infty$ a norm?

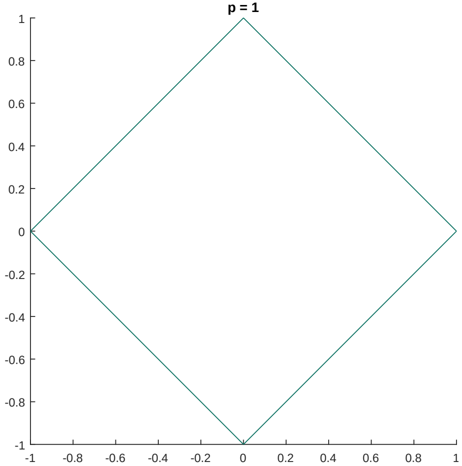
All these norms can be generalized into what we call a **'p-norm'**.

$$\|x\| := (|\alpha_1|^p + |\alpha_2|^p)^{\frac{1}{p}}$$

.

Note that $\lim_{p - \to \infty} \|x\|_p = \|x\|_\infty$.

p-norm:



Normed linear spaces

Examples: On lecture notes...

Example

Let $V = \mathbb{R}^{n\times m}$, and $A = [a_{ij}]$,

$$\|A\| = \max_{i,j} |a_{ij}| \text{ is a norm.}$$

Example

Let $V = \mathbb{R}^{n\times m}$, and $A = [a_{ij}]$,

$$\|A\| = \max_{i} \sum_{j=1}^{n} |a_{ij}| \text{ (abs sum of rows)}$$

Exercise

Show that this is a norm.

Definition

$A : \mathbb{R}^n \to \mathbb{R}^m$ be an $m \times n$ matrix. Let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ denote the norms (vector norms) in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The **induced norm** of a matrix is defined as

$$\|A\| := \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_{\mathbb{R}^m}}{\|x\|_{\mathbb{R}^n}}.$$

Remark:

The induced matrix norm is defined in terms of vector norms. An equivalent definition is:

$$\|A\| := \max_{\|x\|=1} \|Ax\|.$$

Remark:

$$\begin{aligned} \|Ax\| &= \frac{\|Ax\|}{\|x\|}\|x\| \,(\text{ suppose } \ \|x\| \neq 0) \\ &\leq \left(\max_y \frac{\|Ay\|}{\|y\|}\right)\|x\| \\ &= \|A\|\,\|x\| \Rightarrow \|Ax\| \leq \|A\|\,\|x\| \end{aligned}$$

Furthermore, there exists a vector x^* such that $\|Ax^*\| = \|A\|\,\|x^*\|$ which may not be unique.

Example

Choose $\|.\|_2$ in $\mathbb{R}^n$ and $\mathbb{R}^m$,

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \sqrt{(Ax)^T Ax} = \max_{\|x\|=1} \sqrt{x^T A^T Ax}$$

Convergence

Definition

Let $(V, F, \|.\|)$ be a normed space. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of vectors in V. $v_n \in V \;\; n = 1, 2, \ldots$. The sequence is said to be **convergent** to the limit $\bar{v} \in V$ iff $\|v_n - \bar{v}\| \to 0$ as $n \to \infty$

Equivalently, given any $\epsilon > 0$, $\exists N$ $(N$ depends on $\epsilon)$ such that $n > N$ implies $\|v_n - \bar{v}\| \leq \epsilon$.

Remark:

A sequence that is not convergent is called **divergent**.

Example

Given $V = \mathbb{R}$ and $\|v\| = |v|$, consider the sequence $\left\{ \left(\frac{1}{2} \right)^n \right\}_{n=1}^{\infty}$.
The sequence is convergent to $v = 0$.

Example

Consider the sequence $\{(-1)^n\}_{n=1}^{\infty}$

Convergence

In most engineering applications, we are interested in the convergence of an iterative algorithm.

In general, we do not know where!

The given definition of the convergence requires the limiting element $\bar{v} \in V$, an element we may not know, to verify convergence.

There are ways to exclude "$\bar{v} - dependence$".

Cauchy sequence

Definition

Let $(V, F, \|.\|)$ be a normed space. A sequence $\{v_n\}_{n=1}^{\infty}$ in V is said to be a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N$(depending on ϵ) such that $\|v_n - v_m\| < \epsilon$ for all $n, m > N$.

Remark:

Every convergent sequence is a Cauchy sequence. The converse, in general, is not true.

Example

Consider the normed space $(\mathbb{Q}, \mathbb{Q}, |.|)$, (i.e., set of rational numbers over the field rational numbers with norm being the absolute value). Is the sequence $\left\{1 + \sum_{i=1}^{n} \frac{1}{i!}\right\}_{n=1}^{\infty}$ convergent?

Banach Space

Definition

A normed space is said to be **complete** if every Cauchy sequence is convergent. A complete normed space is called a **Banach Space**.

Example

"'A normed space that is not complete"'

Let $V = \{f | f : [-1, 1] \to \mathbb{R},$ f is continuous and $\int_{-1}^{1} |f(t)|dt < \infty\}$.

Define $\|f\|_1 := \int_{-1}^{1} |f(t)|dt$.

Now consider the sequence $\{f_n\}_{n=1}^{\infty}$ defined as follows:

Inner Product Space

An inner product space is a linear space with an additional structure called inner product.

Definition

Let V be a linear space over field F. An inner product is a map of the form $\langle \cdot, \cdot \rangle : V \times V \to F$ that satisfies the following three axioms for all vectors $x, , y, z \in V$ and all scalars $\alpha \in F$.

1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)
   - 2a) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ (linearity in the first argument)
   - 2b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (linearity in the first argument)
3) $\langle x, x \rangle \geq 0$ with equality only for $x = 0$ (positive defineteness)

Notice that

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

Example

$V = \mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

Example

$\dots$

Theorem

Cauchy-Schwarz inequality:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

We have studied

- Sets
- Linear Spaces
- Normed Linear Spaces
- Inner Product Spaces

Sets

Very general concept. We can perform:

- Define subsets
- Take unions, intersections, complements, set subtraction

Summary

Linear Spaces

We defined members of our sets as **vectors** and defined

- Vector addition
- Scalar multiplication

We obtained an **algebraic structure**, where we can

- perform algebraic operations,
- define linear combinations,
- define span, basis, etc. Find representation of vectors wrt to basis,
- define **linear transformations** between vector spaces.

Summary

Normed Linear Spaces

We defined **norms** to incorporate a **geometric structure** on top of the algebraic structure. We can calculate the **distance** between two members of the vector space: $\|x - y\|$ In a normed space we can,

- Define (measure) distance
- Analyse convergence of sequences

Normed spaces have a major shortcoming. The direction cannot be characterized. The direction, or rather **relative direction**, can be studied by the help of a tool we call the **inner product**.

Summary

Inner Product Spaces

Enhancement of the geometric structure of a normed space

Example

Let u an v be two unit vectors in $\mathbb{R}^n$

- $\langle u, v \rangle = 0$ if they are orthogonal
- $\langle u, v \rangle$ is maximum when u and v point in the same direction
- $\langle u, v \rangle$ is minimum when u and v point in the opposite direction

Linear Transformation

Definition

Let V and W be linear spaces over the same field F. A linear transformation T is a mapping $T : V \to W$ satisfying
$T(a_1x_1+a_2x_2) = a_1T(x_1)+a_2T(x_2) \qquad \forall a_1, a_2 \in F \quad and \quad \forall x_1, x_2 \in V$

Example

- $V = W$ polynomials of degree less than n in S; $T = \frac{d}{ds}$
- $V = W =$\{continuous functions of type $f : [0, 1] \to \mathbb{R}$\};
  $T_f = \int_0^1 f(s)ds$

Definition

Given linear transformation $T : V \to W$, the **null space** of T is the set of all $x \in V$ satisfying $Tx = 0_w$. That is,
$\mathcal{N}(T) := \{x \in V : Tx = 0\}$

Definition

Given linear transformation $T : V \to W$, the **range space** of T the set of all $w \in W$ satisfying $Tv = W$. That is
$\mathcal{R}(T) := \{w \in W : w = Tv \quad \text{for some} \quad v \in V\}$

Remark

For a linear transformation $T : V \to W$, $\mathcal{N}(T)$ is a linear subspace of V.

Remark

$\mathcal{R}(T)$ is a subspace of W.

Linear Transformation

Definition

A function $f : X \to Y$ is one-to-one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Theorem

Let $T : V \to W$ be a linear transformation. Then mapping T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.