Fall 2012

EE 503
Linear Algebra Review
(Not to be collected)

Most of these problems require few operations if you are comfortable with basic linear algebra methods. If you are having excessive difficulty, please consider a ***serious*** linear algebra refreshing; linear algebra is extensively used in all DSP courses. Strang's book is the standard and a good textbook for a review. You can also watch Prof. Strang's teaching Linear Algebra on the web. http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/VideoLectures/index.htm (This is the first link coming up on a google search with the keyword "ocw linear algebra strang")

Reading Assignment: Section 2.3 from Hayes.

1. Show that inverse of a lower triangular matrix is also lower triangular.
2. Show that any matrix **A** can be expressed as **A=LU,** where **L** is a lower triangular matrix and **U** is upper triangular. (Hint: Gaussian elimination)
3. Show that the linear equation system **LUx=b**, where **L** and **U** are lower and upper triangular matrices respectively, can be solved in two steps: Step 1: **Ly=b**; Step 2: **Ux=y.** Show that both steps can be solved by successive substitution.
4. Show that any matrix **A** expressed as **A=QR,** where **Q** is an orthogonal matrix and **R** is upper triangular. (Hint: Gram-Schmid orthogonalization)
5. Show that the linear equation system **QRx=b** can be solved by $\mathbf{Rx=Q^Tb}$ using successive substitution.
6. Show that $tr\{\boldsymbol{A}\}=\sum \lambda_k$ where λ_k is the *k*'th eigenvalue of matrix **A.**
7. Show $\det(A)=\prod \lambda_k$ where λ_k is the *k*'th eigenvalue of matrix **A.**
8. Show that similar matrices have the same eigenvalues. **A** and **B** are similar if there is an **M** matrix such that $\mathbf{A=MBM^{-1}}$**.** (Hint: Write characteristic equation)
9. Show that orthogonal (or unitary) matrices have unit magnitude eigenvalues.
10. Show that symmetric matrices have orthogonal eigenvectors.
11. Use $|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|$ to establish $|\mathbf{I+ABA^{-1}}|=|\mathbf{B+I}|$. **(|A|=det(A))**
12. Establish the identity tr{**AB**}=tr{**BA**}. (Hint: Use summation definition of matrix multiplication).
13. Establish the identity $|\mathbf{I+AB}|=|\mathbf{I+BA}|$. (Not very easy)
14. Show that distinct eigenvectors of a matrix are linearly independent.
15. Show that singular matrices have at least one eigenvalue with the value zero.
16. Show that an eigenvector of **A** with zero eigenvalue is orthogonal to the eigenvectors of $\mathbf{A^T}$ with non-zero eigenvalues. (Circuit theory fans can associate this result with the Tellegen's theorem!)
17. Show that eigenvectors of a matrix with zero eigenvalue form the null space, and eigenvectors with non-zero eigenvalue form the range space.
18. Show that for a $N \ x \ N$ **A** matrix, dim(range(**A**))+ dim(null(**A**)) = N, i.e. dimension of range space plus the dimension of null space is N.
19. Show that range space is orthogonal to null space for *symmetric* matrices. (Hint: use 16, 17 or 10)

20. Show that matrices with the same set of eigenvectors but with different eigenvalues commute. (Two matrices are said to commute if **AB=BA)**
21. Show **AB=BA** then **A** and **B** have a common set of eigenvectors. (This is much more difficult than 20)
22. Show that a rank-1 matrix has a single eigenvector with non-zero eigenvalue.
23. Show that **A+ αI** has the eigenvalues of λ_k + **α,** where λ_k is the *k*'th eigenvalue of matrix **A.**
24. Show that if $\mathbf{AA^H}$ has eigenvectors shown with $\mathbf{e_k}$, then $\mathbf{A^H e_k}$ is an eigenvector of $\mathbf{A^H A}$.
25. Show that $\mathbf{AA^H}$ and $\mathbf{A^H A}$ have the same set of eigenvalues. The eigenvalues of $\mathbf{AA^H}$ or $\mathbf{A^H A}$ called singular values.
26. A square or rectangular matrix can be decomposed into singular value decomposition (SVD) as follows: $\mathbf{A} = \mathbf{U}\sum\mathbf{V^H}$. Here **A** is a $M \ x\ N$ matrix, **U** and **V** are unitary matrices of dimensions $M \ x\ M$ and $N \ x\ N$ respectively and ∑ is a diagonal matrix. The diagonal entries of ∑ are called the singular values. What is the relation between $\mathbf{A^H}$ **U** and **V** ? (According to Strang (see his OCW lecture on SVD), SVD is the single most important linear algebra result.)
27. A lesser known matrix multiplication matrix method is the summation of rank-1 matrices. Show that if **C=AB**, then $\mathbf{C} = \sum a_k^c b_k^r$. Here a_k^c is the k'th column of matrix **A;** b_k^r is the k'th row of matrix **B.**
28. Show that $\mathbf{A=M\,\Lambda\,M^H}$ where $\mathbf{M=[m_1\ m_2 \ldots\ m_N]}$ ($\mathbf{m_k}$ represent k'th column of matrix **M), Λ=diag(**$\lambda_1, \lambda_2, \ldots, \lambda_N$**)** is **A=** $\sum_{k=1}^{N} \lambda_k m_k m_k^H$. (Hint: Use 26)
29. Show that if $\mathbf{A=M\,\Lambda\,M^H}$ then $A^{-1} = \sum_{k=1}^{N} \frac{1}{\lambda_k} m_k m_k^H$ (Hint: Use 27)
30. Show that if *f(x)* is a polynomial in, and $\mathbf{A=M\,\Lambda\,M^{-1}}$, **Λ=diag(**$\lambda_1, \lambda_2, \ldots, \lambda_N$**) t**hen f(**A)=M f(Λ)** $\mathbf{M^{-1}}$ where **f(Λ) =diag(f(**λ_1**), f(**λ_2**),…, f(**λ_N**)).**
31. Show that a matrix **A** satisfies its own characteristic equation. (This result is known as the Cayley-Hamilton theorem.)
32. Solve Hayes 2.4
33. Solve Hayes 2.15
34. Solve Hayes 2.17
35. If **P** is an orthogonal projector and α_1, α_2 are non-zero real numbers; then $(\alpha_1)^2$**P** + $(\alpha_2)^2$ **(I-P)** is invertible.
36. If **S** is real and skew-symmetric, then show that **I+S** is non-singular and the Cayley transform $\mathbf{T=(I-S)(I+S)^{-1}}$ is orthogonal.
37. If **T** is real orthogonal matrix and **(I+T)** is non-singular, prove that we can write $\mathbf{T=(I-S)(I+S)^{-1}}$ where **S** is a skew-symmetric matrix.
38. A matrix **A** is called positive definite, if $\mathbf{x^H A x} > 0$ for all **x.** Show that this definition requires **A** matrix to have positive eigenvalues.
39. Show that a matrix has to have positive terms on its diagonal to be positive definite. Is this condition necessary or sufficient?
40. Let $\mathbf{A} = \mathbf{A_s}+\mathbf{A_{ss}}$ where $\mathbf{A_s}$ is a symmetric matrix ($\mathbf{A_s} = \mathbf{A_s^T}$) and $\mathbf{A_{ss}}$ is a skew-symmetric matrix ($\mathbf{A_{ss}} = -\mathbf{A_{ss}^T}$), then show that $\mathbf{x^T A_{ss} x}=0$ and the positive definiteness of the matrix is determined by the its "symmetric part" $\mathbf{A_s}$.