EE 503
Homework 3
Due: Dec. 13, 2011

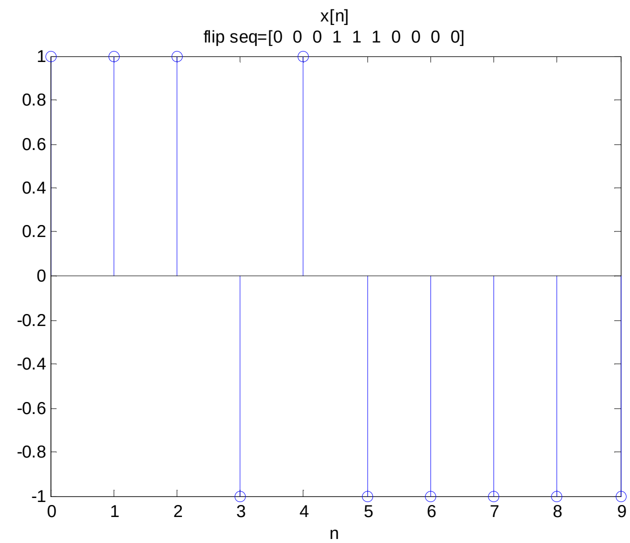
The process x[n] is defined as follows:

i. x[0] is 1 or -1 with probability $\frac{1}{2}$.

ii. $x[n] = \begin{cases} x[n-1] & \text{with prob. } \alpha \\ -x[n-1] & \text{with prob.} (1-\alpha) \end{cases}$

The realizations of the process can be generated as follows:

```
N=10;alpha=0.85;
% prob{ not flipping } = alpha
flip=floor(rand(1,N)./alpha);flip(flip>1)=1;
%flip = 1 ---> flip;
%flip = 0 ---> do not flip
x=filter(1,[1 -1],flip,1);
x=2*rem(x,2)-1;
initval = (-1)^round(rand(1));
x=initval*x;
stem(0:N-1,x);
xlabel('n');
title([ 'x[n]' char(10) ...
        'flip seq=[' num2str(flip) ']']);
```



1. Show that $p\{x[n+k]=1 \mid x[n]=1\} = \frac{1+(2\alpha-1)^k}{2}$, where k>0. (Hint: You can describe the process as a Markov Chain. If you are not familiar with Markov chains (a topic of EE531), then apply induction to show the result.)
2. Is the process strict sense stationary (SSS) in the first order, second order? (considering x[n] for $n \geq 0$.) Is the process SSS for all orders?
3. Analytically calculate the mean and auto-correlation of the process x[n]. Is the process WSS?
4. **Computer Experiment:**
   a. Generate 3 different realizations of process x[n] and plot the realizations in the same figure using the subplot command. Set α=0.85.
   b. **Mean Estimation**
      i. **Ensemble Average:** Set α=0.85 and generate a realization of the process, $x_\zeta[n]$ and sample the process at time instants n={150,200,250}. Repeat the same sampling for 100 realizations and calculate $\hat{\mu}[n] = \frac{1}{100}\sum_{K=1}^{100} x_{\zeta_K}[n]$, where $x_{\zeta_K}[n]$ is the value at the n'th instant of the K'th realization
      ii. **Time Average:** Estimate the mean of the process from a single realization using the estimator, $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} x[n]$. Try N={10,20,50,100} and α=0.85. Repeat the mean estimation procedure for 4 different realizations and present $\hat{\mu}$ in a table.

Repeat the mean estimation procedure for 100 different realizations and calculate the average square error on μ (that is $\frac{1}{100}\sum_{k=1}^{100}(\hat{\mu}-\mu)^2$ where μ is the true mean of process x[n]) and present the results in a table.

Comment on your results.

c. **Auto-correlation Estimation:**

i. **Ensemble Average:** Generate a realization of the process, $x_\zeta[n]$ and evaluate $x_\zeta[n]x_\zeta[n-k], \quad k=\{1,2,3\}$ at the instants n={150,200,250}. Set α=0.85. Repeat the same sampling for 100 realizations and calculate $\hat{r}_x[k]=\frac{1}{100}\sum_{K=1}^{100}x_{\zeta_K}[n]x_{\zeta_K}[n-k]$, $k=\{1,2,3\}$. Report your results in a table.

ii. **Time Average:** Estimate the first three auto-correlation lags of the process from a single realization, i.e. $\hat{r}_x[k]=\frac{1}{N}\sum_{n=1}^{N}x[n]x[n-k], \quad k=\{1,2,3\}$ for N={10,20,50,100} and α=0.85. Repeat the procedure for 4 different realizations and present $\hat{r}_x[k]$ for each realization in a table. Repeat the procedure for 100 different realizations and present the average square error on $r_x[k]$ that is $\frac{1}{100}\sum_{k=1}^{100}(\hat{r}_x[k]-r_x[k])^2$ in a table.

Comment on your results.

5. The process x[n] is filtered with $H(z)=\frac{1}{2}(1+z^{-1})$. The output of this filter is called as $x_2[n]$. Using the results of Question 3, analytically find the mean and auto-correlation of the process $x_2[n]$.
6. Find the pdf of $x_2[n]$. Find the joint pdf of $x_2[n]$ and $x_2[n-1]$. Evaluate $E\{x_2[n]x_2[n-1]\}$ using the joint pdf and compare your result with the first auto-correlation lag found in Question 5.
7. **Computer Experiment:**
   a. Using α=0.85, generate $x_2[n]$ and estimate its mean and first 3 auto-correlation lags (as discussed previously) using a sample length N={20,50,100}. Repeat the same procedure for 100 realizations and report the estimation error variance for N={20,50,100}.
   b. Using α=0.35, generate $x_2[n]$ and estimate its mean and first 3 auto-correlation lags (as discussed previously) using a sample length N={20,50,100}. Repeat the same procedure 100 realizations and report the estimation error variance for N={20,50,100}.
8. Determine the type of the process for $x_2[n]$ (MA, AR, ARMA, Periodic Process).
9. Find an LTI filter generating a Gaussian process having an auto-correlation and mean values identical to the ones of $x_2[n]$ for α=0.85. (Hint: Spectral Factorization) Experimentally verify first few lags of the auto-correlation and present 4 realizations as in Question 4a.

<u>Reading Assignments:</u>

1) You can examine Therrien page 99 (and forward) for an introduction to the Markov chains.
2) The process x[n] is closely related to well known random telegraph signal. A description of this process is given on page 291 of Papoulis. The process x[n] can be considered as the sampled version of the random telegraph signal.
3) As shown in this homework, the time average of a single realization can be identical to the ensemble average for some processes. This feature is called ergodicity. Ergodicity is discussed at a later time in the course. Interested students can examine the ergodicity discussion given in Hayes.