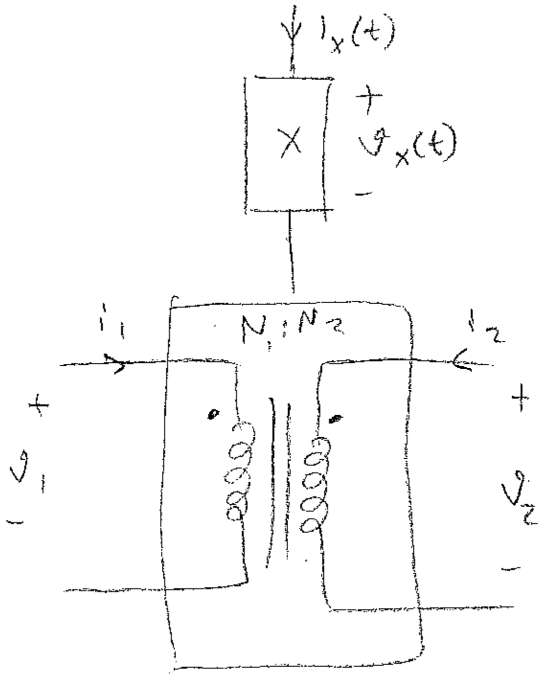
EE 202

EE 201 Review

Components: R, L, C, diode, op-amp, transformer etc.



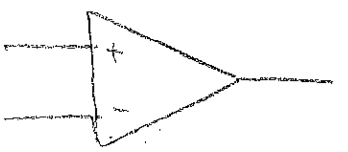
$$\boxed{i_c(t)=C\frac{d}{dt}v_c(t)}$$

capacitance

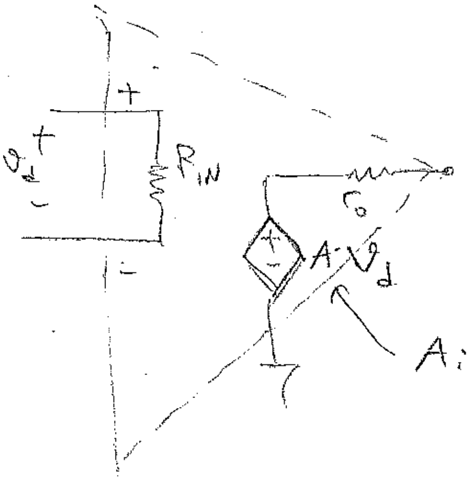
$$\boxed{v_c(t)=v_c(0^-)+\frac{1}{C}\int_{0^-}^{t}i_c(\tau)d\tau}$$

Tau

$$\frac{v_1}{v_2}=\frac{N_1}{N_2}\ ;\ \frac{i_1}{i_2}=\frac{-N_2}{N_1}$$

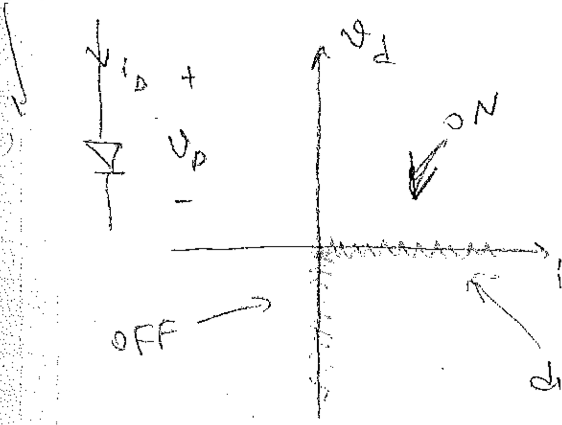


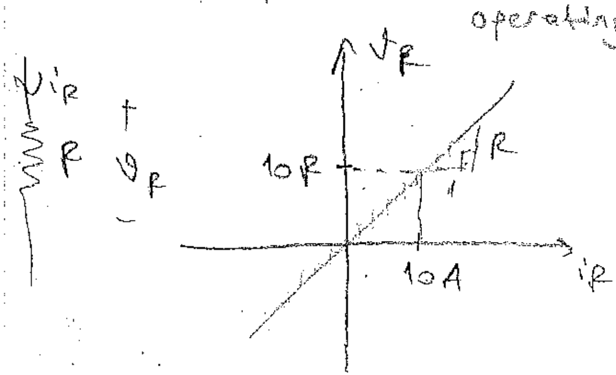
→ Op-amp → Amplifier



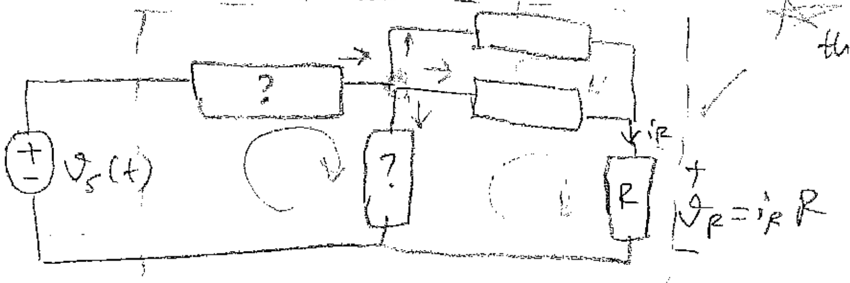
A: open-loop gain

ideal diode





Circuits as Network of Components

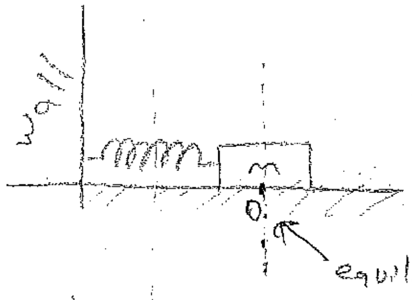


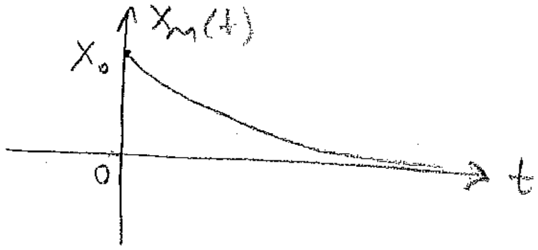
Don't forget the passive sign convention

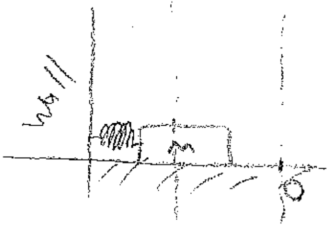
External input: microphone, output of another system, thermometer reader...

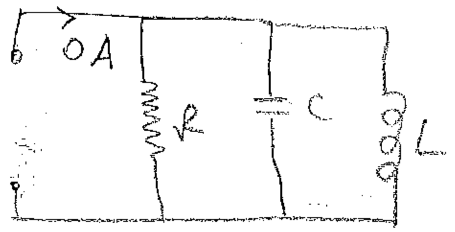
Analysis problems: we need to satisfy

1. Component equation (terminal equation) of every branch.
2. KVL - (Energy conservation)
3. KCL (Charge conservation)



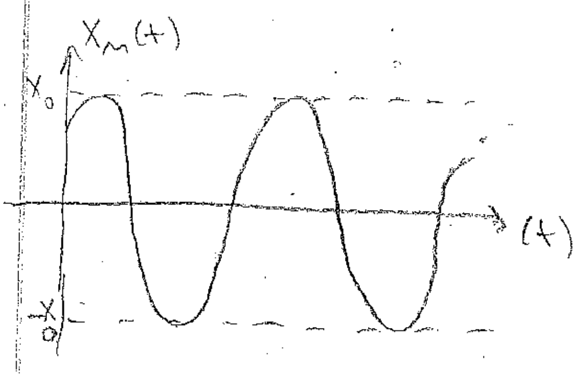






$i_L(0^-) = I_0$

$v_c(0^-) = V_0$



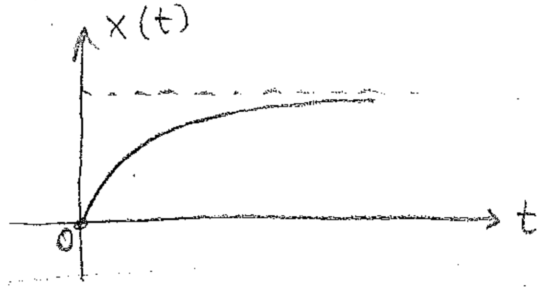
$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) v_c(t) = \frac{D}{LC}\{v_s(t)\}$$

$F_{net} = m a$

$M \frac{dx(t)}{dt}$

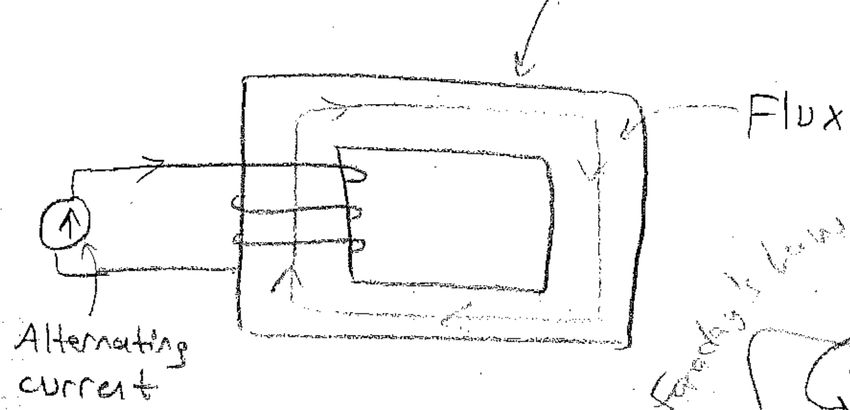
$$-F_{spring} + F_{ext} - F_{fric} = m \cdot \frac{d^2 x(t)}{dt^2}$$

$k \cdot x(t)$



Mutual Inductors

Iron core (μ: permeability is high)



flux

$$\phi(t) = L i_L(t)$$

$$\frac{d}{dt}$$

Faraday's law

$$\dot{\phi}(t) = \frac{d}{dt}\left(L(t) i_L(t)\right)$$

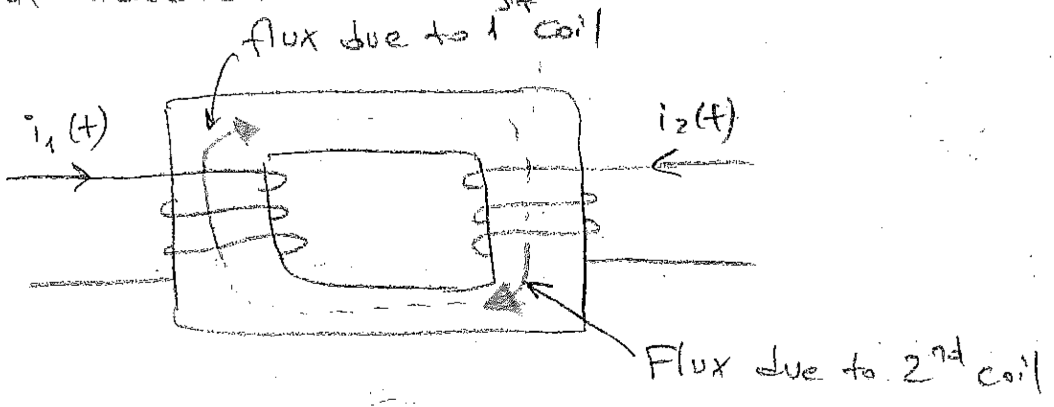
$$\rightarrow v_L(t) = \left(\frac{d}{dt} L(t)\right) i_L(t) + L(t) \frac{d}{dt} i_L(t)$$

Special case: LTI inductor, $L(t) = L$

$$\boxed{v_L(t) = L \frac{d}{dt} i_L(t)}$$

← Self-inductor: (classical inductor definition)

Mutual Inductor:



$$\phi_1 = L_1 \cdot i_1(t) + M i_2(t)$$

Total flux intercepted by 1st coil

Self inductance (H)

Mutual inductance (H)

the same number

$$\phi_2(t) = M \cdot i_1(t) + L_2 i_2(t)$$

Total flux intercepted by 2nd coil

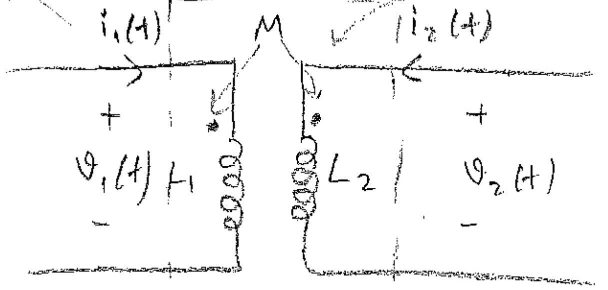
$$\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

$\frac{d}{dt}$

$$\begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_1(t) \\ \frac{d}{dt} i_2(t) \end{bmatrix}$$ ← Mutual inductor terminal equation

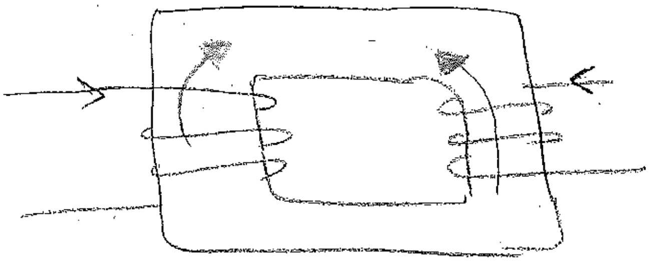
Not a transformer !!!

it is mutual inductors



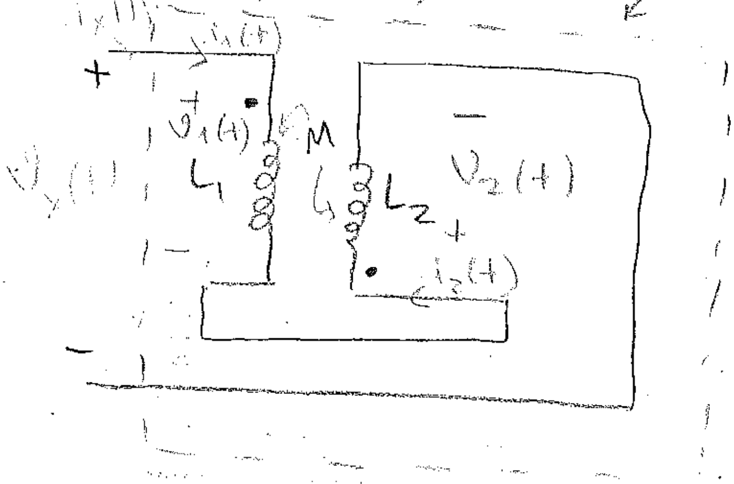
$$\begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_{L_1}(t) \\ \frac{d}{dt} i_{L_2}(t) \end{bmatrix}$$

Positive polarity of v_{L_1} and v_{L_2} is alligned with dot and $i_{L_1}(t)$, $i_{L_2}(t)$ are entering to the dot.



Ex: (Series combination) ← Passive sign convention!!

M: mutual inductance



$$\begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_1(t) \\ \frac{d}{dt} i_2(t) \end{bmatrix} \quad (*)$$

Note! ① $v_x(t) = v_{L_1}(t) + v_{L_2}(t)$

② $i_x(t) = i_1(t)$

$i_x(t) = i_2(t)$

Then multiply from left of (*) by [1.1]

$$\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_{L_1}\\ v_{L_2}\end{bmatrix} = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}\frac{d}{dt}i_x(t)\\ \frac{d}{dt}i_x(t)\end{bmatrix}$$

$i_1(t)$

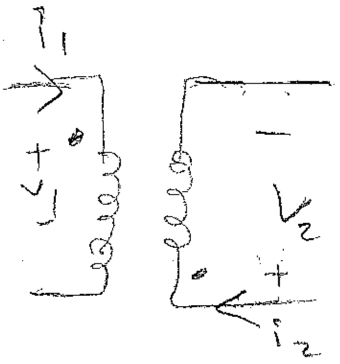
$i_2(t)$

The 1-port is equivalent to an inductor with

(L_1+L_2+2M) theory → ???

$$v_x(t) = (L_1+L_2+2M)\frac{d}{dt}i_x(t)$$

$$\underbrace{v_{L_1}+v_{L_2}}_{v_x(t)} = (L_1+L_2+2M)\frac{d}{dt}i_x(t)$$



$$\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} = \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}i_1(t)\\ i_2(t)\end{bmatrix}$$

- If the dot were on the other side:

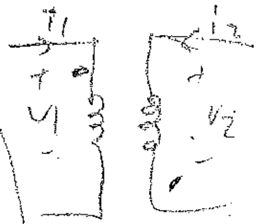
$$v_x(t) = v_{L_1}(t) - v_{L_2}(t)$$

$$i_x(t) = -i_2(t)$$

$\vdots$

$$v_x(t) = (L_1+L_2-2M)\frac{d}{dt}i_x(t)$$

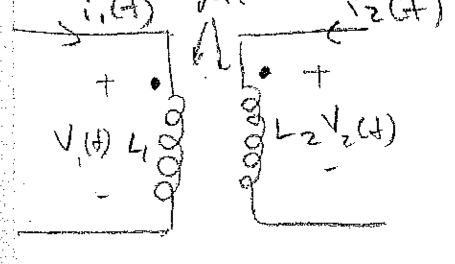
$$\underbrace{v_{L_1}+v_{L_2}}_{v_x(t)} = (L_1+L_2-2M)\frac{d}{dt}i_x(t)$$

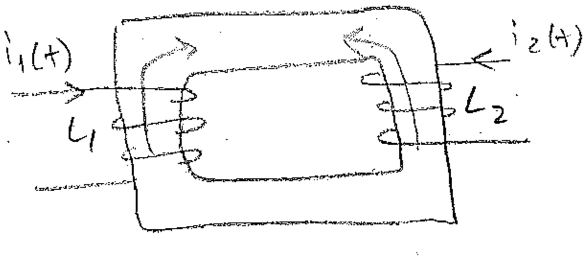


$$\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}L_1 & -M\\ -M & L_2\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix}$$

Tekrar

Mutual Inductors





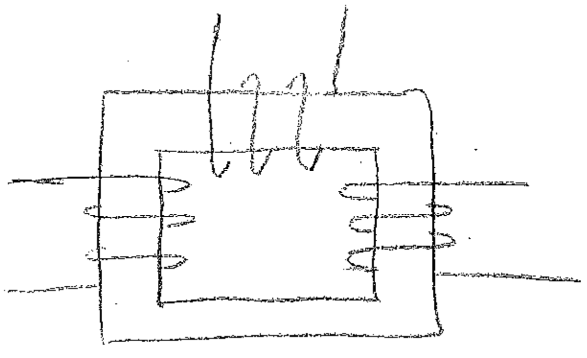
Circuit symbol

The dots are related to the winding direction: If the dots are on the same side the fluxes are constructive ($L_2 = L_1 \ldots + M \ldots$)
If dots are on opposite direction the fluxes are destructive ($L_2 = L_1 - \ldots - M \ldots$)

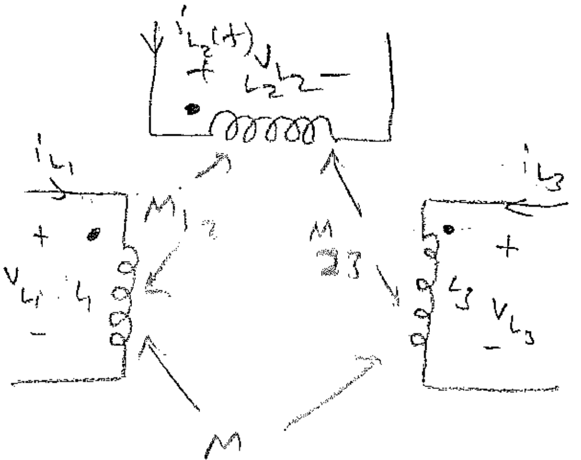
$$\hookrightarrow \begin{bmatrix} V_{L_1}(t) \\ V_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_{L_1}(t) \\ \frac{d}{dt} i_{L_2}(t) \end{bmatrix}$$

↳ only valid if incoming currents are coming to the dots.

3-Port Mutual Inductance



3 coils

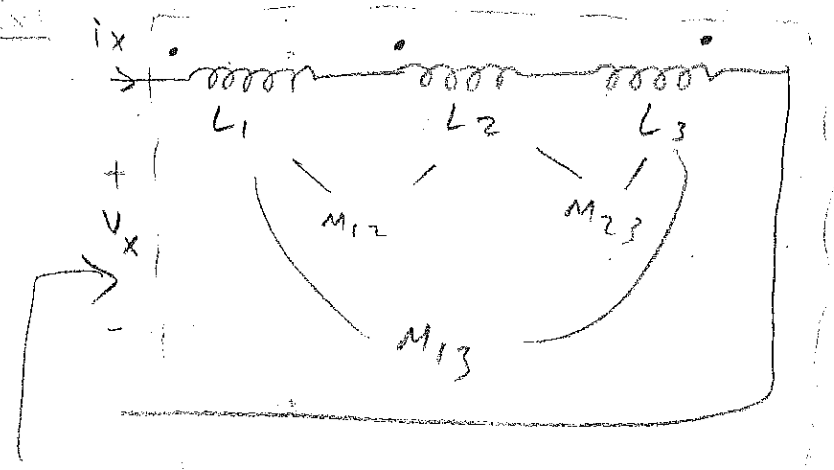


$$\begin{bmatrix} V_{L_1}(t) \\ V_{L_2}(t) \\ V_{L_3}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} L_1 & \overset{\text{equal}}{M_{12}} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix}}_{\underline{\underline{L}}} \begin{bmatrix} \frac{d}{dt} i_{L_1}(t) \\ \frac{d}{dt} i_{L_2}(t) \\ \frac{d}{dt} i_{L_3}(t) \end{bmatrix}$$

$\underline{\underline{L}}$: symmetric matrix

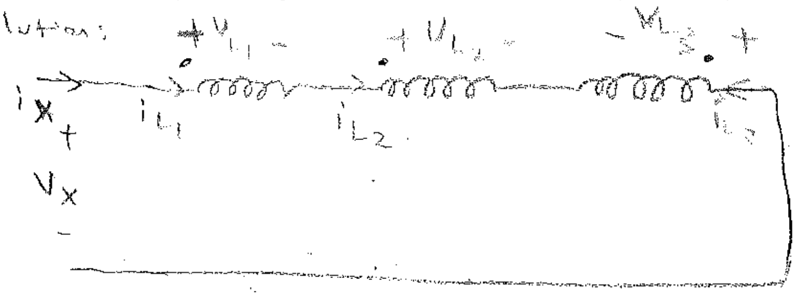
$(\underline{\underline{L}}^T = \underline{\underline{L}})$

Ex:



What is the relation between V_x and i_x?

Solution:



$$v_x = -v_{L_3} + v_{L_2} + v_{L_1}$$

$$= \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_{L_1} \\ v_{L_2} \\ v_{L_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \underline{\underline{L}} \underbrace{\begin{bmatrix} \frac{d}{dt} i_{L_1}(t) \\ \frac{d}{dt} i_{L_2}(t) \\ \frac{d}{dt} i_{L_3}(t) \end{bmatrix}}_{\text{Mutual inductance terminal equations}}$$

i_x

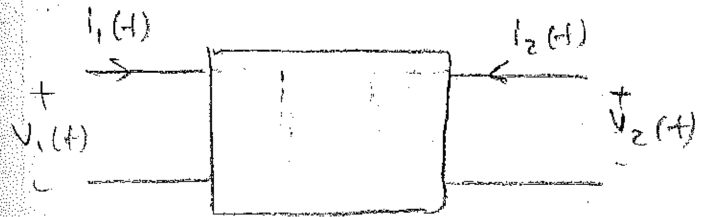
$$v_x = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \underline{\underline{L}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{d}{dt} i_x(t)$$

$$v_x = \Big[\underbrace{(L_1 + M_{21} - M_{31}) + (L_2 + M_{12} - M_{32}) + (L_3 - M_{13} - M_{23})}_{\text{equivalent inductance value}} \Big] \frac{d}{dt} i_x(t)$$

☆ If all mutual inductance $(M_{12}, M_{13}, M_{23} \ldots)$ are 0 the answer must be equal to!

$(L_1 + L_2 + L_3) \ldots$

Power and Energy Relations



General formula for 2-ports

$$P(t) = i_1(t) v_1(t) + i_2(t) v_2(t)$$

$$= \underbrace{\begin{bmatrix} i_1(t) & i_2(t) \end{bmatrix}}_{\underline{i}(t)} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}}_{\underline{v}(t)}$$

$$= \underline{i}(t)^T \underline{v}(t)$$

Note: $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\underline{\underline{A}} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

Under bar denotes a column vector

matrix

For mutual inductor!

$$P = (\underline{i}(t))^T \underline{v}(t) \leftarrow \text{terminal equation for mutual inductors}$$

$$= \underline{i}(t)^T \cdot \underline{\underline{L}} \frac{d}{dt} \underline{i}(t)$$

$$\underline{\underline{L}} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \text{ case: } \quad P(t) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_1 \\ \frac{d}{dt} i_2 \end{bmatrix}$$

$$P(t) = L_1 i_1(t) \frac{d}{dt} i_1(t) + L_2 i_2(t) \frac{d}{dt} i_2(t)$$
$$+ M i_1(t) \frac{d}{dt} i_2(t) + M i_2 \frac{d}{dt} i_1(t)$$

$$P(t) = L_1 i_1(t) \frac{d}{dt} i_1(t) + L_2 i_2(t) \frac{d}{dt} i_2(t) + M \frac{d}{dt} (i_1 i_2)$$

Energy: work $= \Delta E = \int_{-\infty}^{t} p(\tau)\,d\tau$ $\quad p(t) = \frac{d}{dt}E(t)$

Rate of change of $E(t)$

Watts

Joule/sec

Case of $L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$

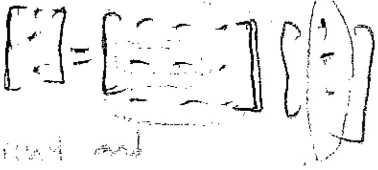
$$E(t) = \int_{-\infty}^{t} p(\tau)\,d\tau) = \frac{1}{2}L_1\,(i_{L_1}(t))^2 + \frac{1}{2}L_2\,(i_{L_2}(t))^2 + M i_1(t)\,i_2(t)$$

at $-\infty$ we assume the energy is 0 (zero)

Joules

General case for $N \times N$ $\underline{\underline{L}}$ matrix:

$$\boxed{E(t) = \frac{1}{2}\,(\underline{i}(t))^T\,\underline{\underline{L}}\,\underline{i}(t) \text{ Joules}}$$



Properties of $\underline{\underline{L}}$ matrix, coupling coefficient and [illegible] coupled inductors)

① Mutual inductor is a <u>passive</u> component.

$$\downarrow$$

$$E(t)_{\text{mut ind.}} \geq 0 \quad , \forall(t)$$

$$\frac{1}{2}\,(\underline{i}(t))^T\,\underline{\underline{L}}\,i(t) \geq 0 \; , \; \forall(t) \text{ and } \forall\,\underline{i}(t) \text{ vector function.}$$

In mathematics,

If $\underline{x}^T \underline{\underline{A}}\,\underline{x} \geq 0 \;\; \underline{\forall}_x$, condition is satisfied

$\xrightarrow{\quad}$ then $\underline{\underline{A}}$ matrix is called <u>positive semi definite</u> matrix.

Then, $\underline{\underline{L}}$ is a positive semi definite matrix.

For 2×2 L matrices, that is $\underline{\underline{L}} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$,

positive semi-definiteness is equivalent to
① $L_1 \geq 0$
② $L_2 \geq 0$
③ $\det(\underline{\underline{L}}) \geq 0$

these 3 conditions are satisfied.

$$\rightarrow L_1 L_2 - M^2 \geq 0$$

$$\boxed{M \leq \sqrt{L_1 L_2}}$$

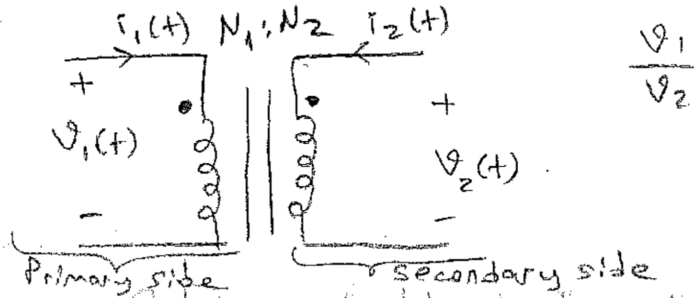
Coupling coefficient : $\boxed{k = \frac{M}{\sqrt{L_1 L_2}}}$, clearly $0 \leq k \leq 1$

Coupling coefficient

Mutual inductors are said to be fully coupled if $\boxed{k=1}$.

Ideal Transformers

Ideal transformers are mutual inductors with fully coupled inductors $(k=1)$ and there is no ohmic loss in the ideal transformer.



$$\frac{\vartheta_1}{\vartheta_2} = \frac{N_1}{N_2} \qquad \frac{i_1}{i_2} = \frac{-N_2}{N_1}$$

Power Relation of Ideal Transformer

$$\boxed{P(t) = \sum_{k=1}^{2} i_k(t)\,\vartheta_k(t)} = i_1\vartheta_1 + i_2\vartheta_2$$

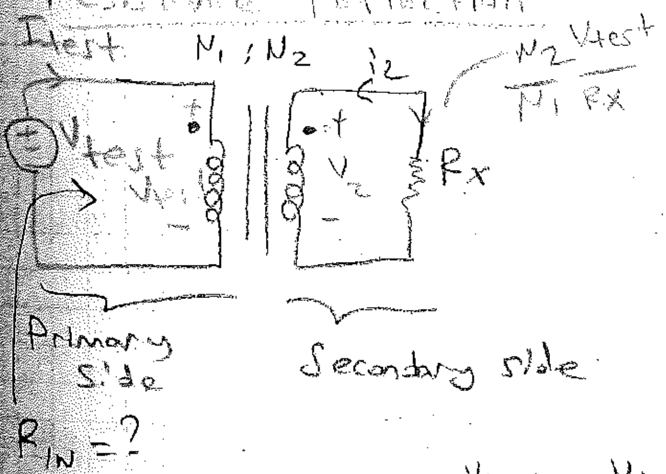
$$= i_1\vartheta_1 + \left(-\frac{N_1}{N_2} i_1\right)\left(\frac{N_2}{N_1}\vartheta_1\right)$$

$$= 0$$

$\boxed{P(t) = 0 \text{ Watts}}$

for ideal transformer!

Resistance Reflection



R_{IN} : Resistance seen from primary side.

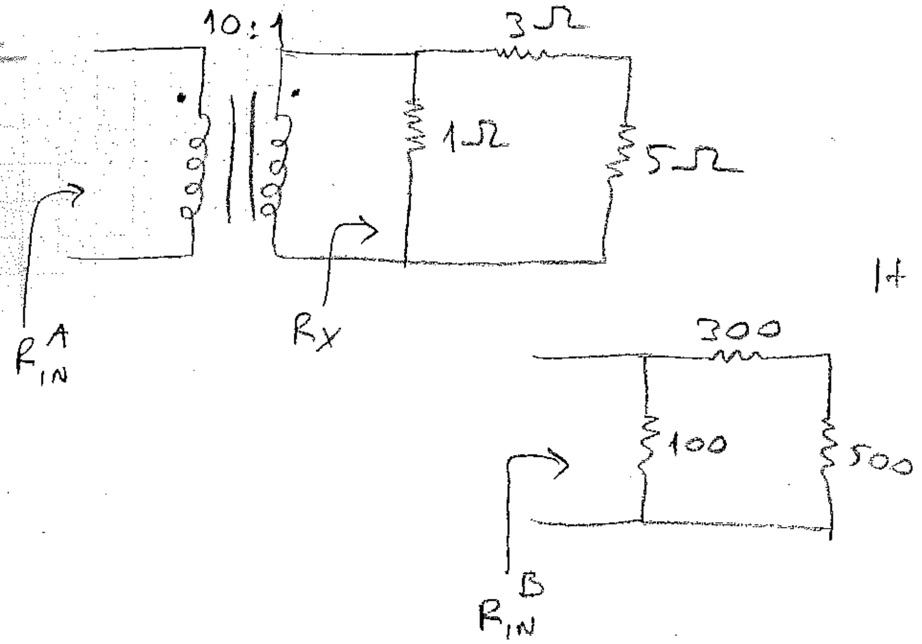
$$R_{IN} = \frac{V_{Test}}{I_{Test}}$$

$$V_2 = \frac{N_2}{N_1} V_{test} \qquad i_2 = \frac{-N_2}{N_1}\frac{V_{Test}}{R_X}$$

$$R_{IN} = \frac{V_{test}}{I_{test}} = \frac{V_{test}}{-\frac{N_2}{N_1}\cdot\left(\frac{-N_2}{N_1}\frac{V_{Test}}{R_X}\right)} \qquad i_1 = \frac{-N_2}{N_1} i_2$$

$$\boxed{R_{IN} = R_X\left(\frac{N_1}{N_2}\right)^2}$$

Ex:

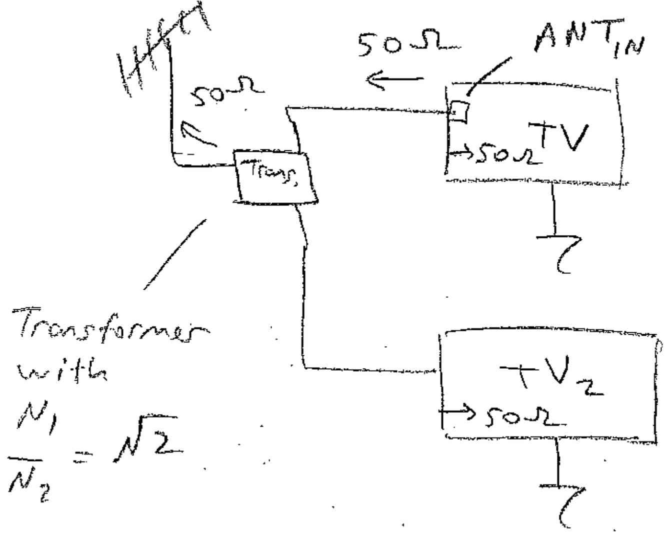


It should be clear that $R_{IN}^A = R_{IN}^B$

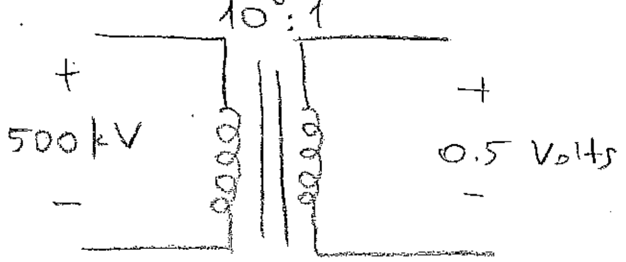
Use of Transformers

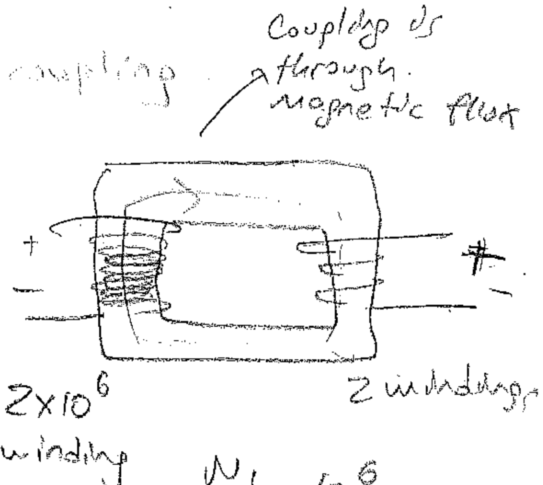
Pls. check Sadiku's book on transformers/mutual inductors and their applications

1) Impedance / Resistance Matching (Achieve max. power transfer)



② Electrical and Magnetic Decoupling





$\frac{N_1}{N_2} = 10^6$

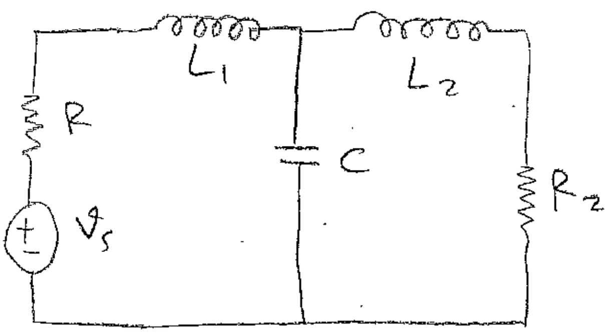
There is no electrical connection between two sides

☆ Check Sadiku's book for more uses of transformers.

State Equations

The order of the circuit $\leq$ the number of dynamic elements.

Goal: Analysis of N^{th} order circuits.



Assume that the variable of interest is $\vartheta_c(t)$, then given the input $(\vartheta_s(t))$ and initial conditions I need to find the solution of a differential equation for the unknown $\vartheta_c(t)$.

For a 3^{rd} order circuit, a possible differential equation is

Scalar differentiation for $\vartheta_c(t)$

$$(D^3+3D^2+2D+1)\,\vartheta_c(t) = 4\,\vartheta_s(t) \leftarrow \text{differential equation for } \vartheta_c(t)$$

$$\left.\begin{aligned} \vartheta_c(0^-) &= V_0 \\ \dot{\vartheta}_c(0^-) &= \dot{V}_0 \\ \ddot{\vartheta}_c(0^-) &= \ddot{V}_0 \end{aligned}\right\} \text{Initial condition set}$$

derivative

State equation: $\underline{X} = \begin{bmatrix} \vartheta_c(t) \\ I_{L_1}(t) \\ I_{L_2}(t) \end{bmatrix}$ → state vector

derivative

1st order matrix differential equation

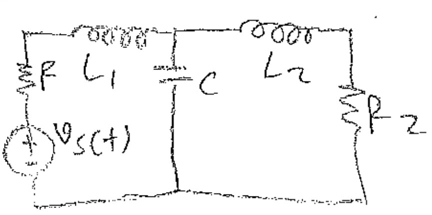
$$\underbrace{\begin{bmatrix} \dot{\vartheta}_c(t) \\ \dot{I}_{L_1}(t) \\ \dot{I}_{L_2}(t) \end{bmatrix}}_{\dot{X}(t)} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{A}{}_{3\times 3} \underbrace{\begin{bmatrix} \vartheta_c(t) \\ I_{L_1}(t) \\ I_{L_2}(t) \end{bmatrix}}_{X(t)}{}_{3\times 1} + \underbrace{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}}_{b}{}_{3\times 1} \vartheta_s(t)$$

$$\begin{bmatrix} \vartheta_c(0^-) \\ I_{L_1}(0^-) \\ I_{L_2}(0^-) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_{L_1} \\ I_{L_2} \end{bmatrix} \leftarrow \text{Set of I.C. (initial conditions)}$$

Goal: Expressing the circuit as

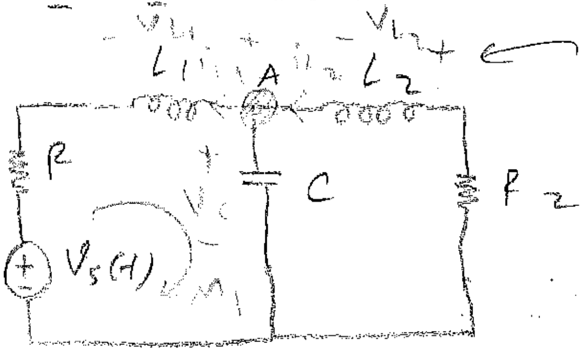
$$\left.\begin{aligned} \dot{\underline{X}}(t) &= \underline{\underline{A}}\,x(t) + \underline{b}\,\vartheta_s(t) \\ \underline{X}(0^-) &= X_0 \end{aligned}\right\} \text{State equation}$$

Ex: Let's find the state equation for the given circuit.



State variables = $\{\vartheta_c(t), I_{L_1}(t), I_{L_2}(t)\}$

1^{st} row of $\underline{\dot{X}}(t) = \underline{\underline{A}}\, x(t) + \underline{b}\, \vartheta_s(t)$ $\longrightarrow$ $\dot{\vartheta}_c(t) = ?\ \vartheta_c(t) + ?\ i_{L_1}(t) + ?\ i_{L_2}(t) + ?\ \vartheta_s(t)$



$\longleftarrow$ We assign the directions randomly.

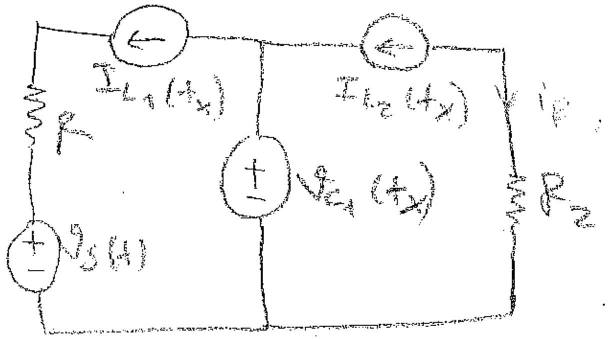
KCL at A: $\underbrace{C\, \dot{\vartheta}_c(t)}_{\text{state variable derivative}} = \underbrace{-i_{L_1}(t) + i_{L_2}(t)}_{\text{a linear combination state variable}}$

KVL at $\circlearrowright M_1$: $-\vartheta_s(t) + (-R I_{L_1}) \underbrace{- \vartheta_{L_1}(t)}_{L_1 \dot{I}_{L_1}(t)} + \vartheta_C = 0$

Mesh $\circlearrowright M_2$: $-\vartheta_c - \underbrace{\vartheta_{L_2}}_{L_2 \dot{I}_{L_2}(t)} + R_2(-i_{L_2}) = 0$

$$\begin{bmatrix} \dot{\vartheta}_c(t) \\ \dot{I}_{L_1}(t) \\ \dot{I}_{L_2}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1/C & 1/C \\ \frac{1}{L_1} & \frac{-R}{L_1} & 0 \\ -\frac{1}{L_2} & 0 & \frac{-R_2}{L_2} \end{bmatrix}}_{A} \begin{bmatrix} \vartheta_c(t) \\ I_{L_1}(t) \\ I_{L_2}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{L_1} \\ 0 \end{bmatrix}}_{b} \vartheta_s(t)$$

Note: If the value of the state variables at time "t" is known, then all other circuit variables related related with all other branches are a linear combination of state variables and input! for time "t"



At $t = t_x$

$$\underline{X}(t_x) = \begin{bmatrix} 1\ \text{Volt} \\ 2A \\ 3A \end{bmatrix}$$

$$i_{R_2}(t_x) = \alpha_1 \vartheta_{C_1}(t_x) + \alpha_2 I_{L_1}(t_x) + \alpha_3 I_{L_2}(t_x) + \beta\, \vartheta_s(t_x)$$

linear combination coefficients

How to write state equations

① Select a proper tree (including dependent and independent sources)

i) Put all voltage sources on the tree and Put all current sources on the co-tree.

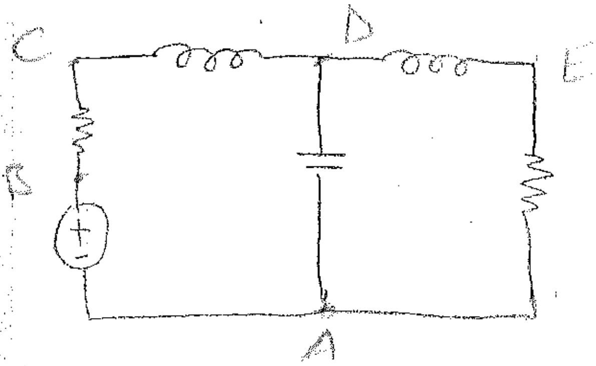
ii) If there's a transformer, place one port in the tree, the other port in the co-tree.

iii) Put maximum number of cap. on the tree. Put max. number of ind. on co-tree.

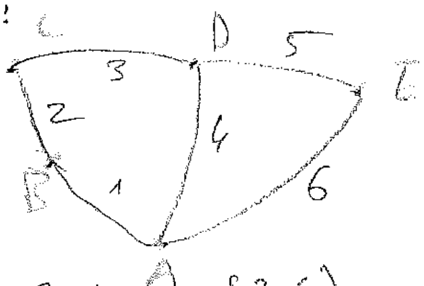
② Write the fundamental loop equations for each inductor in co-tree.

③ Write the fun. cut-set equations for each cap. in the tree.

State variables = { Tree cap. voltages, Co-tree inductor currents }



Graph:



Tree: A set of branches that
① does not form a loop
② reaches all the nodes.
③ Connected ← single piece

Co-tree = {2,6}
Tree = {4,5,3,1}

(another) tree = {1,2,3,5}
Co-tree = {4,6}

Fundamental loop: A loop whose all branches are from tree and a single branch from co-tree.

Link: A branch of co-tree

Co-tree: Remaing branches from tree.

tree branches

Tree = {4,5,3,1}, Co-tree = {2,6}

For co-tree branch 2: Fun-loop: {1,2,3,4}

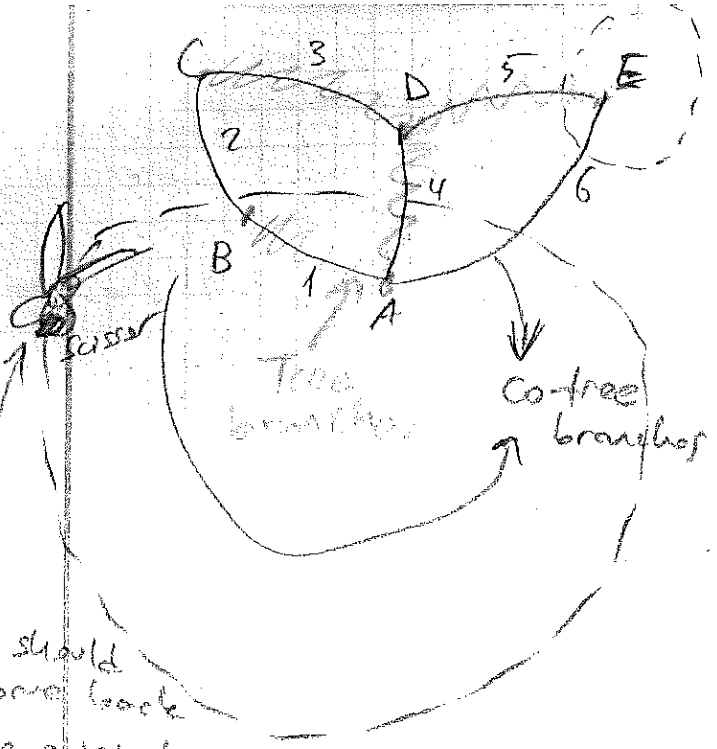
" " " 6: " " : {4,5,6}

tree branch, Co-tree branch

Co-tree branch

Fundamental cut-set: A cut-set whose elements are all from co-tree except a single element from tree.

Cut-set: A set of branches whose removal seperates the graph into two.



Fun. cut-set: Tree: $\{1,3,4,5\}$

Fun. cut-set for branch 1: $\{1, 2\}$

" 3: $\{2, 3\}$

" 4: $\{2, 4, 6\}$

" 5: $\{5, 6\}$

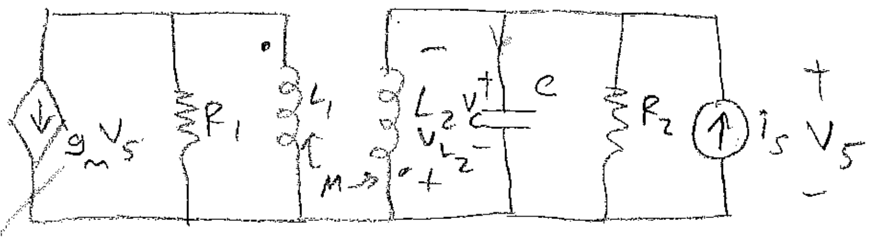
Tree branches

Co-tree branches

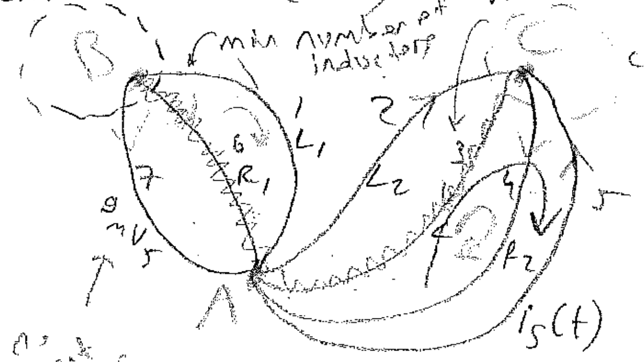
It should come back to original plane.

It shouldn't cut the same branch twice.

Ex:



g_m: conductance (A/V)



Fun. cut-set for capacitor

State variables $= \{v_c, I_{L_1}, I_{L_2}\}$

3rd order circuit

Fund. cut-set for cap. branch:

$$c\,\dot{v}_c(t) + \underbrace{i_{R_2}}_{} - \underset{\checkmark}{i_s(t)} - \underset{\checkmark}{i_{L_2}} = 0$$

↑ state variable

↳ Not a state variable

$$i_{R_2} = \frac{v_{R_2}}{R_2} = \frac{v_c}{R_2}$$

Fun. loop

1st state equation →
$$\dot{v}_c(t) = \frac{1}{c}\left[-\frac{v_c}{R_2} + I_{L_2} + i_s(t)\right] \checkmark$$

State equations for I_{L_1} and I_{L_2} $\Big\}$ $\begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}}\begin{bmatrix} \frac{d}{dt} I_{L_1}(t) \\ \frac{d}{dt} I_{L_2}(t) \end{bmatrix}$

$$\begin{bmatrix} \dot{I}_{L_1}(t) \\ \dot{I}_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}^{-1} \begin{bmatrix} v_{L_1}(t) \\ v_{L_2}(t) \end{bmatrix}$$

Not a State Variable

$$v_{L_1}(t) \overset{F.L.}{=} v_{R_1} = R_1 \cdot i_{R_1} \overset{F.C.}{=} R_1(-g_m V_5 - i_{L_1}) = R_1(-g_m V_C - i_{L_1})$$

Component eqn. (Ohm's Law)

$V_5 = V_C$ fun loop

V_{L_1}: in terms of state variables

$$V_{L_2}(t) \overset{F.loop}{=} -V_C$$

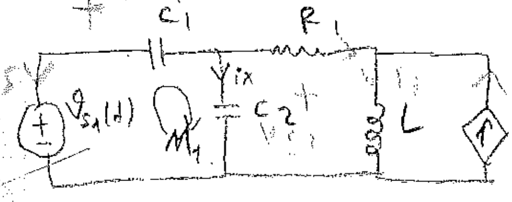
$$\begin{bmatrix} \dot{V}_C(t) \\ \dot{I}_{L_1}(t) \\ \dot{I}_{L_2}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{CR_2} & 0 & 1/C \\ -\Gamma_1 R_1 g_m - \Gamma_2 & -\Gamma_1 R_1 & 0 \\ -\Gamma_3 R_1 g_m - R_4 & -\Gamma_3 R_1 & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ I_{L_1}(t) \\ I_{L_2}(t) \end{bmatrix} + \begin{bmatrix} 1/C \\ \\ \end{bmatrix} i_5(t)$$

$$\begin{bmatrix} \dot{I}_{L_1}(t) \\ \dot{I}_{L_2}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}^{-1}}_{2\times 2} \begin{bmatrix} -R_1(g_m V_C + I_{L_1}) \\ -V_C \end{bmatrix}$$

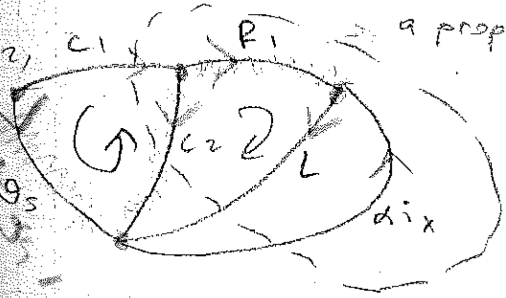
$$= \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} \begin{bmatrix} -R_1(g_m V_C + I_{L_1}) \\ -v_C \end{bmatrix}$$

State Equations (Cont'd)

Ex.



→ Number of dynamic components = 3 = max. possible order of the circuit



⇒ a proper tree (for state equation)

If we know c2, c1 isn't arbitrary

We can see it by giving $V_{s_1}(t) = 0V$

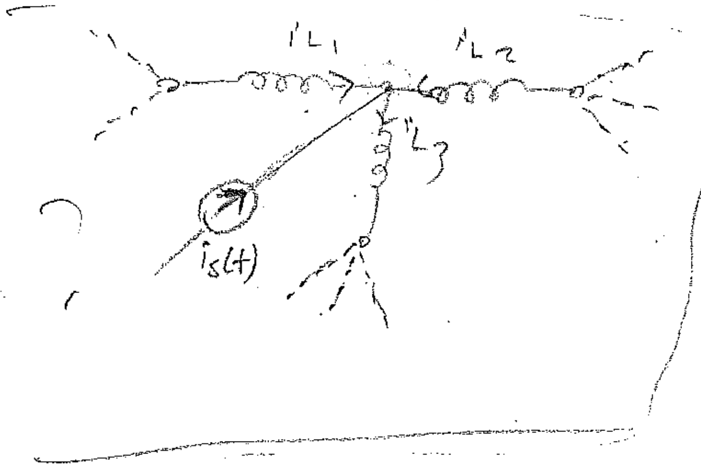
Another proper tree = $\{v_s, C_1, R_1\}$

State variables = $\{v_{C_2}, i_L\}$

$$\underline{X}(t) = \begin{bmatrix} v_{C_2}(t) \\ i_L(t) \end{bmatrix}$$

$$\dot{\underline{X}}(t) = \underline{\underline{A}}\, x(t) + \underline{b}\, v_s(t) + \underline{\gamma}\, \dot{v}_s(t)$$

Capacitive loops (such as in Mesh M_1) and inductive cut-set results in the combination dynamic components and order of the circuit is related with the number of dynamic components after combination.



Finding $\dot{\vartheta}_{C_2}(t)$ in terms of state variables and input :

Fun. Cut-set for C_2 branch

$$i_{C_2} = i_{C_1} - i_L + \alpha i_x$$

$C_2 \dot{\vartheta}_{C_2}$

$\hookrightarrow i_x = C_2 \cdot \dot{\vartheta}_{C_2}$ ✓

$\hookrightarrow i_{C_1} = C_1 \dot{\vartheta}_{C_1} = C_1 (\dot{\vartheta}_s - \dot{\vartheta}_{C_2})$

Component equation

Fun. loop

$$C_2 \dot{\vartheta}_{C_2} = C_1 \dot{\vartheta}_S - C_1 \dot{\vartheta}_{C_2} - i_L + \alpha C_2 \dot{\vartheta}_{C_2}$$

$$\boxed{\dot{\vartheta}_{C_2} = \frac{1}{(1-\alpha)C_2 + C_1} \left[-i_L + C_1 \dot{\vartheta}_S(t) \right]}$$

→ First equation

?

we need one for i_L too

Finding $\dot{I}_{L_1}(t)$!

Fund. loop for L branch :

$$\vartheta_L = -V_{R_1} + \vartheta_{C_2} \quad ✓$$

$L \cdot \dot{I}_L(t)$

Fun. cut-set

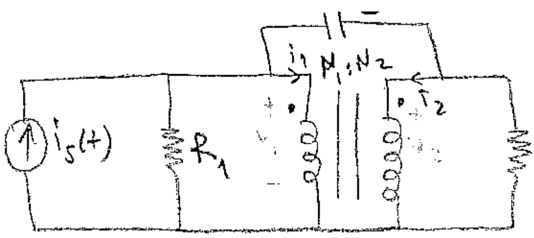
$\hookrightarrow V_{R_1} = R_1 \cdot i_{R_1} = R_1 (i_L - \alpha I_x)$

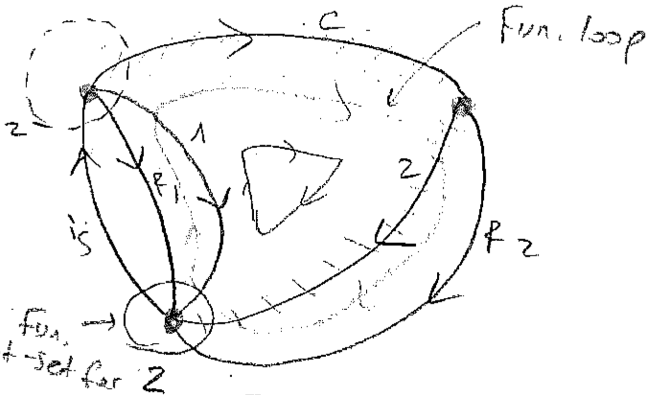
Comp. Law.

$$L\dot{I}_L = -R_1 I_L + \alpha R_1 C_2 \dot{\vartheta}_{C_2} + \vartheta_{C_2}$$

After substitution → $\dot{I}_L(t)$ is expressed in terms of state variables (and input)

Ex:





State variable = $\{ V_c \}$

Fun. cut-set for C branch to find $\dot{\vartheta}_c(t)$

$$\underset{C\dot{\vartheta}_c}{i_c} = i_s - i_{R_1} - i_1$$

$i_{R_1} = ?$ (in terms of state variables)

$$i_{R_1} = \frac{\vartheta_{R_1}}{R_1} \underset{\text{Fun. loop}}{=} \frac{\vartheta_c + \vartheta_2}{R_1}$$

$$\frac{\vartheta_2}{\vartheta_1} = \frac{N_2}{N_1}$$

$$\frac{i_2}{i_1} = \frac{-N_1}{N_2}$$

$$\vartheta_2 = \frac{N_2}{N_1}\vartheta_1 \underset{\text{Fun. loop}}{=} \frac{N_2}{N_1}(\vartheta_c + \vartheta_2) \rightarrow \vartheta_2 = \frac{N_2}{N_1 - N_2}\vartheta_c$$

$$\boxed{i_{R_1} = \frac{\vartheta_c}{R_1} + \left(\frac{N_2}{N_1 - N_2}\right)\frac{1}{R_1}\cdot \vartheta_c}$$

$i_1 = ?$ (in terms of ϑ_c)

$$i_1 \underset{\text{comp. law}}{=} \frac{-N_2}{N_1} i_2 = -\frac{N_2}{N_1}\left(-i_{R_2} + i_s - i_{R_1} - i_1\right)$$

$$(N_1 - N_2)\, i_1 = -N_2 (i_s - i_{R_2} - i_{R_1})$$

↳ already expressed in terms of ϑ_c

$$i_{R_2} \underset{\text{comp. eq.}}{=} \frac{V_{R_2}}{R_2} = \frac{\vartheta_2}{R_2} = \frac{N_2}{(N_1 - N_2)R_2}\vartheta_c$$

$$C\dot{\vartheta}_c = i_s + \frac{-N_1}{N_1 - N_2}\left(\frac{\vartheta_c}{R_1} + \left(\frac{N_2}{N_1 - N_2}\right)\frac{1}{R_1}\vartheta_c\right) + \frac{N_2}{N_1 - N_2}\left(i_s - \frac{N_2}{(N_1 - N_2)R_2}\vartheta_c\right)$$

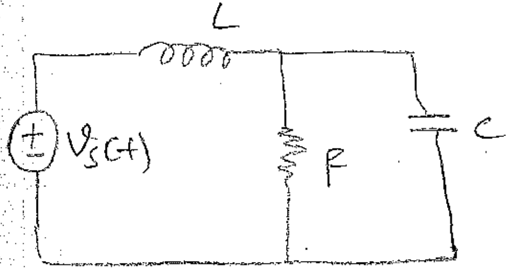
1^{st} order dif. eqn.

Suggested Problems

ZPS-1 : State equations: 1a, 1b, 1c, 1d

Solution of State Equations

Ex:



$R = \frac{1}{3}\,\Omega$, $C = 1F$, $L = \frac{1}{2}\,H$

$$\begin{bmatrix} \dot{\vartheta}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \vartheta_s(t)$$

Note: The scalar differential equation for v_c can be written from state equations as:

$$\ddot{v}_c = -3\dot{v}_c + \dot{I}_L$$

$$\hookrightarrow -2v_c + 2v_s(t)$$

$$\boxed{\begin{array}{ll} (\ddot{v}_c + 3\dot{v}_c + 2v_c) = 2v_s(t) & \\ v_c(0^-) = --- & 2^{nd}\text{ order scalar} \\ \dot{v}_c(0^-) = ... & \text{dif. eq. for } v_c \end{array}}$$

Similarly

$$\ddot{I}_L = -2\dot{v}_c + 2\dot{v}_s$$

$$= -2(-3v_c + I_L)$$

$$\hookrightarrow v_c = -\frac{\dot{I}_L}{2}$$

2^{nd} state eqn.

So, I can also find a 2^{nd} order dif. eqn. for I_L.

Case of zero-input solution:

$v_s(t) = 0$ (External input is not present anymore)

So, there is only initial energy stored in the circuit, and responses is due to the initial energy.

Given: $$\begin{bmatrix} v_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} \xrightarrow[\text{guess}]{} \underbrace{\begin{bmatrix} v_c(t) \\ I_L(t) \end{bmatrix}}_{\underline{X}(t)} = \begin{bmatrix} \alpha_1 e^{\lambda t} \\ \alpha_2 e^{\lambda t} \end{bmatrix}$$

For the guess to be correct, it should satisfy differential equation:

$$\dot{\underline{X}}(t) = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \underline{X}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v_s(t)$$

$$\downarrow$$

$$\lambda \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\lambda t}$$

$$e^{\lambda t} > 0$$

$$\downarrow$$

$$\lambda \underline{\underline{I}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = A \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(\lambda I - A)\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Note: If $(\lambda \underline{\underline{I}} - \underline{\underline{A}})$ matrix is invertible.

$(\lambda \underline{\underline{I}} - \underline{\underline{A}})^{-1}$ exists, then

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = (\lambda \underline{\underline{I}} - \underline{\underline{A}})^{-1}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}v_C(t)\\ I_L(t)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} e^{\lambda t} = \begin{bmatrix}0\\0\end{bmatrix}$$

trivial soln. ← does not satisfy I.C. unless $V_0 \neq 0$, $I_0 \neq 0$

For non-trivial soln:

$(\lambda \underline{\underline{I}} - \underline{\underline{A}})$ shouldnot be invertible.

Then $\det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = 0 \rightarrow |\lambda \underline{\underline{I}} - \underline{\underline{A}}| = 0$

$$\left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\right| = 0 \rightarrow \left|\begin{bmatrix}\lambda+3 & -1\\ 2 & \lambda\end{bmatrix}\right| = 0$$

$\lambda^2 + 3\lambda + 2 = 0$ ← characteristic polynomial

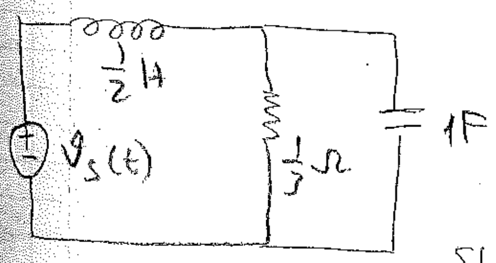
$\lambda = \{-1, -2\}$

Then non-trivial soln. exists for only special values of λ, which are called natural frequencies.

$$\boxed{\underline{X}(t) = \text{span}\left\{\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} e^{-t} + \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} e^{-2t}\right.}$$

Solution of State Eqn: (continued)

a) zero-input (continued)



State equations

$$\begin{cases}\begin{bmatrix}\dot{v}_C(t)\\ \dot{I}_L(t)\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}v_C(t)\\ I_L(t)\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix} v_S(t)\\ \begin{bmatrix}v_C(0^-)\\ I_L(0^-)\end{bmatrix} = \begin{bmatrix}V_0\\ I_0\end{bmatrix}\end{cases}$$

f(t): forcing term

State variables

Assume $v_s(t) = 0$ (zero-input) $\underline{\underline{A}}$

then:

We need to solve this

$$\left\{\begin{aligned} &\begin{bmatrix}\dot{v}_C(t)\\ \dot{I}_L(t)\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}v_C(t)\\ I_L(t)\end{bmatrix}\\ &\begin{bmatrix}v_C(0^-)\\ I_L(0^-)\end{bmatrix} = \begin{bmatrix}V_0\\ I_0\end{bmatrix}\end{aligned}\right.$$

Solution:

Guess: $\begin{bmatrix}v_C^{z.i.}(t)\\ I_L^{z.i.}(t)\end{bmatrix} = \begin{bmatrix}k_1\\ k_2\end{bmatrix}e^{\lambda t}$

$$\lambda\begin{bmatrix}k_1\\ k_2\end{bmatrix}\cancel{e^{\lambda t}} = \underline{\underline{A}}\begin{bmatrix}k_1\\ k_2\end{bmatrix}\cancel{e^{\lambda t}}$$

$$(A-\lambda I)\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$\rightarrow \det(A-\lambda I) \neq 0 \rightarrow \underset{\substack{\uparrow\\ \text{trivial solution}}}{\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}}$ i.e. $(A-\lambda I)$ is invertible

$\rightarrow \begin{bmatrix}v_C^{zi}(t)\\ I_L^{zi}(t)\end{bmatrix} = \begin{bmatrix}k_1\\ k_2\end{bmatrix}e^{\lambda t} = \begin{bmatrix}0\\ 0\end{bmatrix}$

So trivial solution cannot be the solution we are looking for unless initial conditions are all zero.

$$\rightarrow \det(A-\lambda I) = 0 \rightarrow \underbrace{\lambda^2+3\lambda+2=0}_{\text{characteristic polynomial}} \rightarrow \underset{\substack{\uparrow\\ \text{natural frequency}}}{\lambda=\{-1,-2\}}$$

Then for $\lambda=\{-1,-2\}$, a non-trivial solution can exist.
that is:

$$(\underline{\underline{A}} - \underset{\substack{\uparrow\\ \lambda=\{-1,-2\}\\ \nearrow\\ \text{for our problem}}}{\lambda}\underline{\underline{I}})\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

has a solution $\begin{bmatrix}k_1\\ k_2\end{bmatrix} \neq \begin{bmatrix}0\\ 0\end{bmatrix}$

$$\underline{\underline{A}}\begin{bmatrix}k_1\\ k_2\end{bmatrix} = \lambda\begin{bmatrix}k_1\\ k_2\end{bmatrix} \swarrow$$ eigenvalue-eigenvector equation system

$$\lambda = -1 \longrightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \begin{bmatrix} k_1 \\ k_2^{\lambda=-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \beta \in \mathbb{R}$$

$$\underbrace{\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}}_{\substack{(A-\lambda I) \\ \uparrow \\ \lambda=-1}} \begin{bmatrix} k_1^{\lambda=-1} \\ k_2^{\lambda=-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \underset{\text{Proportional to}}{\propto} \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \rightarrow \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -2 \rightarrow \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} k_1^{\lambda=-2} \\ k_2^{\lambda=-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \alpha \in \mathbb{R}$$

$$(*) \boxed{\begin{bmatrix} v_c^{zi}(t) \\ I_L^{zi}(t) \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} \quad \begin{matrix} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{matrix}}$$

any α and β satisfies the dif. eqn. given.

Guess: $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\lambda t}$ $\downarrow$ $\lambda = -1$

Now, only thing that we need to check whether I.Cs are satisfied or not.

$$\begin{bmatrix} v_c^{zi}(0^+) \\ I_L^{zi}(0^+) \end{bmatrix} = \begin{bmatrix} v_c^{zi}(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

Then inserting $t = 0^+$ into $(*)$

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \alpha + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \beta \rightarrow \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Inverse

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} I_0 - V_0 \\ 2V_0 - I_0 \end{bmatrix}$$

Finally,

$$\boxed{\begin{bmatrix} v_c^{z.i}(t) \\ I_L^{z.i}(t) \end{bmatrix} = (I_0 - V_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + (2V_0 - I_0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \ t \geq 0}$$

what we've learned:

① From state equations

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\, x(t) + b\, V_s(t)$$

The natural frequencies (λ) are eigenvalues of $\underline{\underline{A}}$ matrix.

② The zero-input solution is in the form:

zero input solution $\longleftarrow$ $$\overset{zi}{x}(t) = \text{span}\left\{ e_1 e^{\lambda_1 t}, e_2 e^{\lambda_2 t}, \dots, e_N e^{\lambda_N t} \right\}$$

e_k : k^{th} eigenvector of the matrix $\underline{\underline{A}}$

that is

$$\underline{\underline{A}}\, e_k = \lambda_k e_k$$

In our example

$$\begin{bmatrix} V_C^{zi} \\ I_L^{zi} \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} \right\}$$

$\alpha, \beta \in \mathbb{R}$

③ The components zero-input solution such as

$e_k e^{\lambda_k t}$ $\longleftarrow$ k^{th} component in the span expansion

is called the mode of the circuit.

In our example, the circuit has two modes:

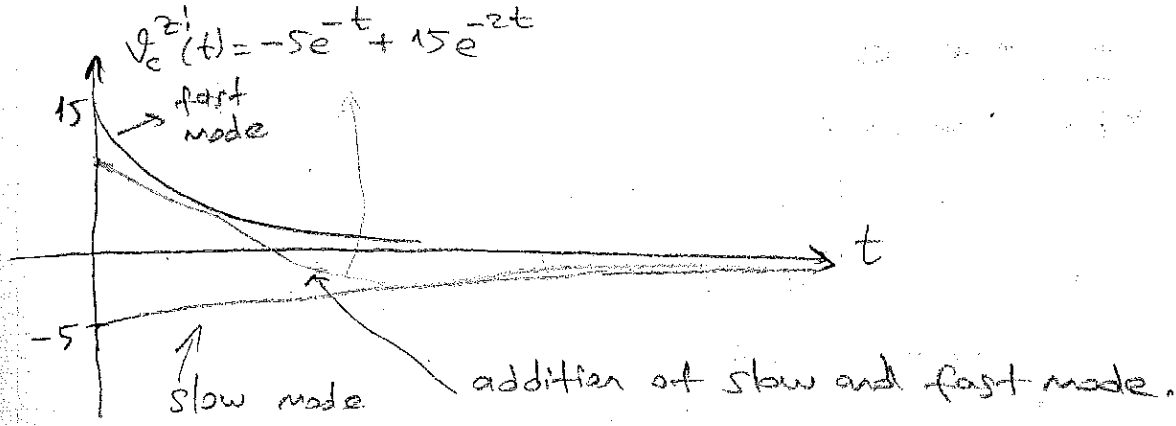
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Hence we can repeat note ② as the zero input solution is a linear combination of the modes.

In our examples let $V_0 = 10V$, $I_0 = 5A$

$$\begin{bmatrix} V_C^{zi}(t) \\ I_L^{zi}(t) \end{bmatrix} = \underbrace{-5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}}_{\text{slow mode}} + \underbrace{15 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}}_{\text{fast mode}} = \begin{bmatrix} -5e^{-t} + 15e^{-2t} \\ -10e^{-t} + 15e^{-2t} \end{bmatrix}$$

(since e^{-2t} decays much faster than e^{-t}.)



④ Mode excitation

The I.C.'s resulting in a single mode for the zero input solution are called mode exciting initial conditions.

$$\underline{\dot{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{b}\, v_s(t)^{\nearrow 0} \quad \text{(zero-input)}$$

$$\underline{X}(0) = X_0$$

↖ a special I.C. such that

$$\underline{X}^{zi}(t) = \underbrace{\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}}_{} e^{\lambda t}$$

↑ only a single mode present in the zi solution.

Conclusion: If the initial condition vector is proportional to an eigenvector of $\underline{\underline{A}}$, say e_k, then that I.C. excites k^{th} mode, i.e.

$$\begin{bmatrix} v_c^{z.i.}(t) \\ I_L^{zi}(t) \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} \Bigg\} \begin{bmatrix} v_c^{zi}(0) \\ I_L^{zi}(0) \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

✓ for eigenvalue $\lambda = -2$

I.C. exiting fast mode. (initial condition)

⑤ State Transition Matrix and Mode Excitations

Let's assume that the initial cond. is expressed as a linear combination of eigenvalues of $\underline{\underline{A}}$.

$$\underline{X_0} = \alpha_1 \underline{e_1} + \alpha_2 \underline{e_2} + \cdots + \alpha_n \underline{e_N}$$

I.C. ↗

↑ ↗ eigenvectors

$$\underline{\underline{A}}\, \underline{e_k} = \lambda_k \underline{e_k}$$

Then

$$\underline{x}(t) = \alpha_1 e_1 e^{\lambda_1 t} + \alpha_2 \underline{e_2} e^{\lambda_2 t} + \cdots + \alpha_N e_N e^{\lambda_N t}$$

↖ the zero-input solution of $\dot{x}(t) = A\,x(t)$

$$\underline{x}(t) = \underbrace{[\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_N]}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_N t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}}_{\begin{bmatrix} e^{\lambda_1 t}\alpha_1 \\ e^{\lambda_2 t}\alpha_2 \\ \vdots \\ e^{\lambda_N t}\alpha_N \end{bmatrix}}$$

$$\underline{x}(t) = \underbrace{\underline{\underline{E}}\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_N t})\,\underline{\underline{E}}^{-1}}_{\Phi(t,0)} x(0) \qquad \underline{\underline{E}}^{-1}x(0) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}$$

$$\boxed{\underline{x}(t) = \underline{\underline{\Phi}}(t,0)\,\underline{x}(0)}$$

$$\underline{\underline{\Phi}}(t,0) = \underline{\underline{E}}\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_N t})\,\underline{\underline{E}}^{-1}$$

$\Phi(t,0)$ is called state-transition matrix

This is a powerful result, since

$\dot{\underline{x}}(t) = \underline{\underline{A}}\,x(t)$ diff. eqn. has the solution

$$\underline{x}(t) = \underbrace{\underline{\underline{\Phi}}(t,0)}_{\text{State transition matrix}}\ \overset{\text{I.C.}}{x_0}$$

State transition matrix which may be calculated from $\underline{\underline{A}}$ matrix.

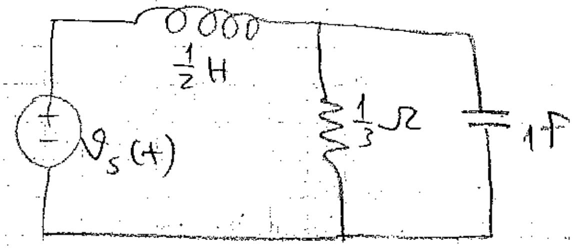
$$\underline{\underline{\Phi}}(t,0) = \underline{\underline{E}}\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_N t})\,\underline{\underline{E}}^{-1}$$

$\lambda_1, \ldots, \lambda_N$: eigenvalues

$\underline{\underline{E}}$ has columns as the eigenvectors of A.

Particular Solution of State Equation System

Ex:



$v_s(t) = e^{s_0 t}$ $s_0 \neq$ natural frequency of the circuit

s_0 is a complex number

Q: Find the particular solution.

$$\begin{bmatrix} \dot{v}_c \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} v_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v_s(t) = e^{s_0 t}$$

(Char. eqn: $\lambda^2 - \mathrm{trace}\{A\}\lambda + |A| = 0$)

$$\begin{bmatrix} V_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

method of undetermined coefficients

Particular solution:
Guess: $$\begin{bmatrix} \vartheta_c(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{s_0 t} \xrightarrow[\text{Eqn.}]{\text{Insert into}}$$

$$s_0 \underline{\underline{I}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{s_0 t} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{s_0 t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{s_0 t}$$

$$\left(s_0 \underline{\underline{I}} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} s_0+3 & -1 \\ 2 & s_0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{s_0^2+3s_0+2} \begin{bmatrix} s_0 & 1 \\ -2 & s_0+3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{s_0^2+3s_0+2} \begin{bmatrix} 2 \\ 2s_0+6 \end{bmatrix}}$$

k_1 and k_2 values required to finalize the particular solution to the input $e^{s_0 t}$.

$j = i = \sqrt{-1}$

Particular Sol. of State Equations (Continued)

$$\begin{cases} \underline{\dot{X}}(t) = \underline{\underline{A}} x(t) + b \vartheta_s(t) \\ \underline{X}(0) = X_0 \end{cases}$$

Assume $\vartheta_s(t) = M e^{s_0 t}$, s_0 can be $\{1, -2, j, 1+j, \dots\}$

an exponential input

i.e. $s_0 \in \mathbb{C}$ complex field

Goal: Finding the particular solution to the exponential input.

★ s_0 is not a natural frequency of the circuit.

Note: $$A e^{s_0 t} \xrightarrow[s_0=0]{} A$$

(or $Me^{s_0 t}$) $$A e^{s_0 t} \xrightarrow[s_0=-1]{} A e^{-t}$$

$$A e^{s_0 t} \xrightarrow[s_0=2j]{} A e^{2jt} = A[\cos(2t) + j \sin(2t)]$$

$j = \sqrt{-1}$

So exponential input family is quite general and includes many inputs of interest

$$\underset{N \times 1}{\underline{X}^p(t)} = \underbrace{\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}}_{\underline{k}} e^{s_0 t} \leftarrow \text{Guess for particular solution}$$

For the guess to be correct, the following dif. eqn. should be satisfied:

$$\underline{\dot{X}}^p(t) = \underline{\underline{A}}\, \underline{x}^p(t) + b \underline{\vartheta_s}(t)$$

Identity matrix

$$s_0 \underline{k} e^{s_0 t} = \underline{\underline{A}}\, \underline{k} e^{s_0 t} + \underline{b} M e^{s_0 t} \rightarrow (s_0 \underline{\underline{I}} - \underline{\underline{A}}) k = bM$$

$$\underline{k} = \left[(s_0 \underline{\underline{I}} - A)^{-1} b \right] M$$

$\uparrow$ unknown k vector is found $\nearrow$ since $(s_0 \underline{\underline{I}} - \underline{\underline{A}})$ is invertible, I have calculated $\underline{k}$ by simply inverting the matrix.

The earlier numerical example:

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} V_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v_s(t)$$

$\underline{v_s(t) = e^{s_0 t}}$

$\hookrightarrow$ Guess: $\begin{bmatrix} v_c^P(t) \\ I_L^P(t) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{s_0 t} \longrightarrow \left(s_0 \underline{\underline{I}} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \right) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

?

$$\begin{bmatrix} s_0+3 & -1 \\ 2 & s_0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{(s_0^2+3s_0+2)} \begin{bmatrix} s_0 & 1 \\ -2 & s_0+3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\det(s_0 \underline{\underline{I}} - \underline{\underline{A}})$

Since $s_0 \neq$ nat. freq.,

Remember nat. freq. for this example $= \{-1, -2\}$

$\rightarrow s_0^2 + 3s_0 + 2 \neq 0$

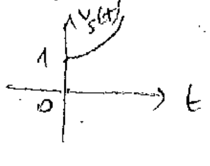
$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{s_0^2+3s_0+2} \begin{bmatrix} 2 \\ 2(s_0+3) \end{bmatrix}$$

$$\hookrightarrow \boxed{\begin{aligned} \begin{bmatrix} v_c^P(t) \\ I_L^P(t) \end{bmatrix} &= \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{s_0 t} \\ &= \frac{1}{s_0^2+3s_0+2} \begin{bmatrix} 2 \\ 2(s_0+3) \end{bmatrix} e^{s_0 t} \end{aligned}}$$

$$\boxed{v_c^P(t) = \frac{2}{s_0^2+3s_0+2} e^{s_0 t}}$$

$\downarrow$

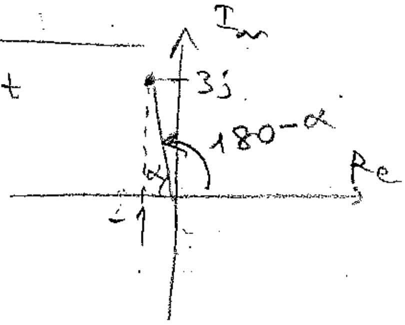
$s_0 = 0$

$v_s(t) = e^{s_0 t}$	$v_c^P(t)$
$s_0 = 0 \rightarrow v_s(t) = 1$	$v_c^P(t) = 1$
$s_0 = 1 \rightarrow v_s(t) = e^t$ 	$v_c^P(t) = \frac{1}{3} e^t$

$\vartheta_s(t) = e^{s_0 t}$	$\vartheta_c^P(t)$
$s_0 = 2j \rightarrow \vartheta_s(t) = e^{2jt}$	$\vartheta_c^P(t) = \frac{2}{(2j)^2 + 3(2j) + 2} e^{2jt}$

$$\vartheta_s(t) = \begin{pmatrix} \cos(2t) \\ + \\ j\sin(2t) \end{pmatrix}$$

$$= \frac{1}{-1+3j} e^{2jt}$$



$$= \frac{1}{\sqrt{10}\ \angle 180^\circ - \tan^{-1}(3)} e^{2jt} = \frac{1}{\sqrt{10}} \angle \tan^{-1} 3 - 180^\circ e^{2}$$

$$= \frac{1}{\sqrt{10}} \angle -180^\circ + \tan^{-1}(3)\, e^{2jt}$$

$= cis(180^\circ - \tan^{-1}(3)$

$$= \frac{1}{\sqrt{10}} e^{j(180^\circ - \tan^{-1}(3))} e^{2jt}$$

$$= \frac{1}{\sqrt{10}} e^{j(2t - 180^\circ + \tan^{-1} 3)}$$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} \cos(2t - 180^\circ + \tan^{-1} 3) \\ + \\ j\sin(2t - 180^\circ + \tan^{-1} 3) \end{bmatrix}$$

$\vartheta_c^P(t)$ for e^{2jt} input

$$\frac{1}{1+j} = \frac{1-j}{1+1} = \frac{1-j}{2}$$

$(1-j)$

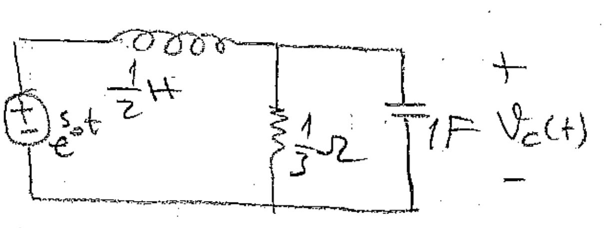
$$\frac{1}{1+j} = \frac{1}{\sqrt{2}\angle 45^\circ} = \frac{1}{\sqrt{2}} \angle -45^\circ$$

$$= \frac{1}{\sqrt{2}} (\cos 45^\circ - j\sin 45^\circ)$$

$$= \frac{1}{2} - \frac{j}{2}$$

$$\boxed{\angle\theta = cis\,\theta = \cos\theta + j\sin\theta)}$$

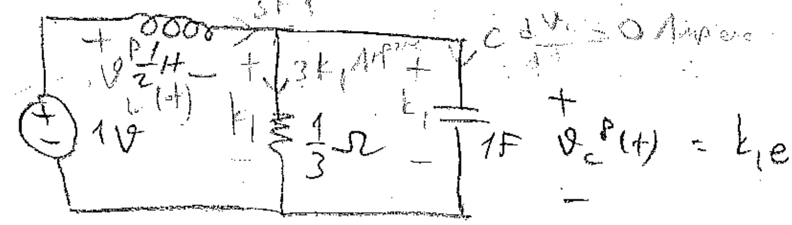
Now, let's go back and examine the circuit one more time:



Let's try to find particular solution from the circuit:

I know that $\vartheta_c^P(t) = \frac{2}{s_0^2 + 3s_0 + 2} e^{s_0 t}$ ✓ ← we will verify this result one more time.

Before general verification, Let's do $s_0 = 0$ Special case:

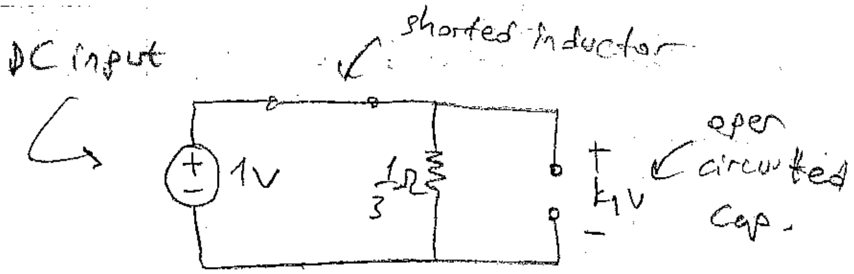


$\vartheta_L^P(t) = L\frac{di_L}{dt} = 0$

$$\vartheta_c^P(t) = k_1 e^{s_0 t} \Big\downarrow_{s_0=0} = k_1 \quad ? \quad \boxed{k_1 = 1V}$$

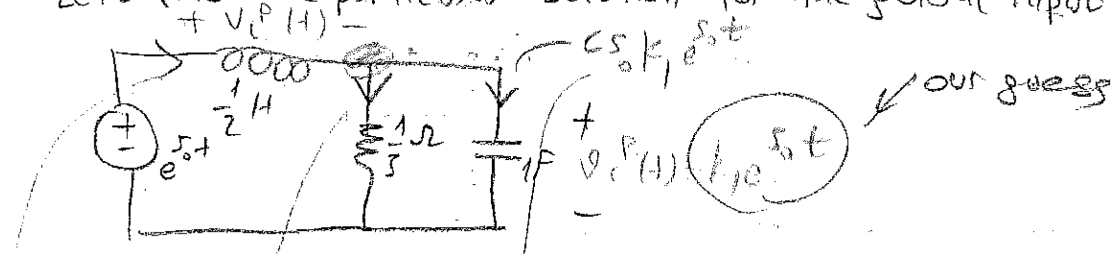
↑ by KVL

$V_s(t) = 1V$ is DC input



$t \rightarrow \infty$

Let's find the particular solution for the general input $e^{s_0 t}$.



$= I_c^P(t) = C\dot{V}_c^P(t)$

by KCL

$3 \cdot k_1 \cdot e^{s_0 t} = I_R^P(t)$

$k_1 e^{s_0 t}(3 + cs_0) \qquad V_L^P(t) = L\frac{d}{dt} I_L^P(t)$

$= Lk_1(3+s_0 C)s_0 e^{s_0 t}$

Finally, KUL: $-e^{s_0 t} + V_L^P(t) + V_R^P(t) = 0$

$$-\cancel{e^{s_0 t}} + Lk_1(3+s_0 c)s_0 \cancel{e^{s_0 t}} + k_1 \cancel{e^{s_0 t}} = 0$$

$$k_1 = \frac{1}{s_0^2 CL + L3s_0 + 1}$$

$$k_1 = \frac{1}{s_0^2 \frac{1}{2} + \frac{3s_0}{2} + 1} \qquad \boxed{k_1 = \frac{2}{s_0^2 + 3s_0 + 2}}$$

$$\boxed{V_c^P(t) = \frac{2}{s_0^2 + 3s_0 + 2} e^{s_0 t}}$$ ← Final answer for particular solution

Notes: ① For the exponential input in the form $Ke^{s_0 t}$ the branch voltages and currents have the particular solution in the form:

k^{th} branch variables $\begin{cases} V_k = A_k e^{s_0 t} \\ i_k = B_k e^{s_0 t} \end{cases}$

The goal of particular soln. is to find the coefficients A_k and B_k for the k^{th} branch and all the other branches.

② The general solution of an LTI circuit is in the form:

$$\vartheta_k(t) = \underbrace{\vartheta_k^h(t)}_{\text{homogeneous}} + \underbrace{\vartheta_k^p(t)}_{\text{particular}}$$

Homogeneous solution: $\vartheta_k^h(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \dots + \alpha_N e^{\lambda_N t}$

$$\left[\begin{array}{c} \lambda_k : \text{natural frequency} \\ \text{char. eqn: } \det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = 0 \\ \text{roots of char. eqn. are nat.} \\ \text{frequency} \end{array}\right]$$

Particular solution (for $Ae^{s_0 t}$ input): $\vartheta_k^p(t) = \underset{\nwarrow ?}{k_1} e^{s_0 t}$

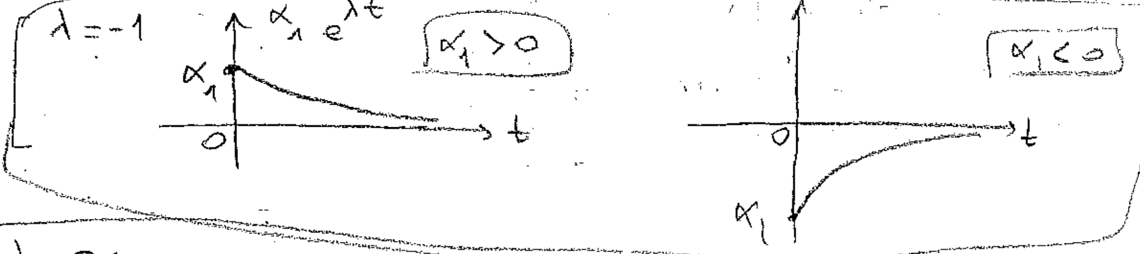
$$\vartheta_k^{\text{complete}}(t) = \vartheta_k^h(t) + \vartheta_k^p(t)$$

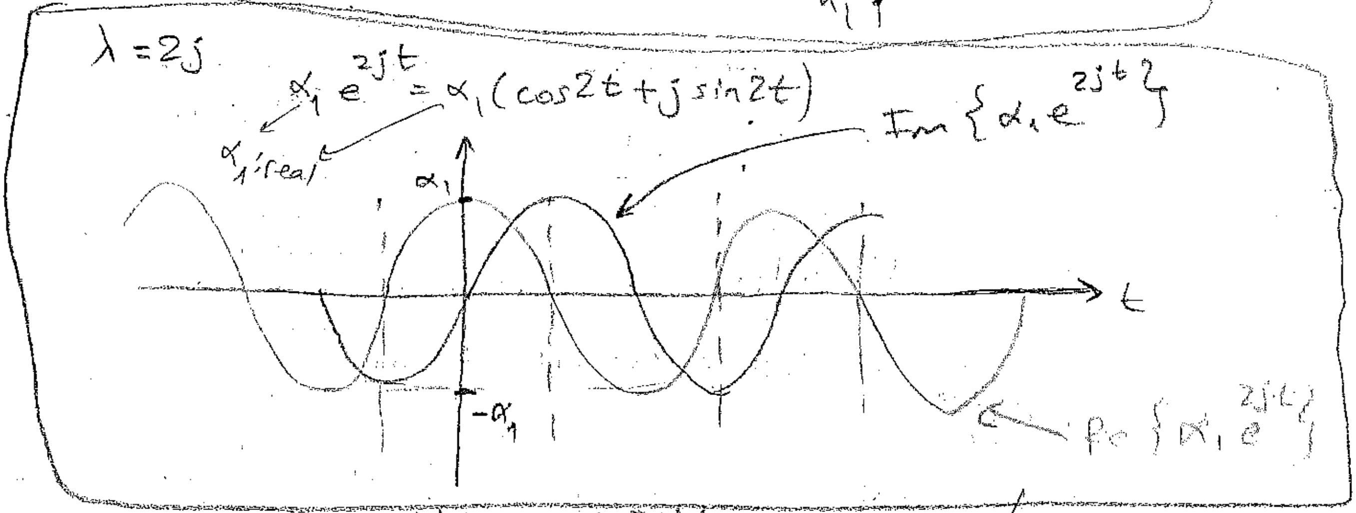
$$= \left(\sum_{k=1}^{N} \alpha_k e^{\lambda_k t}\right) + k_1 e^{s_0 t}$$

If all λ_k's are real and negative valued, then as $t \to \infty$,

$\vartheta_k^h(t) \longrightarrow 0$ (homogeneous solution vanishes as t increases)

A circuit is called stable if $\mathrm{Re}\{\lambda_k\} < 0$ $k = \{1, \dots, N\}$
that is, real part of natural frequencies should be negative.





Clearly the function does not decay

$\lambda = -1 + 2j$, $\alpha_1 e^{\lambda t} = \alpha_1 e^{(-1+2j)t}$

$$= \alpha_1 e^{-t} \cdot e^{2jt}$$

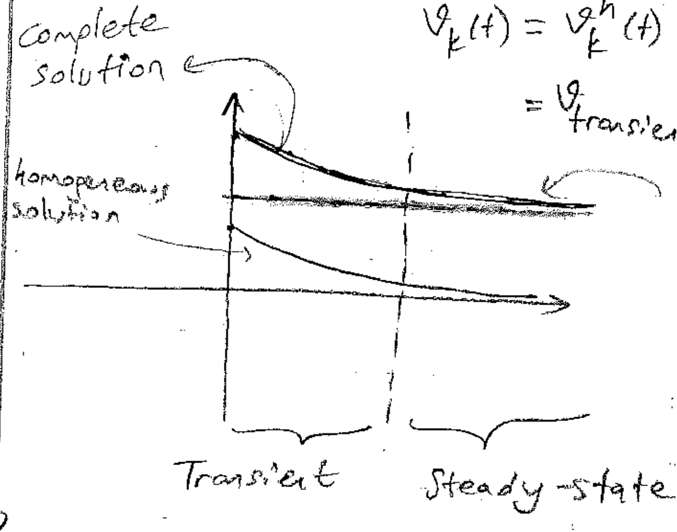
$$= \underbrace{\alpha_1 e^{-t} \cdot (\cos 2t + j\sin 2t)}$$

← clearly we see a decaying function

So for stable circuits as t → ∞ the complete solution → particular solution,
(approaches)

In other words,

$$v_k(t) = v_k^h(t) + v_k^p(t)$$
$$= v_{transient}(t) + v_{steady\text{-}state}(t)$$



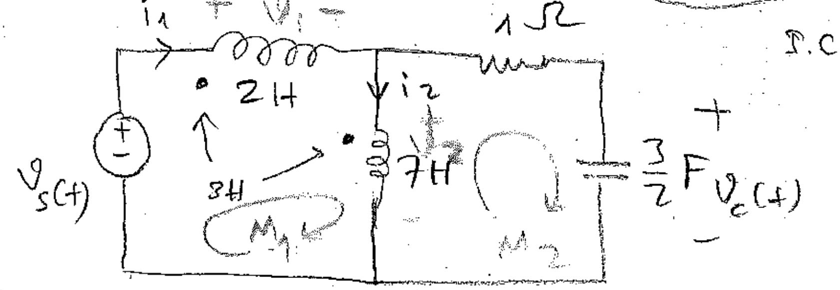
Transient ~~part~~ of the solution has the non-diminished effects of homogeneous solution,

as t → ∞ the complete solution becomes identical to the particular solution

Zero Input ile homogeneous arasındaki farkı ~~anlatacak~~ ///

Other Analysis Methods for Dynamic Circuits

① Mesh Analysis → Only for planar circuits ☆



I.C.s $\begin{cases} i_1(0^-) = I_1 \\ I_2(0^-) = I_2 \\ v_c(0^-) = V_0 \end{cases}$

the order of the circuit : 3 ??

$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} \dot{I}_1(t) \\ \dot{I}_2(t) \end{bmatrix}$$

$\dot{I}_x$ $\quad \frac{d}{dt} i_1(t)$ $\quad I_x - I_y$

KVL @ M_1: $-v_s(t) + v_1 + v_2 = 0$

$-v_s(t) + 5\dot{I}_1(t) + 10\dot{I}_2(t) = 0$

$$\boxed{-v_s(t) + 15\dot{I}_x(t) - 10\dot{I}_y(t) = 0}$$

→ mesh equation aims to do analysis with mesh currents, so every equation should have mesh currents i_x and i_y as unknowns.

Mesh eq #1: ↗

KVL @ M_2: $-v_2 + 1 \cdot I_y + v_c(t) = 0$ $\rightarrow v_c(0^-) + \frac{1}{c}\int_{0^-}^{t} i_{cap}(\tau)\, d\tau$

$\boxed{-10\dot{I}_x(t) + 7\dot{I}_y(t)}$ $\quad i_y(\tau)$

$$D = \frac{d}{dt}, \quad D^{-1} = \int_{0^-}^{t} (\;)\, d\tau \qquad D^{-1}\{f(t)\} = \int_{0^-}^{t} f(\tau)\, d\tau$$

Mesh eq #2: $\boxed{-10\dot{I}_x + 7\dot{I}_y + I_y + V_0 + \frac{1}{c} D^{-1}\{I_y\} = 0}$

We can ~~take~~ take derivative for all t>0

$$\begin{bmatrix} 15D & -10D \\ -10D & 7D+\frac{2}{3}D^{-1}+1 \end{bmatrix} \begin{bmatrix} Ix(t) \\ Iy(t) \end{bmatrix} = \begin{bmatrix} v_s(t) \\ -V_0 \end{bmatrix}$$

→ we can take derivative of this side

Mesh eq. set

Finding Natural Frequencies

$$\underbrace{v_c(t)}_{\text{complete solution (for capacitor)}} = \underbrace{\alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \alpha_3 e^{\lambda_3 t}}_{\text{homogeneous solution}} + \text{particular solution}$$

Find $\lambda_1, \lambda_2, \lambda_3$:

Assume $v_s(t) = 0 \rightarrow \underbrace{\begin{bmatrix} I_x \\ I_y \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{\lambda t}}_{\text{Guess for zero-input solution}}$

→ Insert guess into differential equation

$$\rightarrow \overbrace{\begin{bmatrix} 15\lambda & -10\lambda \\ -10\lambda^2 & 7\lambda^2+\frac{2}{3}+\lambda \end{bmatrix}}^{M} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ trivial soln. $\begin{bmatrix} I_x(t) \\ I_y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

→ Non-trivial solution:

$\det(\underline{\underline{M}}(\lambda)) = 0 \rightarrow \boxed{15\lambda \cdot (7\lambda^2 + \frac{2}{3} + \lambda - 100\lambda^3 = 0}$

↳ roots of this polynomial = nat frequencies

$\lambda = \{0, -1, -2\}$

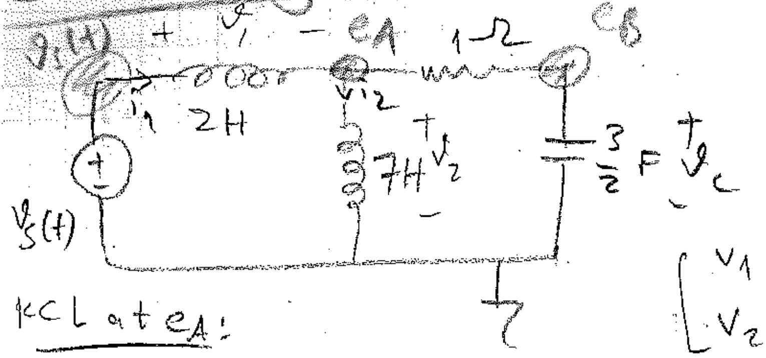
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{\underline{A}}\ \underline{x} = \underline{b} \qquad \underline{\underline{A}}^{-1} \underline{\underline{A}}\ \underline{x} = A^{-1} \underline{b}$

If $\det\{A\} = 0$ $\qquad \boxed{\underline{x} = \underline{A}^{-1} \underline{b}}$

→ there is no inverse

② Node Analysis



I.C.s $\begin{cases} i_1(0^-) = I_1 \\ i_2(0^-) = I_2 \\ v_c(0^-) = V_0 \end{cases}$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_1 \\ \frac{d}{dt} i_2 \end{bmatrix}$$

Di_1

Di_2

KCL at e_A:

$$\frac{e_A - e_B}{1} + i_2 - i_1 = 0$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}^{-1}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix}$$

$$D^{-1} \frac{1}{5} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = D^{-1} \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} \rightarrow \begin{bmatrix} i_1 - i_1(0^-) \\ i_2 - i_2(0^-) \end{bmatrix} = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} - \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

① $D \cdot D^{-1} \{f(t)\} = \frac{d}{dt} \int_{0^-}^{t} f(\tau) d\tau = f(t)$

② $D^{-1} \cdot D \{f(t)\} = \int_{0^-}^{t} \dot{f}(\tau) d\tau = f(t) - f(0^-)$

$$\frac{e_A - e_B}{1} + I_2 - I_1 + \frac{D^{-1}}{5} (-10 v_1 + 5 v_2) = 0$$

e_A $v_s - e_A$

Node eq. #1.
$$\boxed{\frac{e_A - e_B}{1} + I_2 - I_1 - 2D^{-1}\{v_s\} + 2D^{-1}\{e_A\} + D^{-1}\{e_A\} = 0}$$

KCL at e_B: $\frac{e_B - e_A}{1} + \frac{3}{2} \dot{e}_B = 0$

Node eq. #2
$$\boxed{e_B - e_A + \frac{3}{2} D\{e_B\} = 0}$$

$$\begin{bmatrix} 1 + 3D^{-1} & -1 \\ -1 & 1 + \frac{3}{2}D \end{bmatrix} \begin{bmatrix} e_A(t) \\ e_B(t) \end{bmatrix} = \begin{bmatrix} 2D^{-1}\{v_s(t)\} + I_1 \cdot I \dots \\ 0 \end{bmatrix}$$

↗ Integral differential equation

Finding Natural Frequencies

(1) Take derivative of 1^{st} Node equation to make it D^{-1} free

$$\left[\begin{array}{c|c} (D+3) & -D \\ \hline -1 & 1+\frac{3}{2}D \end{array}\right]\begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} 2\,v_s(t) \\ 0 \end{bmatrix}$$

Make a guess for homogeneous solution $(v_s(t)=0)$

$$\begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{\lambda t}$$

trivial solution

non-trivial soln.

$$\left|\begin{array}{c|c} \lambda+3 & -\lambda \\ \hline -1 & 1+\frac{3}{2}\lambda \end{array}\right| = 0 \longrightarrow (\lambda+3)\left(\frac{3\lambda}{2}+1\right)-\lambda = 0$$

If we take derivative one more time λ will be added, $\boxed{\lambda=0}$ will be a solution.

$$\frac{3\lambda^2}{2}+\frac{9\lambda}{2}+\cancel{\lambda}+3-\cancel{\lambda}=0$$

$$\lambda^2+3\lambda+2=0$$

$$\lambda = \{-1,-2,0\}$$

Comment: ① State equations result in correct number of nat. frequencies and their values.

② Mesh/Node Analysis results in all natural frequency; but the ones with $\underline{\lambda=0}$ can be missing or can appear without any physical meaning.

Modified Node Analysis (MNA)

Node or Mesh analysis may result in a set of integro-differential equations, that is an equation system containing both $D=\frac{d}{dt}$ and $D^{-1}=\int_0^t(\ )d\tau$ operators.

MNA aims to write a differentiation, as in state equations, describing the circuit.

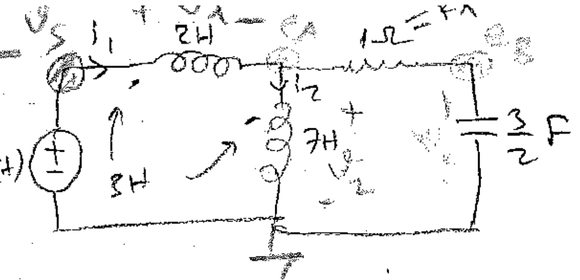
In general MNA equations are much simpler to write, but they may contain many more equations than Node/Mesh analysis.

MNA recipe

Auxiliary variables

1) Introduce current variables for inductor currents.
2) Introduce current variables for transformer primary/secondary side currents.
3) If there's a need introduce current variables for voltage source currents.
4) Write KCL equations as in Node Analysis.
5) Write a component equation for each introduced auxiliary variables such that at the end, we have "N" equations for "N" unknowns.

Ex:



$v_S - e_A$

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} D i_1 \\ D i_2 \end{bmatrix}$$

e_A

Note that I've only $D = \frac{d}{dt}$ in my equations!
↓
So, difn. equation

$$\begin{array}{l} \text{KCL @ } e_A: \\ \text{KCL @ } e_B: \\ \text{Mut.} \\ \text{Inductors} \end{array} \underbrace{\begin{bmatrix} 1/R_1 & -1/R_1 & -1 & 1 \\ -1/R_1 & \frac{1}{R_1}+\frac{3}{2}D & 0 & 0 \\ 1 & 0 & 2D & 3D \\ -1 & 0 & 3D & 7D \end{bmatrix}}_{\underline{\underline{M}}} \begin{bmatrix} e_A \\ e_B \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_S(t) \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}}(D) \begin{bmatrix} e_A \\ e_B \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_S(t) \\ 0 \end{bmatrix}$$

To find nat. freqs:

Assume $v_S(t) = 0$

$$\text{Guess: } \begin{bmatrix} e_A \\ e_B \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} e^{\lambda t}$$

Insert Guess into difn. eqn:

$$\underline{\underline{M}}(D) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We substitute λ into D.

$$\underline{\underline{M}}(\lambda) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

→ trivial

→ non-trivial solution

For non-trivial solution

$$\det(M(\lambda)) = 0$$

↑ char poly.! and roots of char. poly. are the nat. freq.

Check the course website!

Solution of Dif Eqn. Systems with Laplace Transform

$f(t) \longleftrightarrow F(s)$

Laplace Transform

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

s: a complex number selected such that the integral does not diverge!

$F(s) = \mathcal{L}\{f(t)\}$ another notation

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds \text{, where } \sigma \in \text{R.O.C. of } F(s)$$

$j = i = \sqrt{-1}$

Inverse Laplace transform

We won't use inverse Laplace transform formula in this course,

region of convergence

Laplace Transform Pairs

① $f(t) = u(t)$

Step function

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} 1 \cdot e^{-st} dt$$

$$= \frac{e^{-st}}{-s}\Bigg|_{t=0}^{t=\infty} = \cancel{e^{-s\infty}} + \frac{1}{s}$$

for $s > 0 \rightarrow$ (this term) vanishes

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}}$$

② $f(t) = e^{\alpha t} u(t) \longleftrightarrow \frac{1}{s-\alpha} = \left[\int_{0^-}^{\infty} \underbrace{e^{\alpha t} e^{-st}}_{1 \cdot e^{-(s-\alpha)}} dt = \frac{1}{s-\alpha} \right]$

Property

If $f(t) \longleftrightarrow F(s)$

then $f(t) e^{\alpha t} \longleftrightarrow F(s-\alpha)$ ☆

③ $f(t) = \sin\omega t \longleftrightarrow$

Euler's formula

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$-\quad e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin\omega t \longleftrightarrow F(s) = \mathcal{L}\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\}$$

$$= \frac{1}{2j}\left[\mathcal{L}\{e^{j\omega t}\} - \mathcal{L}\{e^{-j\omega t}\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

④ $\cos\omega t \longleftrightarrow \dfrac{s}{s^2+\omega^2}$

⑤ $te^{-at} \longleftrightarrow \dfrac{-d}{ds}\mathcal{L}(e^{-at}) = -\dfrac{d}{ds}\left(\dfrac{1}{s+a}\right) = \dfrac{1}{(s+a)^2}$

Property if $f(t) \longleftrightarrow F(s)$

then $tf(t) \longleftrightarrow \dfrac{-d}{ds}F(s)$ — $-\dfrac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt$

$\mathcal{L}\{tf(t)\}$ — $\int f(t)\dfrac{d}{ds}e^{-st}dt$

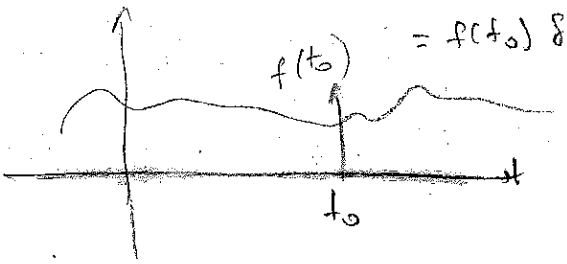
$\int tf(t)e^{-st}$

⑥ $\delta(t) \longleftrightarrow 1$

$$\int_{0^-}^{\infty}\delta(t)e^{-st}dt = \int_{0^-}^{\infty}\delta(t)e^{-s.0}dt = \int_{0^-}^{\infty}\delta(t)\cdot 1\cdot dt = 1.\int_{0}\delta(t)dt$$

$= 1$

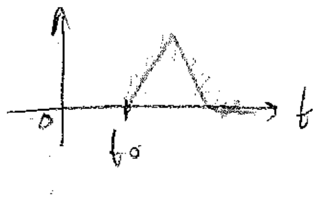
since $f(t)\,\delta(t-t_0)$
$= f(t_0)\,\delta(t-t_0)$



⑦ $\delta(t-t_0) \longleftrightarrow ?$

Property $f(t)u(t) \longleftrightarrow F(s)$

$f(t-t_0)u(t-t_0) \longleftrightarrow e^{-st_0}F(s)$



$$\mathcal{L}\{\overbrace{f(t-t_0)u(t-t_0)}^{g(t)}\} = \int_{0^-}^{\infty}\overbrace{f(t-t_0)u(t-t_0)}^{g(t)}e^{-st}dt$$

$$= \int_{t=t_0}^{\infty} f(t-t_0)e^{-st}dt = e^{-st_0}\int_{\tau=0}^{\infty} f(\tau)e^{-s\tau}d\tau$$

$\tau+t_0$

$= e^{-st_0}.F(s)$

⑧ Integration and Differentiation Property

Property:

$$f(t) \longleftrightarrow F(s)$$

$$\frac{d}{dt}f(t) \longleftrightarrow sF(s) - f(0^-)$$

Proof: Put this and use integration by parts

$$D^{-1} = \int_{0^-}^{t} (\ )\,d\tau \qquad f(t) \longleftrightarrow F(s)$$

$$\int_{0^-}^{t} f(\tau)\,d\tau \longleftrightarrow \frac{F(s)}{s}$$

!!

function of time

⇓

area between $[0^-, t]$ of argument

Ex: Use Laplace Transform to solve:

$$(D^2+3D+2)\,v_c(t) = f(t) \quad (*)$$

$$v_c(0^-) = V_0$$

$$\dot{v}_c(0^-) = \dot{V}_0$$

$t > 0^-$ Let $\mathcal{L}\{v_c(t)\} = V_c(s)$

Sliler büyük harf

then $\mathcal{L}\left\{\frac{d}{dt}v_c(t)\right\} = s\mathbf{V}_c(s) - v_c(0^-)$

$$\mathcal{L}\left\{\frac{d^2 v_c(t)}{dt^2}\right\} = s^2\mathbf{V}(s) - s\mathbf{V}_c(0^-) - \dot{v}_c(0^-)$$

Known I.C. Values

If $(*)$ holds for all $t > 0^-$ then I can multiply both LHS and RHS by e^{-st} and then integrate from $t=0^-$ to $t=\infty$ → after this operation equality should still hold.

$$\mathcal{L}\{D^2 v_c(t)\} + 3\mathcal{L}\{D v_c(t)\} + 2\mathcal{L}\{v_c(t)\} = \mathcal{L}\{f(t)\}$$

$$V_c(s)\,(s^2+3s+2) - sV_0 - \dot{V}_0 - 3V_0 = F(s)$$

$$\mathbf{V}_c(s) = \frac{F(s) + (s+3)V_0 + \dot{V}_0}{(s^2+3s+2)}$$

$$\underbrace{\mathbf{V}_c(s)}_{\text{complete solution in Laplace domain}} = \underbrace{\frac{(s+3)V_0 + \dot{V}_0}{(s+2)(s+1)}}_{\text{part related with I.C.}} + \underbrace{\frac{F(s)}{(s+2)(s+1)}}_{\text{part related with } f(t)}$$

part related with I.C. = zero input part

part related with $f(t)$ = zero-state part

$f(t)$ ⟵ external input

↑ forcing term

Let's focus on

external input

① zero-input part $(f(t) = 0)$

$$V_c^{z.i.}(s) = \frac{(s+3)V_0 + \dot{V}_0}{(s+2)(s+1)}$$

I need $v_c^{z.i.}(t)$ ← time domain

$$v_c^{z.i.}(t) = \mathcal{L}^{-1}\left\{V_c^{z.i.}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{(s+3)V_0 + \dot{V}_0}{(s+2)(s+1)}\right\}$$

$$\frac{A}{s+2} + \frac{B}{s+1}$$

$$\frac{(-2+3)V_0 + \dot{V}_0}{(-2+1)} = \frac{V_0 + \dot{V}_0}{-1}$$

$$(s+1)\frac{(s+3)V_0 + \dot{V}_0}{(s+2)(s+1)} = \frac{A}{(s+2)}(s+1) + \frac{B}{(s+1)}(s+1)$$

Insert $s = -1$

$$\frac{2V_0 + \dot{V}_0}{1} = \frac{A\cancel{0}}{\cancel{2}} + B$$

$$\boxed{B = \frac{2V_0 + \dot{V}_0}{1}}$$

$$V_c^{z.i.}(s) = \frac{-V_0 - \dot{V}_0}{s+2} + \frac{2V_0 + \dot{V}_0}{s+1}$$

$$\mathcal{L}^{-1}\{\ \} \rightarrow \boxed{v_c^{z.i.}(t) = (-V_0 - \dot{V}_0)e^{-2t}u(t) + (2V_0 + \dot{V}_0)e^{-t}u(t)}$$

Check Sadiku's book
Laplace transform
partial frac.
expressions.

② zero-state solution

→ state-variables take 0 value, that is all I.C. are zero, or zero initial energy.

$$V_c^{z.s.}(s) = \frac{F(s)}{(s+2)(s+1)}$$

Case of $f(t) = u(t) \longrightarrow F(s) = 1/s$, $V_c^{z.s.}(s) = \frac{1/s}{(s+2)(s+1)}$

Step response

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$A = \frac{1}{2}$; $B = -1$ $C = \frac{1}{2}$

$$V_c^{z.s.}(s) = \frac{1}{s(s+1)(s+2)} \doteq \frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}$$

$$\mathcal{L}^{-1}\{\} \rightarrow \boxed{v_c^{step}(t) = \frac{1}{2}u(t) - e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t)}$$

Case of $f(t) = \delta(t)$ (impulse response) $\rightarrow F(s) = 1$

$$V_c^{z.s.}(t) = \frac{1}{(s+2)(s+1)} = \frac{-1}{s+2} + \frac{1}{s+1}$$

$$\boxed{v_c^{impulse}(t) = (-e^{-2t} + e^{-t})u(t)}$$

Case of $f(t) = 3e^{-t}u(t)$

$$\rightarrow F(s) = \frac{3}{s+1} \rightarrow V_c^{z.s.}(t) = \frac{3}{(s+1)^2(s+2)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Note: When we write this equation we implicittly assume that A, B, C are scalars, i.e. numbers!

Important Note: Make sure that A, B, C is indeed a scalar, not something else.

$$A = \frac{3}{(-1)^2} = 3$$

$$B = ---$$

$$C = \frac{3}{1} = 3$$

$$\frac{C(s+D)}{(s+1)^2}$$

$$\frac{C(s+1-1)+D}{(s+1)^2}$$

$$\frac{C(s+1)}{(s+1)^2} + \frac{C-1+D}{(s+1)^2}$$

Cs+D / deneye gerek yok

☆ How to find B (when we have repeated poles in Laplace domain)

i.e. singularities

$$\frac{3}{(s+1)^2(s+2)} = \frac{3}{s+2} + \frac{B}{s+1} + \frac{3}{(s+1)^2}$$

$(s+1)^2 \quad (s+1)(s+2) \quad (s+2)$

$B = -3$

~~2nd way~~ or

put $s = 0$ (or any number) $\frac{3}{2} = \frac{3}{2} + \frac{B}{1} + \frac{3}{1}$

~~3rd way~~ or Steps

① Multiply by $(s+1)^2$ $\longrightarrow \frac{3}{s+2} = A\frac{(s+1)^2}{s+2} + B(s+1) + C$

② Take derivative of both parts $\rightarrow \frac{-3}{(s+2)^2} = A\frac{d}{ds}\left(\frac{(s+1)^2}{s+2}\right) + B$

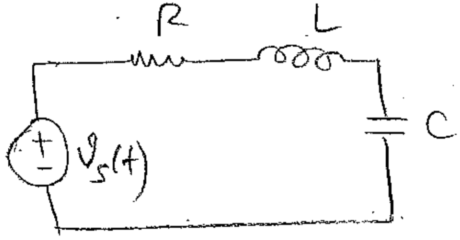
③ Evaluate at $s = -1$ $\rightarrow \frac{-3}{1} = A0 + B$

$\rightarrow \boxed{B = -3}$

Pls check Sadiku's book notes are also on the web!

$$v_c^{z.s.}(t) = (3e^{-2t} - 3e^{-t} + 3te^{-t})u(t)$$

Laplace Domain Analysis of State Equation System



$$\begin{bmatrix} \dot{v}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_s(t)$$

$$\begin{bmatrix} v_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} \qquad t \geq 0^-$$

Apply Laplace transfer to both parts

$$\begin{bmatrix} sV_c(s) - V_0 \\ sI_L(s) - I_0 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} & \\ \hline & \end{array}\right]}_{A} \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} + \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{b} V_s(s)$$

capital letter

$$s\underline{\underline{I}} \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} - A \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = bV_s(s) + \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$(s\underline{\underline{I}} - \underline{\underline{A}}) \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = \underline{b}\, V_s(s) + \begin{bmatrix} V_0 \\ I_0 \end{bmatrix}$$

$$\begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix} + (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} b V_s(s)$$

$L = 1\,H$
$R = 6\,\Omega$
$C = \frac{4}{100}\,F$

$$[s\underline{\underline{I}} - \underline{\underline{A}}]^{-1} = \left[\begin{array}{c|c} s & -25 \\ \hline 1 & s+6 \end{array}\right]^{-1} = \frac{1}{\Delta} \begin{bmatrix} s+6 & 25 \\ -1 & s \end{bmatrix}$$

$\Delta = s(s+6) + 25$
$= s^2 + 6s + 25$

Let's focus on zero-input solution ($v_s(t) = 0$)

$$\begin{bmatrix} V_c^{z.i}(s) \\ I^{z.i}(s) \end{bmatrix} = \frac{1}{s^2+6s+25} \begin{bmatrix} s+6 & 25 \\ -1 & s \end{bmatrix} \begin{bmatrix} V_0 \\ I_0 \end{bmatrix},$$

$\mathcal{L}\{\}$

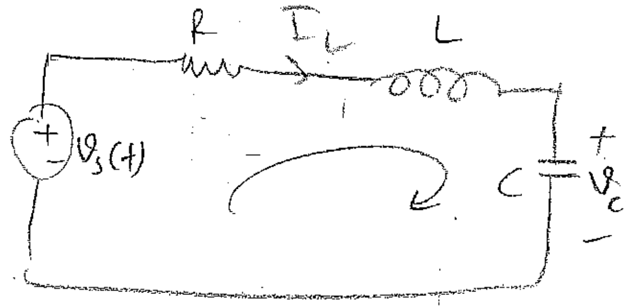
$$\begin{bmatrix} V_c^{z.i}(s) \\ \text{-----} \\ I_L^{z.i}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{sV_0 + 6V_0 + 25I_0}{s^2+6s+25} \\ \text{-----} \\ \dfrac{sI_0 + V_0}{s^2+6s+25} \end{bmatrix} = \begin{bmatrix} \dfrac{(s+3)V_0 + 3V_0 + 25I_0}{(s+3)^2 + 4^2} \\ \text{-----} \\ \dfrac{(s+3)I_0 - 3I_0 - V_0}{(s+3)^2 + 4^2} \end{bmatrix}$$

$$\begin{bmatrix} v_c^{z.i.}(t) \\ ---- \\ i_L^{z.i.}(t) \end{bmatrix} = \begin{bmatrix} V_0 e^{-3t}\cos 4t + \frac{3V_0+25I_0}{4} e^{-5t} \sin 4t \\ I_0 e^{-3t}\cos 4t + \frac{(-3I_0-V_0)}{4} e^{-3t} \sin 4t \end{bmatrix}, \quad t>0$$

$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t$

Ex:



Apply mesh analysis and transform the mesh equations to Laplace domain.

$$\int_{0^-}^{\infty} V_0 e^{-st}\,dt \quad \mathcal{L}\{V_0 u(t)\}$$

$$\text{Mesh 2: } -v_s(t) + R i_L + \underbrace{L\frac{d}{dt} i_L(t)}_{v_L(t)} + \underbrace{v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_L(\tau)\,d\tau}_{v_c(t)} = 0, \quad t>0$$

$$\mathcal{L}\{\ \} \rightsquigarrow -V_s(s) + RI_L(s) + L\left(sI_L(s) - I_0\right) + \frac{V_0}{s} + \frac{1}{C}\frac{I_L(s)}{s} = 0$$

$$I_L(s) = \frac{V_s(s) - V_0/s + I_0}{R + sL + \frac{1}{Cs}} = \underbrace{\frac{\frac{s}{L}\left(-\frac{V_0}{s}+I_0\right)}{s^2+\frac{R}{L}s+\frac{1}{LC}}}_{I_L^{z.i.}(s)} + \underbrace{\frac{\frac{s}{L}V_s(s)}{s+\frac{R}{L}s+\frac{1}{LC}}}_{I_L^{z.s.}(s)}$$

A.C. Circuit Analysis

① Phasor Concept:

$$v(t) = V_m\cos(\omega t+\phi) = \mathrm{Re}\left(V_m e^{(j\omega+\phi)}\right) = \mathrm{Re}\left(V_m \angle\phi\, e^{j\omega t}\right)$$

phasor

Phasers help us to reduce the computations with trigonometric functions.

Euler's Formula: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

$j = \sqrt{-1}$

$$\boxed{\angle\theta = \cos\theta = \cos\theta + j\sin\theta}$$

$A\angle\theta$ $\cos + j\sin$

cos... + sin

phasor: $\mathrm{Re}\{e^{j\theta}\} = \cos\theta$

$$A\cos(\omega t+\phi) = A\,\mathrm{Re}\left\{e^{j(\omega t+\phi)}\right\} \xleftrightarrow[\text{phasor}]{} A\angle\phi$$

Note that frequency (ω) is missing in $A\angle\phi$ representation

Ex: $f_1(t) = A_1\cos(\omega t+\phi_1)$

$f_2(t) = A_2\cos(\omega t+\phi_2)$ Q: $f_1(t)+f_2(t) = ?$

$$\hookrightarrow \mathrm{Re}\left\{A_1 e^{j(\omega t+\phi_1)}\right\} + \mathrm{Re}\left\{A_2 e^{j(\omega t+\phi_2)}\right\} = \mathrm{Re}\left\{A_1 e^{j(\omega t+\phi_1)} + A_2 e^{j(\omega t+\phi_2)}\right\}$$

$= \longrightarrow$

$$= \mathrm{Re}\{(\underbrace{A_1 e^{j\varphi_1}+A_2 e^{j\varphi_2}}_{X}) e^{j\omega t}\} = \mathrm{Re}\{\|X\| e^{j\angle X}\cdot e^{j\omega t}\}$$

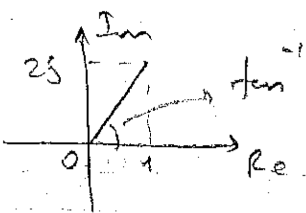
polar coord. rep. of X

$$= \|X\|\cdot \mathrm{Re}\{e^{j(\omega t+\angle X)}\}$$

$$= \|X\|\cos(\omega t+\angle X)$$

$i = 2j = 2i = 2\sqrt{-1}$

$X = 1 + j2 = \sqrt{5}\, e^{j\tan(\frac{2}{1})}$



$(1+2j)+(5+3j) = 6+j5$

$(1+2j)(5+3j) = (5-6)+j(10+3)$

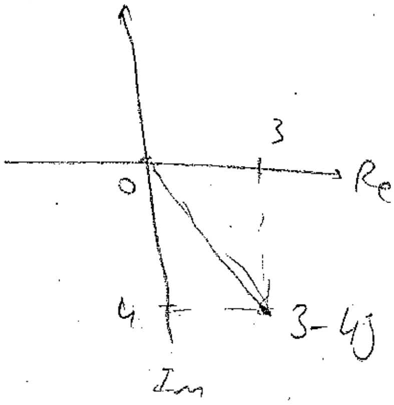
$\sqrt{5}\, e^{j\tan^{-1}(2)} \cdot \sqrt{34}\, e^{j\tan^{-1}(\frac{3}{5})} = \sqrt{170}\cdot e^{j(\tan^{-1}2+\tan^{-1}\frac{3}{5})}$

Exe $3\cos(\omega t) + 4\sin(\omega t) = ?$

$= \mathrm{Re}\{3e^{j\omega t} + 4e^{j(\omega t-90°)}\}$ $= \cos(\omega t - 90°)$

$= \mathrm{Re}\{e^{j\omega t}\underbrace{(3-4j)}_{5e^{-j\tan^{-1}(\frac{4}{3})}}\}$

$= 5\cos(\omega t - \tan^{-1}(\frac{4}{3}))$



$3\cos(2t) + 4\sin(2t) = ?$

phasor $\updownarrow$ phasor

$3\angle 0° + 4\angle -90°$ $(\omega = 2\ \text{rad/sec})$

$= 3-4j$

$= 5\angle -\tan^{-1}(\frac{4}{3})$

$\updownarrow$ phasor

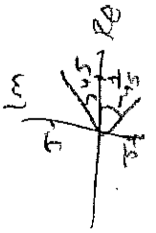
$5\cos(2t - \tan^{-1}\frac{4}{3})$

$e^{j\theta} = 1\angle\theta$

$j+1$

j

1



$3\cos(5t+\theta)$

$3\angle\theta$

$3\cos(5t+10°)$

$2\cos(5t+\theta)$

ph

$2\angle\theta$

$2\mathrm{Re}\{e^{j(5t+\theta)}\}$

$\mathrm{Re}\{2e^{j\theta} e^{j5t}\}$

$\mathrm{Re}\{3e^{j10} e^{j5t}\}$

$=$

$\mathrm{Re}\{\underbrace{(2e^{j\theta}+3e^{j10°})}_{\text{phasor}} e^{j5t}\}$

$e^{j} = \cos(\omega t) + j\sin(\omega t)$

$e^{-j} = \cos(\omega t) - j\sin(\omega t)$

$\frac{e^{j}+e^{-j}}{2} =$

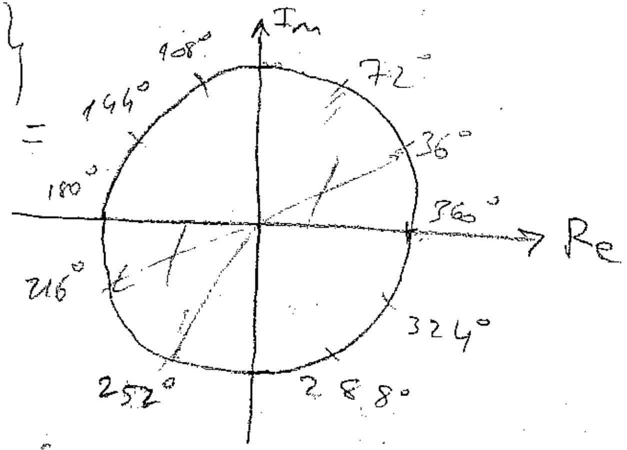
$\frac{e^{j}-e^{-j}}{2j}$

Ex: $\sum_{k=1}^{10} \cos\left(\omega t + \frac{2\pi}{10} k\right) = ?$

$\sum_{k=1}^{N} \mathrm{Re}\left\{ e^{j\omega t} \cdot e^{j\frac{2\pi}{10}k} \right\}$

$= \mathrm{Re}\left\{ e^{j\omega t} \left(\sum_{k=1}^{10} e^{j\frac{2\pi}{10}k} \right) \right\}$

$= 0$



Ex: $f(t) = 3\cos(5t + 30°)$

$\dot{f}(t) = \frac{df(t)}{dt} = -15\sin(5t+30°)$

frequencies are the same!

$\mathrm{Re}\left\{ 3 e^{j5t} \cdot e^{j30°} \right\}$

$\mathrm{Re}\left\{ e^{j5t} \left(3e^{j30°} - 15\, e^{j30°-90°} \right) \right\}$

$f(t) + \dot{f}(t) = ?$

$\updownarrow$

$3\angle 30° - 15\angle 30° - 90° \quad (\omega = 5)$

$90° - (30° + 5t)$

$\cos(60° - 5t) = \cos(5t - 60°)$

$3\angle 30° - 15\angle -60° \quad (\omega = 5)$

$= 3\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) - 15\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$

$= \left(\frac{3\sqrt{3} - 15}{2}\right) + j\left(\frac{3 + 15\sqrt{3}}{2}\right)$

$\boxed{\frac{3\sqrt{3} - 15}{2}} \boxed{\angle 0°} + \left(\frac{3+15\sqrt{3}}{2}\right) \angle 90°$

$\updownarrow$

$\frac{3\sqrt{3}-15}{2} \cos(5t) + \frac{3+15\sqrt{3}}{2} \underbrace{\cos(5t + 90°)}_{-\sin(5t)}$

$= \cos(5t + 0°)$

?

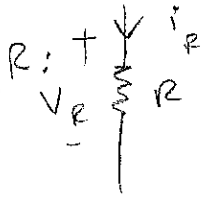
$M\angle\phi \longleftrightarrow \mathrm{Re}\left\{ M e^{j(\omega t + \phi)} \right\} \longleftrightarrow M\cos(\omega t + \phi)$

$M e^{j\phi} = M(\cos\phi + j\sin\phi)$

Euler's formula

Phasor Circuit Analysis and RLC in phasor Domain

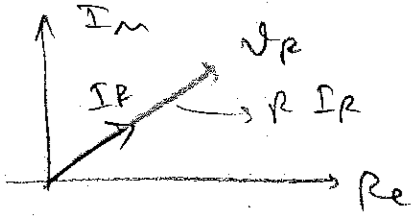
R:

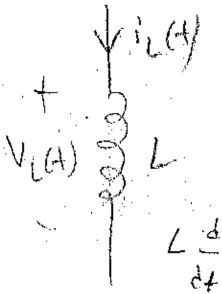


$i_R(t) = I_S \cos(\omega t + \phi)$

Time Domain	Phasor Domain
$i_R = I_S \cos(\omega t + \phi)$	$I_R = I_S \angle \phi$
$V_R = R I_S \cos(\omega t + \phi)$	$V_R = R I_S \angle \phi$

(Resistance scales current phasor by R.)





Time Domain	Phasor Domain
$i_L(t) = I_S \cos(\omega t + \phi)$	$I_L = I_S \angle \phi$
$V_L(t) = -L I_S \omega \sin(\omega t + \phi)$	$V_L = \omega L I_S \angle \phi + 90^\circ$

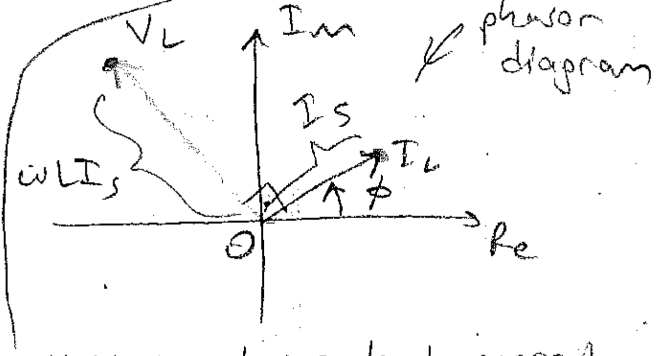
$L \frac{d}{dt} i_L(t) = -L I_S \omega \cos(\omega t + \phi - 90^\circ)$

$= L I_S \omega \cos(\omega t + \phi - 90^\circ + 180^\circ)$

$= \omega L I_S \cos(\omega t + \phi + 90^\circ)$

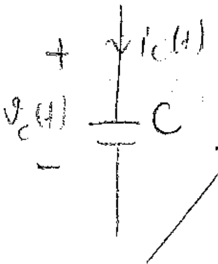
$= j\omega L I_S \angle \phi$

(Remember $j = 1 \angle 90^\circ$)



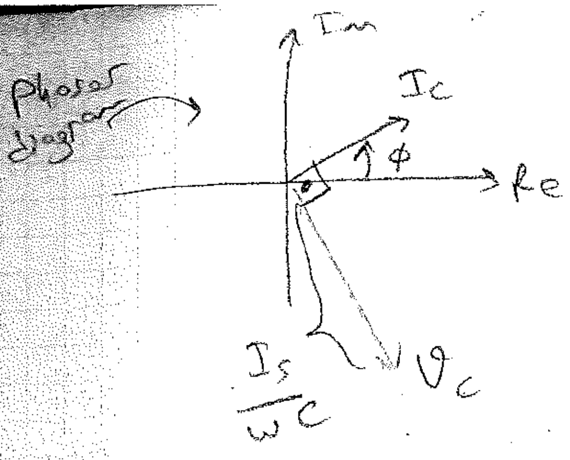
phasor diagram

Voltage phasor leads current phasor by 90° for the inductor.



Time Domain	Phasor Domain
$i_C(t) = I_S \cos(\omega t + \phi)$	$I_C = I_S \angle \phi$
$V_C(t) = \frac{I_S}{C\omega} \sin(\omega t + \phi)$	$V_C = \frac{I_S}{C\omega} \angle \phi - 90^\circ$
$= \frac{I_S}{C\omega} \cos(\omega t + \phi - 90^\circ)$	$= \left(\frac{I_S}{C\omega} \angle \phi\right)(1 \angle -90^\circ)$
	$= \frac{1}{j\omega C} I_S \angle \phi$

$C \frac{dV_C(t)}{dt}$



V_{cap} lags i_{cap} by 90°

Impedence: The ratio of voltage phasor over current phasor.

$$\boxed{Z = \frac{\text{Voltage phasor}}{\text{current phasor}}}$$

Complex number division result

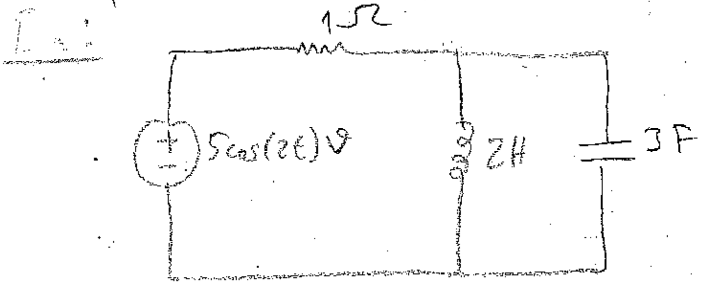
Analog
Digital

Impedence	Phasor domain representation
R ; $Z_R = R$	R
L ; $Z_L = j\omega L$	$j\omega L$
C ; $Z_C = \frac{1}{j\omega C}$	$\frac{1}{j\omega C}$

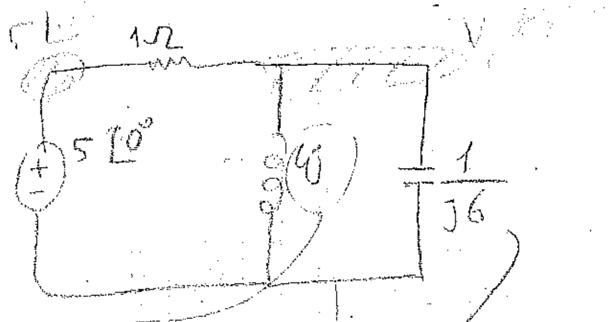
Digital Signal
↓
bitmap

362
Logic
[illegible]

Ex:



Find $i_s(t)$ at steady-state



Phasor domain circuit

(ω=2)

$Z_L = j\omega L = j \cdot 2 \cdot 2$
$= j4$

$Z_C = \frac{1}{j\omega C} = \frac{1}{j2 \cdot 3}$

KCL at V_c node

$$\frac{V_c - 5\angle 0^\circ}{1} + \frac{V_c}{j4} + \frac{V_c}{1/j6} = 0$$

$$V_c\left(1 + \frac{-j}{4} + 6j\right) = 5\angle 0^\circ$$

$$V_c = \frac{5\angle 0^\circ}{1 + j5.75}$$

$$V_c = \frac{5}{1 + j\frac{23}{4}} = \frac{5}{\sqrt{1 + \left(\frac{23}{4}\right)^2}} \angle -\tan^{-1}\left(\frac{23}{4}\right)$$

$$v_c^{s.s.}(t) = \frac{5}{\sqrt{1 + \left(\frac{23}{4}\right)^2}} \cos\left(2t - \tan^{-1}\left(\frac{23}{4}\right)\right)$$

AC Steady State Analysis

Review

Phasor: A complex number refering to amplitude and phase of a cosine waveform.

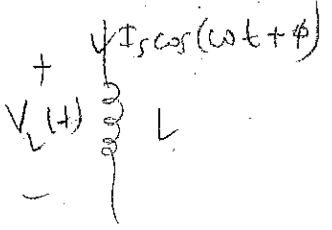
Ex:

$$3\cos(2t + 15^\circ) \xleftrightarrow[\text{domain}]{\text{phasor}} 3e^{j15^\circ} = 3(\cos 15^\circ + j\sin 15^\circ)$$

$\frac{\pi}{12}$ radians

(Another notation: $3\angle 15^\circ$)

Impedance: The ratio of Voltage phasor over current phasor of a circuit component.



$$\text{Impedence of } L = \frac{\text{Voltage phasor}}{\text{Current phasor}}$$

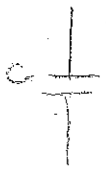
$$= \frac{\omega L I_s \angle 90^\circ + \phi}{I_s \angle \phi}$$

$$= \frac{j\omega L I_s \angle \phi}{I_s \angle \phi}$$

$$1 \cdot e^{j(90^\circ + \phi)} = e^{j90^\circ} \cdot e^{j\phi} = j e^{j\phi}$$

$$\boxed{Z_L = j\omega L}$$

$$\boxed{Z_R = R}$$



$$\boxed{Z_C = \frac{1}{j\omega C}}$$

Comments

① $Z_i = R + jX$ a) Z_L has the units Ω (ohm) as in ordinary resistors.

b) $Re\{Z\} = R$ ⟵ Resistance

$Im\{Z\} = X$ ⟵ Reactance

② $Y = \frac{1}{Z}$ ⟶ $Re\{Y\}$ is called conductance (℧)

$Im\{Y\}$ is called susceptance.

↑ Admittance

<u>A funny mistake</u>: A component has resistance of 4 ohms and and reactance of 5 ohms. What's its conductance and susceptance?

Soln: $Z = 4 + j5$ Then $Y = \frac{1}{Z} = \frac{1}{4+j5} = \frac{4-j5}{41}$

$(4-j5)$

$Re\{Y\} = \text{conductance} = 4/41$

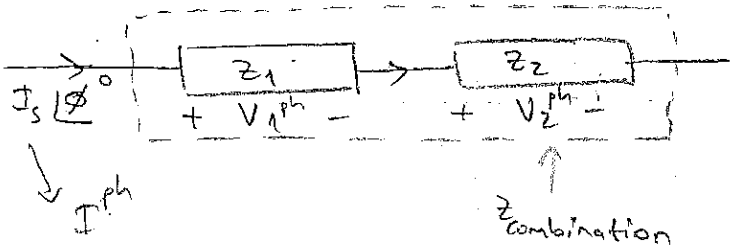
$Im\{Y\} = \text{susceptance} = \frac{-5}{41}$

$\text{conductance} \neq \frac{1}{4}$

$\text{Susceptance} \neq \frac{1}{5}$

Combination of Impedances

① Series Combination



$$V_1^{ph} = (Z_1) I_s \angle \phi$$

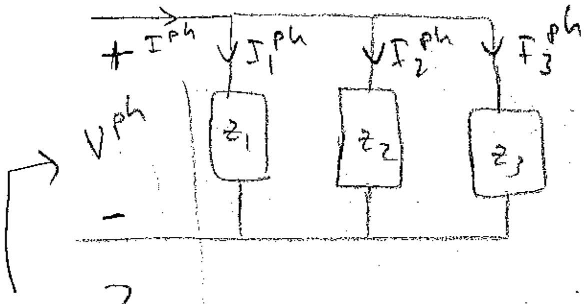
$$+ \; V_2^{ph} = (Z_2) I_s \angle \phi$$

$$V_1^{ph} + V_2^{ph} = (Z_1 + Z_2) I_s \angle \phi$$

$$\rightarrow Z_{combination} = \frac{V^{ph}_{combination}}{I^{ph}_{combination}}$$

$$= Z_1 + Z_2$$

② Parallel Combination



$Z_{eq} = ?$

All are cosine.

$$I^{ph} = I_1^{ph} + I_2^{ph} + I_3^{ph}$$

$$= \frac{V^{ph}}{Z_1} + \frac{V^{ph}}{Z_2} + \frac{V^{ph}}{Z_3}$$

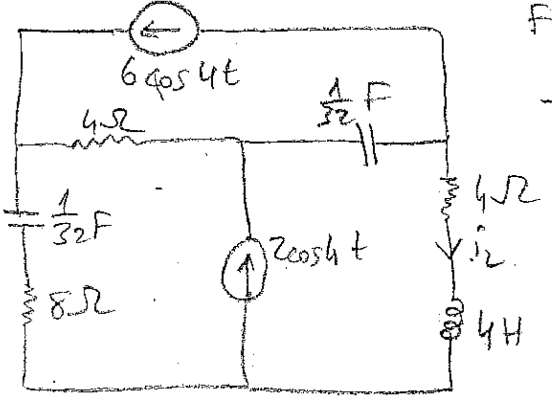
$$= \underbrace{\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)} V^{ph}$$

$$\frac{V^{ph}}{I^{ph}} = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)^{-1}$$

A.C. Steady State Analysis is identical to earlier resistive LTI circuit analysis methods with the Introduction of phasor and Impedence concepts.

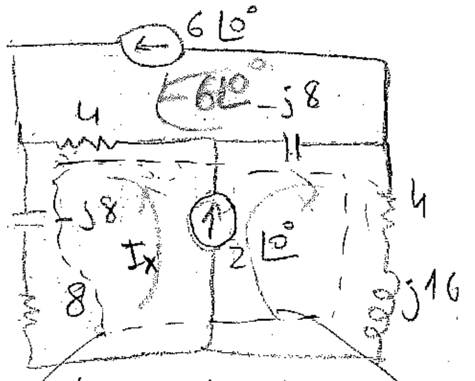
So, Thevenin Equivalents, Δ-Y conversions, Source Transformations, Mesh/Node analysis etc. are applicable as they are with the replacement of resistances with impedences.

Ex: Mesh Analysis



Find $i_L^{ss}(t)$ → steady state current of inductor L.

$\longrightarrow$ phasor domain



phasor domain circuit (ω = 4 rad/sec)

$\rightarrow 2\angle 0^\circ - I_x$

Supermesh

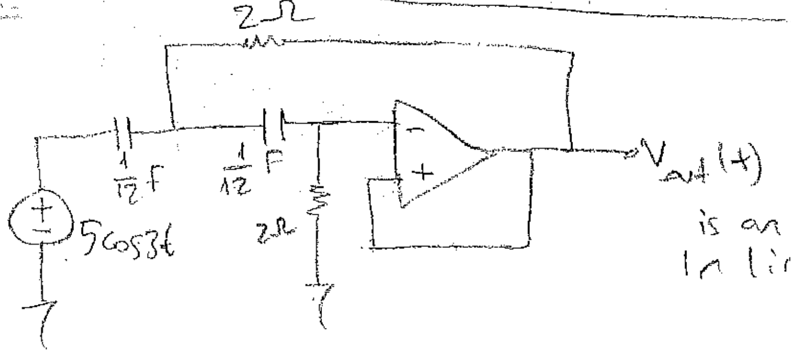
$$\text{KVL: } 4\,(6\angle 0^\circ - I_x) - j8\,(6\angle 0^\circ + 2\angle 0^\circ - I_x) + (4+j16)\,[2\angle 0^\circ - I_x] + (8-j8)(-I_x) = 0$$

$$I_x = \frac{-(24 - j64 + 8 + j32)}{-4 + \cancel{j8} - 4 - \cancel{j16} - 8 + \cancel{j8}} \qquad I_x = \frac{32 - j32}{16}$$

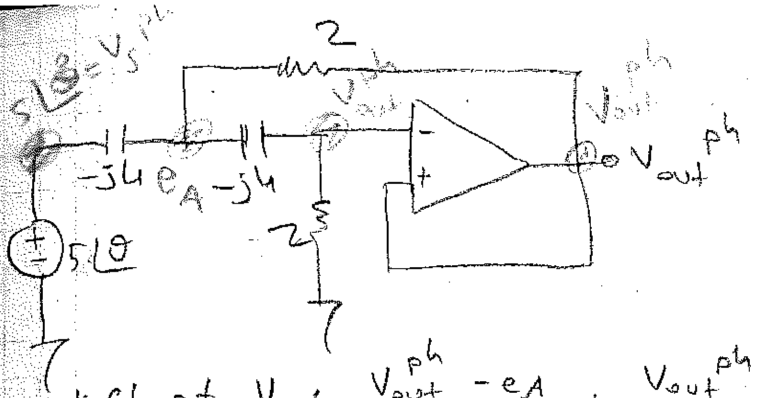
$$I_L = 2\angle 0^\circ - I_x = \frac{j32}{16} = j2$$ ← phasor value of $i_L^{ss}(t)$

$$I_L^{ph} = j2 = 2\angle 90^\circ \longrightarrow i_L^{ss}(t) = 2\cos(4t + 90^\circ)\ A$$

Ex:



Find $V_{out}(t)$ at steady-state (Assume op-amp is an ideal op-amp in linear region)



(ω = 3 rad/sec)

KCL at V_-: $\frac{V_{out}^{ph} - e_A}{-j4} + \frac{V_{out}^{ph}}{2} = 0 \rightarrow$

$$e_A = (1-j2)\,V_{out}^{ph}$$

KCL at e_A: $-j4\,/$ $\frac{e_A - V_s^{ph}}{-j4} + \frac{e_A - V_{out}^{ph}}{2} + \frac{e_A - V_{out}^{ph}}{-j4} = 0$

$$(1-j2+1)\underbrace{e_A}_{(1-j2)V_{out}^{ph}} + (j2-1)\,V_{out}^{ph} = V_s^{ph}$$

$$[(2-j2)(1-j2)+(2j-1)]\,V_{out}^{ph} = 5\angle\theta$$

$$(1-j2)(1-j2) = (1-j2)^2$$

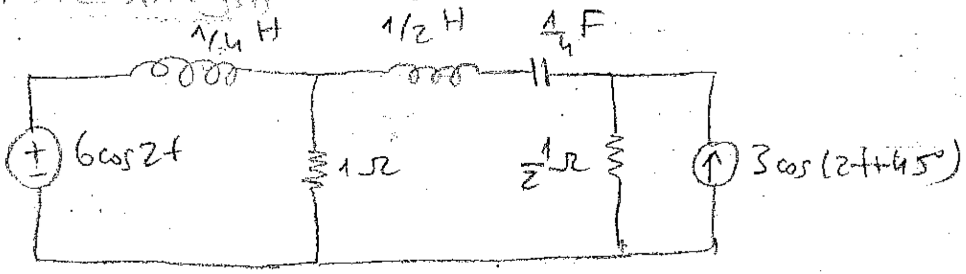
$$= -3-j4 \qquad V_{out}^{ph} = \frac{5\angle\theta}{-3-j4} = \frac{5\angle\theta}{5\angle \tan^{-1}\frac{4}{3}+180°}$$

$$\longrightarrow V_{out}^{ph} = 1\angle\theta - \tan^{-1}\tfrac{4}{3} - 180°$$

$$\boxed{V_{out}^{ss}(t) = 1\cos(3t+\theta-\tan^{-1}\tfrac{4}{3}-180°)\text{ Volts}} \rightarrow \text{the particular solution.}$$

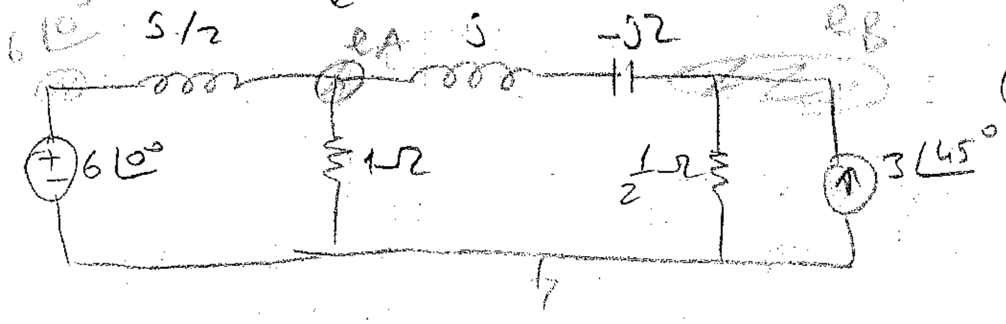
General solution $= V_{out}^{ss}(t) + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

Nodal Analysis



Find s.s. (steady state) node voltages.

phasor



(ω = 2 rad/sec.)

KCL at e_A:

$$\frac{e_A - 6\angle 0^\circ}{j/2} + \frac{e_A}{1} + \frac{e_A - e_B}{-j} = 0$$

KCL at e_B:

$$\frac{e_B - e_A}{-j} + \frac{e_B}{1/2} - 3\angle 45^\circ = 0$$

$$\begin{bmatrix} -j2+1+j & -j \\ -j & j+\frac{1}{2} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} -j/2 \\ 3\angle 45^\circ \end{bmatrix}$$

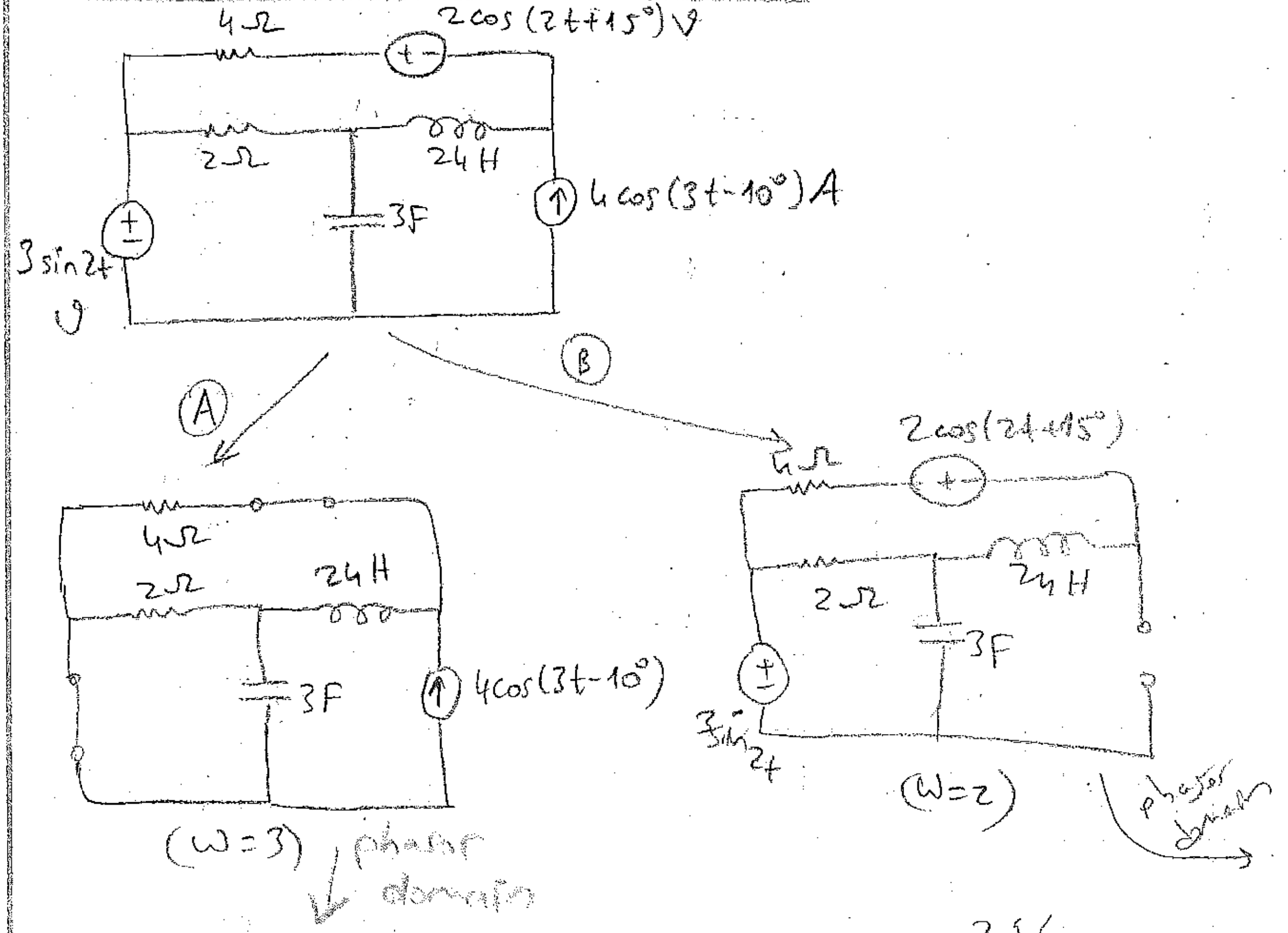
$$\hookrightarrow \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} 5.82\angle -51^\circ \\ 3.93\angle 14^\circ \end{bmatrix}$$

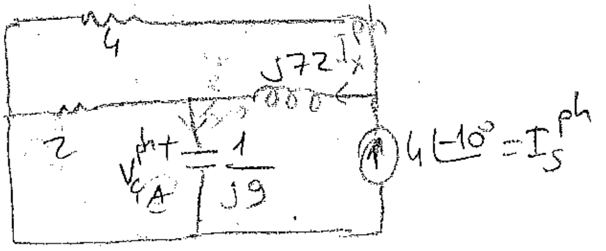
$e_A^{s.s.}(t) = 5.82\cos(2t-51^\circ)$ V

$e_B^{ss}(t) = 3.13\cos(2t-14^\circ)$ V

A.C. Analysis (cont'd)

Sources with different frequencies





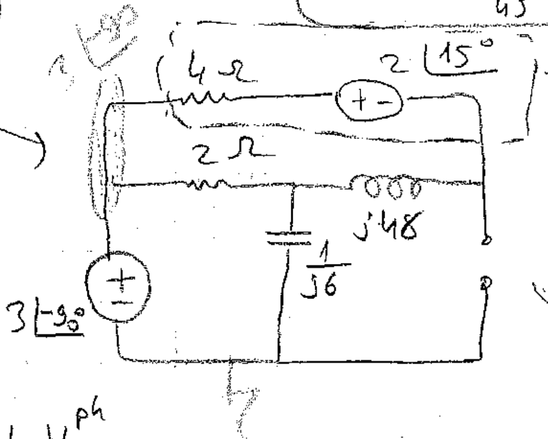
$$2 \,//\, \frac{-j}{9} = \frac{-2j/9}{2 - j/9} = \frac{-2j}{18-j}$$

$I_x^{ph} = I_s^{ph} \dfrac{4}{4 + \frac{-2j}{18-j} + j + 72}$ $\qquad I_y^{ph} = I_x^{ph} \dfrac{2}{2 - \frac{j}{9}}$

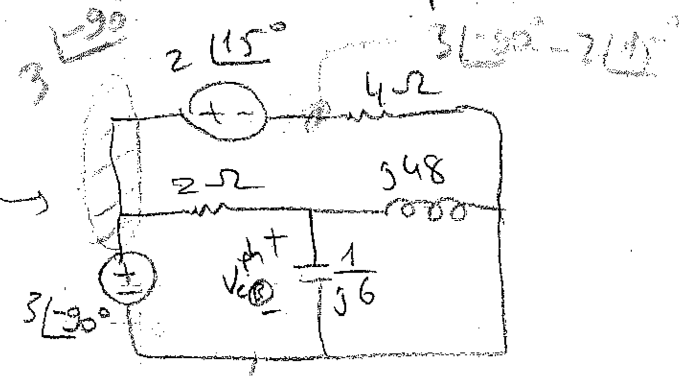
$\hookrightarrow V_{c_{Ⓐ}}^{ph} = I_y^{ph} \cdot \dfrac{1}{j9} = \dfrac{16}{45} \angle -100^\circ$

$\boxed{V^{s.s.}_{c,A}(t) = \frac{16}{45} \cos(3t - 100^\circ) \text{ Volts}}$

phasor domain (B)



→ We can change the places of V.S. and the 4Ω resistor.



KCL at $V_{c_{Ⓑ}}^{ph}$:

$$\frac{V_{c_{Ⓑ}}^{ph}}{\frac{1}{j6}} + \frac{V_{c_{Ⓑ}}^{ph} - 3\angle -90^\circ}{2} + \frac{V_{c_{Ⓑ}}^{ph} - (3\angle -90^\circ - 2\angle 15^\circ)}{4 + j48} = 0$$

$V_{c_{Ⓑ}}^{ph} = M_{Ⓑ} \cdot \angle \theta_{Ⓑ} \longrightarrow \boxed{V^{s.s.}_{c_{Ⓑ}}(t) = M_{Ⓑ} \cos(2t + \theta_{Ⓑ}) \text{ Volts}}$

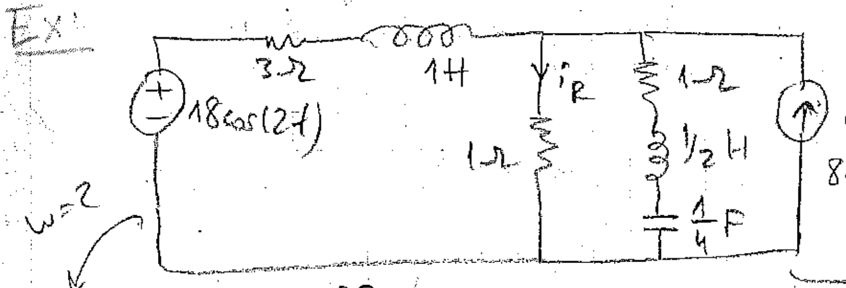
→ Magnitude

Final answer:

$V_c^{s.s.}(t) = V^{s.s.}_{c_{Ⓐ}}(t) + V^{s.s.}_{c_{Ⓑ}}(t)$

$= \frac{16}{45} \cos(3t - 100^\circ) + M_{Ⓑ} \cos(2t + \theta_{Ⓑ})$ Volts //

Ex:



Find $i_R^{s.s.}(t)$

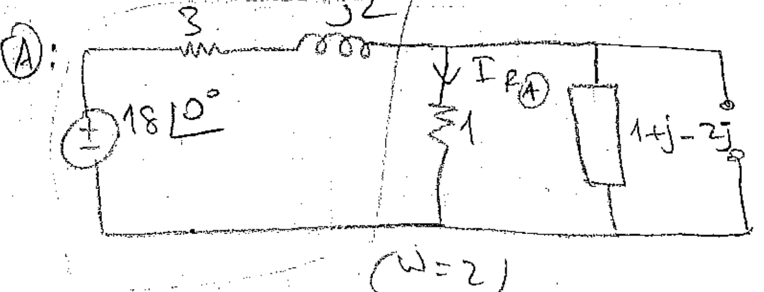
$8\cos(\omega t + 0^\circ) = 8$

ω=0

ω=2

ω = 0 Ⓑ

Ⓐ:



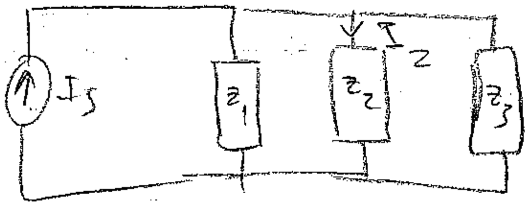
(ω=2)

Source transformation → current division

$\rightarrow I_{R_{Ⓐ}} = \dfrac{18\angle 0^\circ}{3+j2} \dfrac{1/1}{\frac{1}{1} + \frac{1}{(3+j2)} + \frac{1}{(1-j)}}$

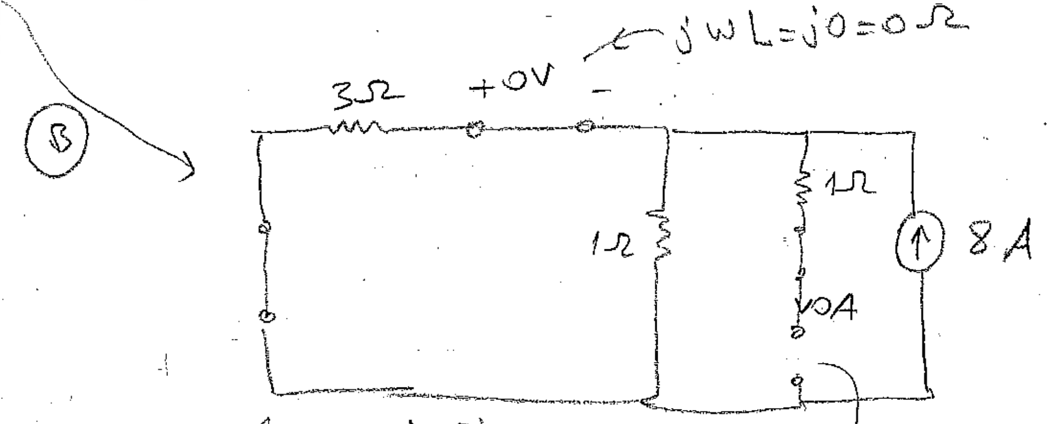
$= 2(1-j)$

$= 2\sqrt{2} \angle -45^\circ$



$$\longrightarrow I_Z = I_S \cdot \frac{1/Z_2}{1/Z_1 + 1/Z_2 + 1/Z_3}$$

$$\boxed{i_{R_{(A)}}^{S.S.}(t) = 2\sqrt{2}\cos(2t-45^\circ)\ A}$$



(ω=0)

$$I_{R_{(B)}} = 8\,\frac{1/1}{\frac{1}{1}+\frac{1}{3}} = 6A$$

$$\text{cap}: Z_c = \frac{1}{j\omega C}\Big|_{\omega=0} = \infty$$

$$i_{R_{(B)}}^{S.S.} = 6A$$

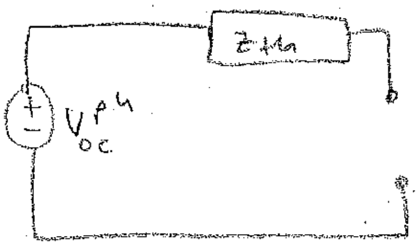
Final answer:

$$i_R^{S.S.}(t) = i_{R_{(A)}}^{S.S.}(t) + i_{R_{(B)}}^{S.S.}(t)$$

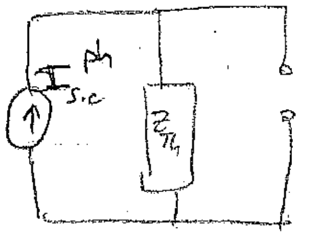
$$\boxed{i_R^{SS}(t) = 2\sqrt{2}\cos(2t-45^\circ) + 6\ ,\ A}$$

Thevenin - Norton Equivalents

The method for Thevenin-Norton equivalent finding applies with no changes in phasor domain. The only difference is, we have Z_{Th} (Thevenin impedance) instead of R_{Th}. So, the Thevenin equivalent in phasor domain is



Thevenin Eq.



Norton Eq.

Sadiku {
Pr. 10.56 p.449
10.25 p.445
10.31 p.446
10.55 p.449
10.62 p.450

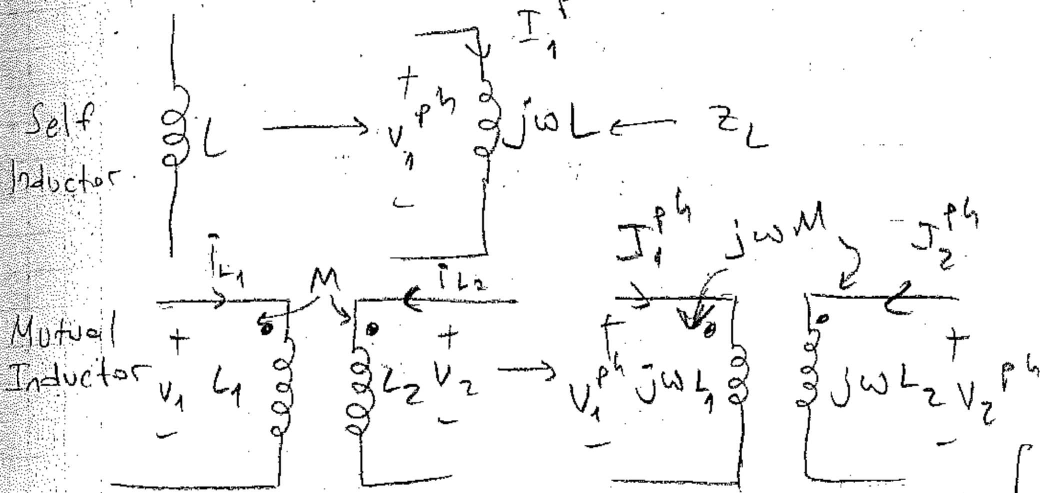
↑ Answers are at the back of the book

~~Homework~~ Please check:

Sadiku 4th edition }
Ex 9.10 p.393
Ex 9.11 p.394
Ex 10.1 p.414
Ex 10.4 p.414
Ex 10.10. p.429

Also check the web-site. HW-3 is on the web!

Mutual Inductor in Phasor Domain



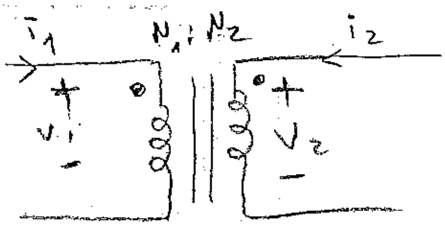
$$\begin{bmatrix} V_{L_1}(t) \\ V_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_{L_1} \\ \frac{d}{dt} i_{L_2} \end{bmatrix} \xrightarrow{\text{phasor}}$$

$$\begin{bmatrix} V_{L_1}^{ph} \\ V_{L_2}^{ph} \end{bmatrix} = \underbrace{\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix}}_{\text{impedence matrix}} \begin{bmatrix} I_1^{ph} \\ I_2^{ph} \end{bmatrix}$$

$i_{L_1}(t) = A\cos(\omega t + \theta_A)$, $i_{L_2}(t) = B\cos(\omega t + \theta_B)$

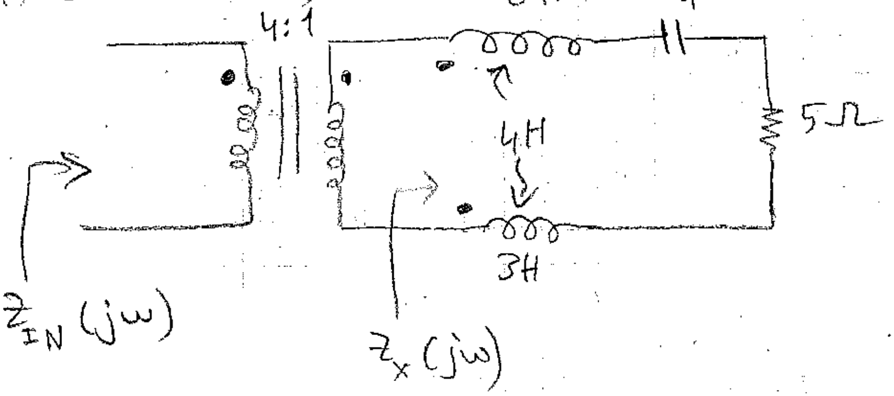
$$\boxed{I_{L_1}^{ph} = A \angle \theta_A} \qquad \boxed{I_{L_2}^{ph} = B \angle \theta_B}$$

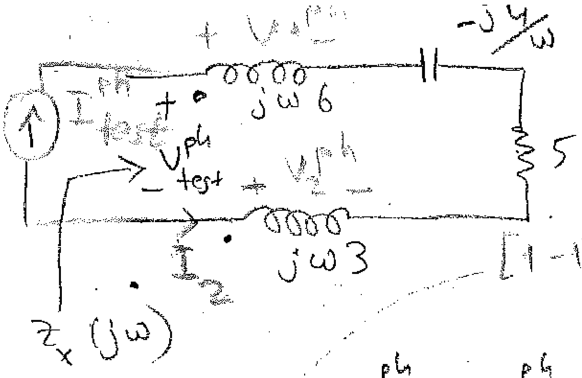
Transformers



$\frac{V_1}{V_2} = \frac{N_1}{N_2}$ | $\frac{i_1}{i_2} = -\frac{N_2}{N_1}$ ← remains exactly the same in phasor domain

Ex: (ZPS-II Pr: 5)





$$Z_x(j\omega) = \frac{V_{test}^{ph}}{I_{test}^{ph}}$$

$$[1\ -1]\begin{bmatrix} V_1^{ph} \\ V_2^{ph} \end{bmatrix} = \begin{bmatrix} j\omega 6 & j\omega 4 \\ j\omega 4 & j\omega 3 \end{bmatrix} \begin{bmatrix} I_1^{ph} \\ I_2^{ph} \end{bmatrix}$$

<u>KVL</u>: $-V_{test}^{ph} + V_1^{ph} + \left(5 - \frac{j4}{\omega}\right) I_{test}^{ph} - V_2^{ph} = 0$

to obtain this

$$V_1^{ph} - V_2^{ph} = [j\omega 2 \quad j\omega] \begin{bmatrix} I_1^{ph} \\ I_2^{ph} \end{bmatrix} = j\omega\, I_{test}^{ph}$$

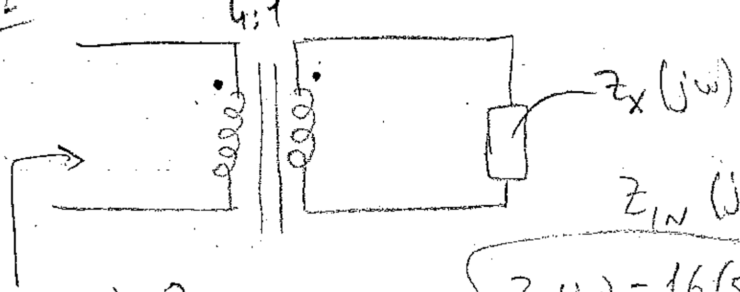
$I_1^{ph} = I_{test}$

$I_2^{ph} = -I_{test}$

$$V_{test}^{ph} = \left(j\omega + 5 - \frac{j4}{\omega}\right) I_{test}^{ph}$$

$$\boxed{Z_x(j\omega) = \frac{V_{Test}^{ph}}{I_{test}^{ph}} = 5 + j\left(\omega - \frac{4}{\omega}\right)}$$

<u>Now</u>:



$Z_{IN}(j\omega) = ?$

Resistance reflection formula for ideal transformers

$$Z_{IN}(j\omega) = \left(\frac{4}{1}\right)^2 \cdot Z_x(j\omega)$$

$$\boxed{Z_{IN}(j\omega) = 16\left(5 + j\left(\omega - \frac{4}{\omega}\right)\right)\ \Omega}$$

Commenting the example

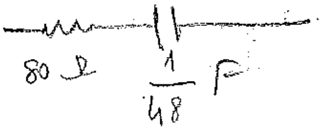
$Z_{IN}(j\omega) = 16\left(5 + j\left(\omega - \frac{4}{\omega}\right)\right)$

$\omega = 2 \longrightarrow Z_{IN}(j2) = 80$

$\omega = 3 \longrightarrow Z_{IN}(j3) = 80 + \frac{j80}{3}$

$\omega = 1 \longrightarrow Z_{IN}(j) = 80 - j48$

equivalent to:



for $\omega = 1$

MT1:
AC analysis (m
Sonora Kader
Power_RMS
got

$(D^2+3D+12)I_L(t) = 2x(t)$
$I_L(0^-) = I_0$
Only responsible for solution of this

04.04.2016

$(D^2+3D+2)\, I_L(t) = 2u(t)$

$I_L(0^-) = I_0$

$\dot{I}_L(0^-) = I_1$

$\hookrightarrow (s^2+3s+2) I_L(s) + (-sI_0 - I_1) - 3I_0 = 2 \cdot \frac{1}{3}$

$\hookrightarrow I_L(s) = \dfrac{2/s + sI_0 + I_1 + 3I_0}{(s^2+3s+2)}$

$\boxed{\lambda = 0}$

$= \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s}$

① Check the number of the natural frequencies by finding the # of state variables (graphical/tree method)

Existence of Steady-state Solutions

Phasor domain analysis gives us the particular solution to an LTI circuit composed of dynamic components.

We should remember that the complete solution of the circuit, say for the k^{th} branch voltage is

$$V_k^{comp}(t) = \underbrace{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \ldots + A_N e^{\lambda_N t}}_{\text{homogeneous solution}} + \underbrace{V_k^{particular}(t)}_{\text{particular solution}}$$

If the circuit is stable, that is λ_k's have negative real parts, then homogeneous solution decays to 0 as $t \to \infty$

$$\boxed{\mathrm{Re}\{\lambda_k\} < 0 \quad \forall k}$$

$(D^2+3D+2) V_k^{ph} = 0$

$\downarrow \lambda = \{-1, -2\}$

Then for stable circuits,

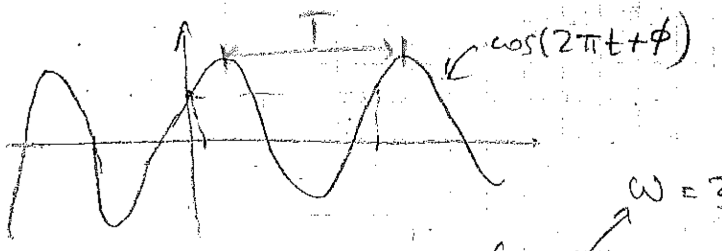
$V_k^{comp}(t) \longrightarrow V_k^{particular}(t)$ as $t \to \infty$

that's the phasor domain solution is the steady state solution ($t \to \infty$) for stable circuits.

RMS Values

Periodic Functions: $f(t) = f(t-T)$ $\forall t$, $T \neq 0$ T: Period (sec)

T: Period of the function.

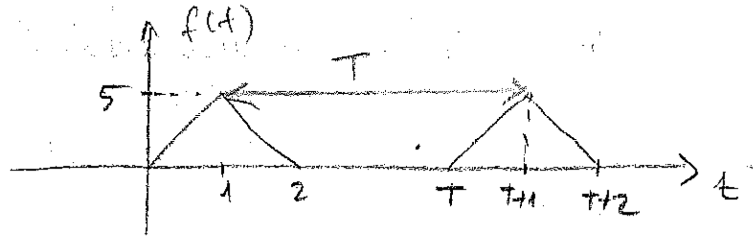


$f = \frac{1}{T}$ $\left(\frac{1}{sec}, Hz\right)$

↑ frequency

$\omega = \frac{2\pi}{T} = 2\pi f$ (rad/sec)

(omega) radial frequency



Simple fact: If $f(t)$ is periodic by T, it is also periodic by $2T$, $3T$, --- in general kT. $\leftarrow k \in \mathbb{Z}$

The smallest period among all periods is called the fundamental period.

In EE301

$$\boxed{f(t) = \sum_{k=0}^{\infty} a_k \cos\left(\frac{2\pi}{T} k t\right) + b_k \sin(\omega_k t)} \rightarrow ?$$

f(t) has fundamental period of T

RMS: (Root Mean Square)

f(t): Periodic by T

$$\boxed{f_{RMS} = \sqrt{\frac{1}{T}\int_0^T (f(t))^2\, dt}}$$

also called effective value

Ex: $f(t) = A\cos(\omega t + \theta)$

$$f_{RMS} = \sqrt{\frac{1}{T}\int_0^T \underbrace{[A\cos(\omega t + \theta)]^2}\, dt}$$

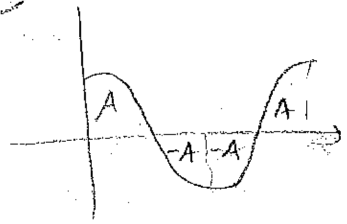
$$A^2\cos^2(\omega t + \theta) = \frac{A^2}{2}\left[\cos(2(\omega t + \theta)) + 1\right]$$

$$= \sqrt{\frac{1}{T}\int_0^T \frac{A^2}{2}\, dt + \underbrace{\frac{1}{T}\int_0^T \frac{A^2}{2}\cos(2\omega t + 2\theta)\, dt}_{0 \text{ (zero)}}} \quad \text{periodic by } \frac{T}{2}$$

$\frac{2\pi}{T}$

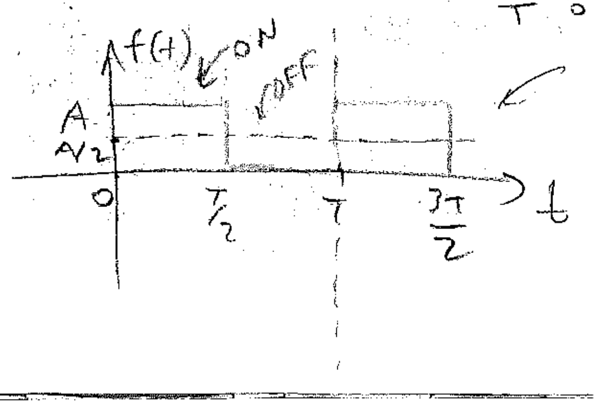
$\sqrt{\frac{A^2}{2T} T}$

$$\boxed{f_{rms} = \frac{A}{\sqrt{2}}}$$



Mean of a Periodic Function

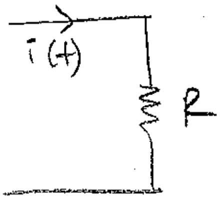
$$\langle f(t) \rangle = \text{mean value} = \frac{1}{T}\int_0^T f(t)\, dt$$



Square wave with %50 duty cycle.

ON interval is %50 of period.

Let's find the mean value of Power dissipated over resistor R whose current is periodic.



$p(t) = R\,[i(t)]^2$

Instanteneous power.

$$\boxed{\langle p(t) \rangle = \frac{1}{T}\int_0^T R\,[i(t)]^2\,dt = R(i_{RMS})^2}$$

< > means average

effective value of periodic i(t)

$$\Delta E = \int_0^{1\,\text{hour}=3600\,\text{sec}} p(t)\,dt = \int_0^{1\,\text{hour}} R\,[i(t)]^2\,dt$$

work (Joule)

≈ 3600 sec

T = 20 msec

f = 50 Hz

$\frac{3600\,\text{sec}}{20\,\text{msec}}$ = 180 000 oscillations

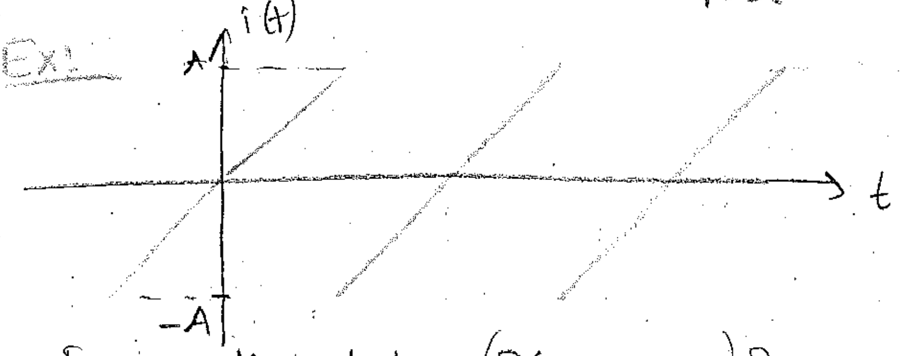
$= 180 \times 10^3$ Energy dissipation per period

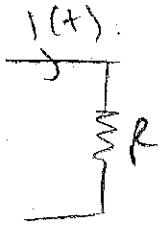
$= 180 \times 10^3\,(T\,P_{AVG})$ Period

$= (180 \times 10^3)\,(20\,\text{msec}\,(R\,i_{rms}^2))$

$= 3600\,\text{sec}\ P_{AVG}$ Joule/sec = (Watt)

Ex!





$\frac{T}{24}$

Energy dissipated in an hour $= (3600\,\text{sec.})\,P_{AVG}$

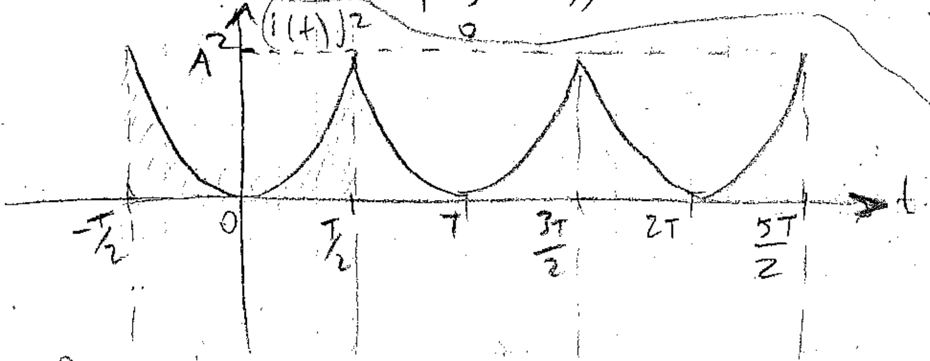
$= (3600\,\text{sec})\,R(i_{rms})^2$

$$i_{rms} = \sqrt{\frac{1}{T}\int_0^T (i(t))^2\,dt} = \sqrt{\frac{1}{T}\int_{-T/2}^{T/2} \left(A\frac{2}{T}t\right)^2 dt}$$

$$= \sqrt{\frac{A^2 4}{T^3}\int_{-T/2}^{T/2} t^2\,dt}$$

$$= \sqrt{\frac{A^2 \cancel{4}}{\cancel{T^3}} \cdot \frac{\cancel{T^3}}{\cancel{12}\,3}}$$

$$\boxed{i_{rms} = \frac{A}{\sqrt{3}}}$$



Since $\langle P \rangle = P_{AVG} = R(i_{Rms})^2$

i_{Rms} can be considered as the equivalent (effective) value of a hypothetical DC source providing the same average power!

Rms values (continued)

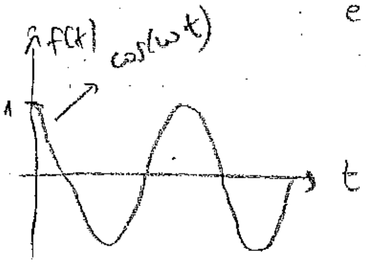
f(t): Periodic waveform with period T.

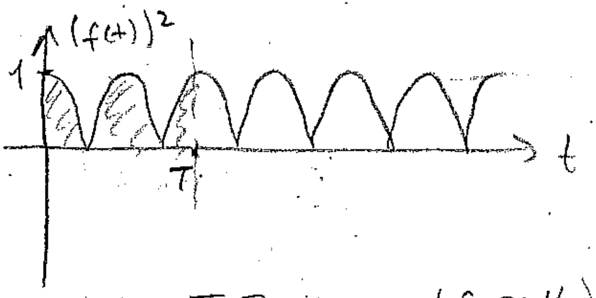
$$f_{RMS} = \sqrt{\frac{1}{T}\int_0^T (f(t))^2 dt} = \sqrt{\lim_{T_x \to \infty} \frac{1}{T_x}\int_0^{T_x} (f(t))^2 dt}$$

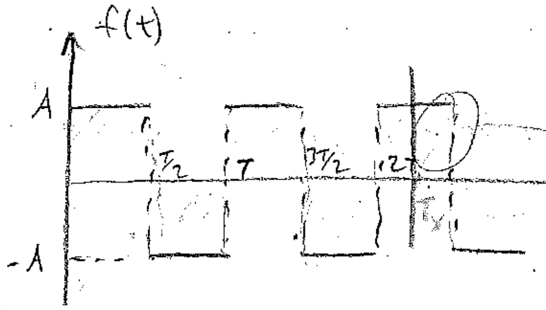
Digital multimeter calculates this.

$$f(t) = A\cos(\omega t) \text{ Volt} \longrightarrow f_{rms} = \frac{A}{\sqrt{2}} \text{ Volts}$$

effective value







T=20 msec (f=50 Hz)

T_x=10 sec → 500 periods

As T_x increases this additional integral ($T_x - 2T$ in this case) becomes insignificant.

Ex: $f(t) = A_1\cos\left(\frac{2\pi t}{T_1} + \theta_1\right) + A_2\cos\left(\frac{2\pi t}{T_1} + \theta_2\right)$ Volts

T = LCM(T₁, T₂)

↳ Lowest common multiple

$$f_{RMS} = \sqrt{\lim_{T_x \to \infty} \frac{1}{T_x}\int_0^{T_x} \underbrace{(f(t))^2}_{} dt}$$

$$A_1^2\cos^2(\omega_1 t + \theta_1) + A_2^2\cos^2(\omega_2 t + \theta_2) + 2A_1A_2\frac{\left[\cos\left((\omega_1+\omega_2)t + \theta_1 + \theta_2\right) + \cos\left((\omega_1-\omega_2)t + \theta_1 - \theta_2\right)\right]}{2}$$

$$A_1^2\left(\frac{1 + \cos(2\omega_1 t + 2\theta_1)}{2}\right)$$

$$A_2^2\left(\frac{1 + \cos(2\omega_2 t + 2\theta_2)}{2}\right)$$

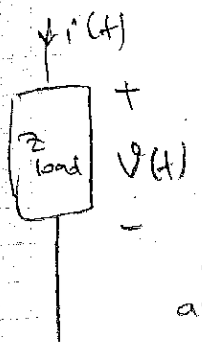
More significant

$$f_{Rms} = \sqrt{\frac{A_1^2}{2} + \frac{A_2^2}{2}}$$

So, if $f(t) = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)$ $(\omega_1 \neq \omega_2)$

$$\boxed{f_{Rms}^2 = f_{Rms_1}^2 + f_{Rms_2}^2} \qquad f_{rms_1} = \frac{A_1}{\sqrt{2}} \qquad f_{rms_2} = \frac{A_2}{\sqrt{2}}$$

Average and Instanteneous Power



$$v(t) = V_m \cos(\omega t + \theta_v) \longrightarrow V^{ph} = V_m \angle \theta_v$$

$$i_m(t) = I_m \cos(\omega t + \theta_i) \longrightarrow I^{ph} = I_m \angle \theta_i$$

$$\boxed{Z_{load} = \frac{V^{ph}}{I^{ph}} = \frac{V_m}{I_m} \angle \theta_v - \theta_i}$$

$$= |Z_{load}| \angle \theta_{Load}$$

Goal: Find instanteneous power and its average $P_{AVG} = \langle p(t) \rangle$

← instanteneous power

average

$$p(t) = v(t) \cdot i(t)$$

$$= V_m I_m \underbrace{\cos(\omega t + \theta_v) \cos(\omega t + \theta_i)}_{\frac{\cos(2\omega t + \overset{\theta_{load}+\theta_i}{\theta_v} + \theta_i) + \cos(\overbrace{\theta_v - \theta_i}^{\theta_{load}})}{2}}$$

$$P_{AVG} = \langle p(t) \rangle = \lim_{T_x \to \infty} \frac{1}{T_x} \int_0^{T_x} p(t)\,dt$$

$$p(t) = \frac{V_m I_m}{2} \left[\cos(\theta_{Load}) + \underbrace{\cos(2\omega t + 2\theta_i + \theta_{Load})}_{\cos(2\omega t + 2\theta_i)\cos(\theta_{Load}) - \sin(2\omega t + 2\theta_i)\sin(\theta_{Load})} \right]$$

Its average approaches to zero.

$$p(t) = \frac{V_m I_m}{2} \cos(\theta_{Load}) \left[1 + \cos(2\omega t + 2\theta_i) \right] - \frac{V_m I_m}{2} \sin(\theta_{Load}) \left[\sin(2\omega t + \theta_i) \right]$$

Its average approaches to zero.

$$\boxed{\langle p(t) \rangle = P_{AVG} = \frac{V_m I_m}{2} \cos(\theta_{Load}) = V^{load}_{Rms} I^{load}_{Rms} \cos(\theta_{Load})}$$ Watts

$$V^{load}_{Rms} \to \frac{V_m}{\sqrt{2}} \qquad I^{load}_{Rms} \to \frac{I_m}{\sqrt{2}}$$

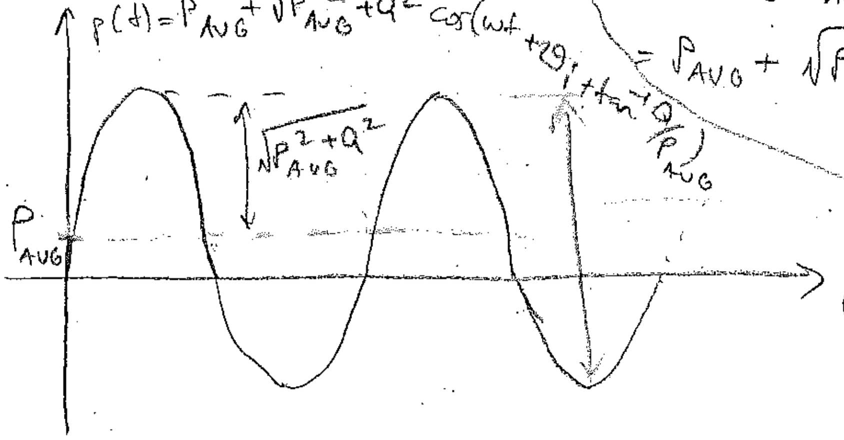
$$\boxed{Q = \frac{V_m I_m}{2} \sin(\theta_{Load}) = V^{load}_{Rms} I^{load}_{Rms} \sin(\theta_{Load})}$$

Reactive Power

$$p(t) = P_{AVG} + P_{AVG} \cos(2\omega t + 2\theta_i) - Q \left[\sin(2\omega t + 2\theta_i) \right]$$

$$= P_{AVG} + \sqrt{P_{AVG}^2 + Q^2} \cos\left(2\omega t + 2\theta_i + \tan^{-1} \frac{Q}{P_{AVG}}\right)$$

$$p(t) = P_{AVG} + \sqrt{P_{AVG}^2 + Q^2} \cos\left(\omega t + 2\theta_i + \tan^{-1} \frac{Q}{P_{AVG}}\right)$$



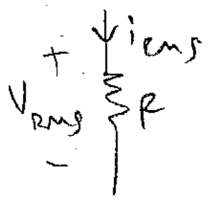
Peak to peak swing

$p(t)$ oscillates with double frequency of voltage and current waveforms (oscillates at 2ω) and its mean/average value is $P_{AVG} = V^{RMS} I^{RMS} \cos(\theta_{Load})$ and it has a peak-to-peak swing value of $2\sqrt{P_{AVG}^2 + Q_{AVG}^2}$ Watts

Notes Q (Reactive Power) affects peak-to-peak swing of instanteneous power around the average value.

Special Cases

① Load is a Resistor



$$Z_L = R \begin{cases} \to P_{AVG} = V_{RMS} I_{RMS} \underbrace{\cos(\theta_{Load})}_{1} = V_{RMS} I_{RMS} \text{ Watts} \\ \to Q = V_{RMS} I_{RMS} \sin(\underbrace{\theta_{Load}}_{0^\circ}) = 0 \text{ VAR's} \end{cases}$$

$|Z_L| = R \cdot 1 \angle \theta = 0^\circ$ ← Load (not inductor)

VAR's: Volt Ampere Reactive

Remember

$$Z_{load} = \frac{V_{load}^{ph}}{I_{load}^{ph}}$$

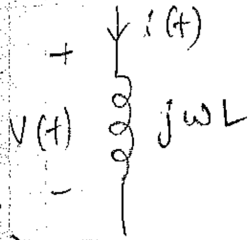
$$|Z_{load}| = \frac{|V_{load}^{ph}|}{|I_{load}^{ph}|} \longrightarrow |V_{Load}^{ph}| = |Z_{Load}| \cdot |I_{Load}^{ph}|$$

$$V_{Load}^{RMS} \triangleq \frac{V_{Load}^{Amplitude}}{\sqrt{2}}$$

$$\boxed{V_{Load}^{RMS} = |Z_{Load}| I_{Load}^{RMS}} \longrightarrow \text{A.C. signals}$$

$$\boxed{\begin{aligned} \to P_{AVG} &= V_{RMS} I_{RMS} \cancel{\cos(\theta_{Load})} \to 1 \text{ for } Z_{Load} = R \\ &= V_{RMS} I_{RMS} \\ &= (R I_{RMS})(I_{RMS}) = R I_{RMS}^2 = \frac{V_{RMS}^2}{2} \end{aligned}}$$

② Load is Inductor



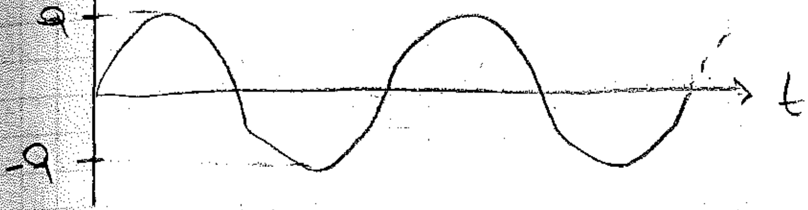
$Z_{load} = j\omega L = \omega L \angle 90^\circ$

$P_{AVG} = V_{RMS} I_{RMS} \cos(\theta_{Load})$

$Q = V_{RMS} I_{RMS} \sin(\theta_{Load}) = V_{RMS} I_{RMS}$

Also denoted "P" by (P)

$p(t) = P_{AVG} + \sqrt{P_{AVG}^2 + Q^2} \cos(2\omega t + 2\theta_i + \tan^{-1}\frac{Q}{P_{AVG}})$

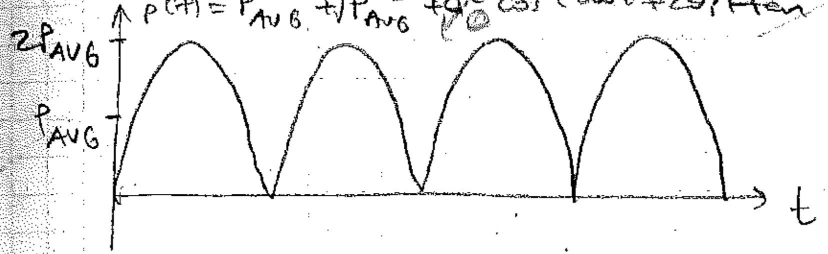


$E_{absorbed\ 0 \to t} = \int_0^{t} p(t)\,dt$

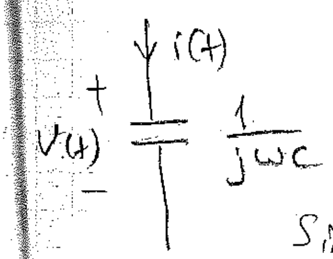
p(t) oscillates between $Q = V_{RMS} I_{RMS}$ and $-Q$

The instanteous power for $Z_L = R$ is:

$p(t) = P_{AVG} + \sqrt{P_{AVG}^2 + Q_0^2} \cos(2\omega t + 2\theta_i + \tan^{-1}\frac{Q}{P})$



③ Load is Capacitor

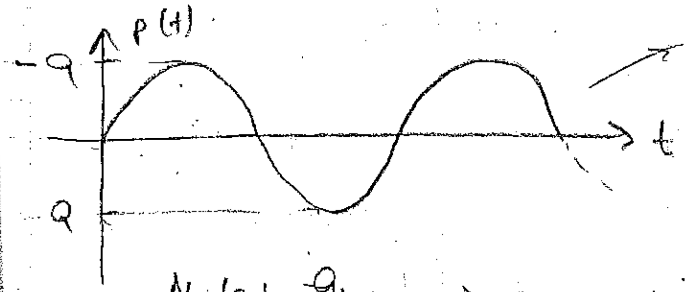


$Z_{Load} = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90^\circ$

$\to P_{AVG} = V_{RMS} I_{RMS} \cos(\theta_{Load}) = 0$

$Q = V_{RMS} I_{RMS} \sin(\theta_{Load}) = -V_{RMS} I_{RMS}$

Similar to the inductor case capacitor does not consume any power on the average but it may absorb power and store it as energy and then release at a later time.



→ instanteous power of capacitor under AC-input at steady state

Note: $Q_{inductor} > 0$

$Q_{capacitor} < 0$

↙

?

$$A\cos(\omega t)+B\sin(\omega t)\overset{?}{=}\sqrt{A^2+B^2}\,\cos\left(\omega t-\tan^{-1}\frac{B}{A}\right)$$

1^{st} way

$$B\cos(\omega t-90^\circ)$$

$$A\angle 0^\circ + B\angle -90^\circ = A+(-jB)$$

$$=\sqrt{A^2+B^2}\,\angle -\tan^{-1}(B/A)$$

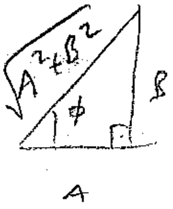
2^{nd} way

$$A\cos\omega t+B\sin\omega t=?$$

$$\sqrt{A^2+B^2}\left[\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos(\phi)}\cos(\omega t)+\underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin(\phi)}\sin\omega t\right]=?$$

$$=\sqrt{A^2+B^2}\left[\underbrace{\cos\phi\cos\omega t+\sin\phi\sin\omega t}_{\cos(\omega t-\phi)}\right]$$

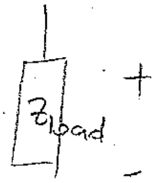
$$\cos(\omega t-\phi)=\cos\left(\omega t-\tan^{-1}\frac{B}{A}\right)$$



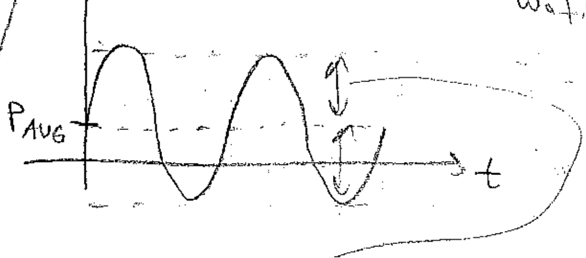
Meaning of Q

Q: Reactive Power, (VAR's) → (Volt-Ampere Reactive)

$$Q=V_{RMS}^{Load}I_{RMS}^{Load}\sin(\theta_{Load})$$



$$p(t)=P_{AVG}+\sqrt{P_{AVG}^2+Q^2}\cos(2\omega t+\text{phase})$$ watts



So Q affects the swing in the instanteous power from the average power level.

Swing from P_{AVG} is $\sqrt{P_{AVG}^2+Q^2}$

Appearant Power $\triangleq V_{RMS}^{Load}I_{RMS}^{Load}=\sqrt{P_{AVG}^2+Q^2}$

Note: The swing amount is identical to the appearent power.

2) Q and Average Stored Energy in a Cap/Inductor

For Capacitor

$$E_{cap}(t)=\frac{1}{2}Cv_c^2(t)$$

$$E_{cap}^{AVG} = \langle E_{cap}(t) \rangle = \frac{1}{2} C \langle \vartheta_c^2(t) = \frac{1}{2} C \left[\frac{1}{T} \int_0^T \vartheta_c^2(t) dt \right] = \frac{1}{2} C (\vartheta_c^{RMS})^2$$

$$\boxed{E_{cap}^{AVG} = \frac{1}{2} C (\vartheta_c^{RMS})^2} \quad (1)$$

$$Q_{cap} = ? \longrightarrow$$

$$z_{cap} = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90° \longrightarrow Q_{cap} = V_{RMS} I_{RMS} \sin(\vartheta_{Load})$$

$$= -V_{RMS}^{cap} I_{RMS}^{cap} = -V_{RMC}^{cap} (V_{RMS}^{cap} \omega C)$$

$$Q_{cap} = -\omega \cdot C (\vartheta_{RMS}^{cap})^2 \quad (2)$$

Then combining (1) with (2)

we get $E_{cap}^{AVG} = \frac{-Q^{cap}}{2\omega}$

$$\boxed{Q^{cap} = -2\omega E_{cap}^{AVG}}$$

$$E = \frac{1}{2} \cancel{C} \frac{Q}{-\omega \cancel{C}}$$

Q of capacitor is proportional to the average energy stored in the capacitor.

for Inductor

$$E_{Ind}^{AVG} = \langle \frac{1}{2} L I_L^2(t) \rangle = \boxed{\frac{1}{2} L (I_{RMS}^{Ind})^2} \quad (1)$$

$$Q_{Inductor} = V_{RMS}^{Ind} I_{RMS}^{Ind} \sin(\vartheta_{Ind}) \quad 90° = V_{RMS}^{Ind} I_{RMS}^{Ind}$$

$$= (|Z_{Ind}| I_{RMS}^{Ind}) I_{RMS}^{Ind}$$

$$= |Z_{Ind}| I_{RMS}^2$$

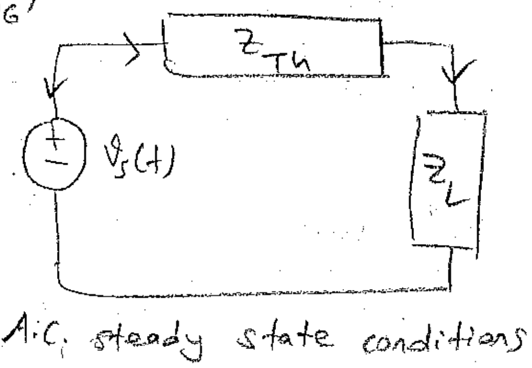
$$\boxed{Q_{Ind} = \omega L I_{RMS}^2} \quad (2)$$

Combine (1) and (2);

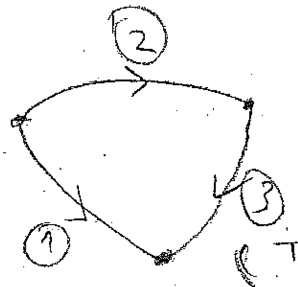
$$\boxed{Q^{Ind} = 2\omega E_{Ind}^{Avg}}$$

Conservation of P and Q

$(P = P_{AVG})$



A.C. steady state conditions



(Tellegen's Theorem)

$$\sum_{k=1}^{3} P_k(t) = 0$$

Conservation of power

$$\sum_{k=1}^{3} \vartheta_k(t) i_k(t) = 0$$

$$\sum_{k=1}^{3} \vartheta_k(t) \hat{i}_k(t) = 0$$

$$\hat{\vartheta}_k i_k(t) = 0$$

Another circuit with the same graph

$$\underbrace{\vartheta_1(t) i_1(t)}_{\text{absorbed}} + \underbrace{\vartheta_2(t) i_2(t)}_{\text{absorbed}} + \underbrace{\vartheta_3 i_3(t)}_{\text{absorbed}} = 0$$

$$\vartheta_2(t) i_2(t) + \vartheta_3(t) i_3(t) = \underbrace{-\vartheta_1(t) i_1(t)}_{\text{delivered power}}, \forall t$$

(with minus sign)

$$p_k(t) = v_k(t)\, i_k(t) = P_k + \sqrt{P_k^2 + Q_k^2}\cos\left(2\omega t + \theta_i + \tan\frac{Q_k}{P_k}\right)$$

$$= P_k\left(1 + \cos(2\omega t + 2\theta_1)\right) - Q_k \sin(2\omega t + 2\theta_i)$$

$$\begin{tikzcd}[column sep=small]
& P_2(t) + P_3(t) = -P_1(t) \arrow[dl] \arrow[d] \arrow[dr] & k = \{1,2,3\} \ \forall t \\
P^{(2)}_{AVG} + P^{(2)}_{AVG}\cos(2\omega t + 2\theta_i) - Q^{1}\sin(2\omega t + 2\theta_i) & P^{(3)}_{AVG} + P^{(3)}_{AVG}\cos(2\omega t + 2\theta_i) - Q^{(3)}\sin(2\omega t + 2\theta_i) & -P^{(1)}_{AVG} + \text{—} \ \text{— —}
\end{tikzcd}$$

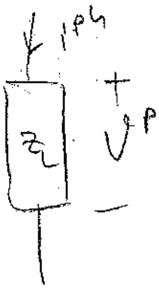
$$P^{(2)}_{AVG} + P^{(3)}_{AVG} = -P^{(1)}_{AVG}$$

$$Q^{(2)} + Q^{(3)} = -Q^{(1)}$$

So, not only instantereous power but its average P_{AVG} and Q is also conserved.

Complex Power

$$S = P + jQ$$



$$P = \frac{|V^{ph}| \cdot |I^{ph}|\cos(\theta_L)}{\underset{\Leftarrow (\sqrt{2}\cdot\sqrt{2})}{2}} = V_{RMS} I_{RMS}\cos(\theta_L)$$

$$Q = \frac{|V^{ph}| \cdot |I^{ph}|}{2} \cdot \sin(\theta_L) = V_{RMS} I_{RMS} \sin(\theta_L)$$

$$S = P + jQ = V_{RMS} I_{RMS}\cos(\theta_L) + jV_{RMS} I_{RMS}\sin(\theta_L)$$

complex power

Clearly, complex power S is also conserved that is

$$\sum_{k=1}^{\#\text{ branches}} S_k = 0$$

S has the unit of VA (Volt-Ampere)

$$\begin{tikzcd}
& S = P + jQ \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{VA} & \text{watts} & \text{VAR's}
\end{tikzcd}$$

Alternative forms of Complex Power:

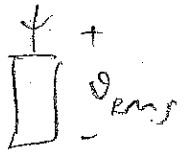
① $S = V_{RMS} I_{RMS} (\cos \theta_{Z_L} + j \sin \theta_{Z_L}) = V_{RMS} I_{RMS} e^{j\theta_{Z_L}}$

② $S = \frac{1}{2} V^{ph} (I^{ph})^{*}$ ← Conjugate

③ $S = I_{RMS}^{2} Z_L$

④ $S = \frac{V_{RMS}^{2}}{(Z_L)^{\circledast}}$ ← Conjugate!

$v_{RMS} = |Z_L| I_{RMS}$

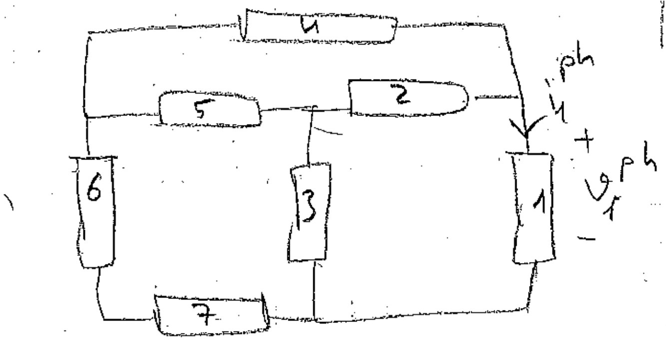


Complex Power (continued)

$S = P + jQ$

Complex Power (VA) → S; Average Power (Watts) → P; Reactive Power (VAR's) → Q

$\boxed{\sum_{k=1}^{7} P_k = 0}$ $\boxed{\sum_{k=1}^{7} Q_k = 0}$ $\boxed{S_k = P_k + jQ_k}$



How to Calculate Complex Power

$P = V_{RMS} I_{RMS} \cos(\theta_{Load})$ $Q = V_{RMS} I_{RMS} \sin(\theta_{Load})$

① $\boxed{S = V_{RMS} I_{RMS} e^{j\theta_{Load}}}$

(since $S = \underbrace{V_{RMS} I_{RMS} \cos(\theta_L)}_{P} + j \underbrace{V_{RMS} I_{RMS} \sin(\theta_L)}_{Q}$)

② $\boxed{S = \frac{1}{2} V^{ph} (I^{ph})^{*}}$ ← conjugate

(Since $S = \frac{1}{2} |V^{ph}| e^{j\angle V^{ph}} \cdot (|I^{ph}| e^{j\angle I^{ph}})^{*} = V_{RMS} I_{RMS} e^{j(\angle V^{ph} - \angle I^{ph})}$

$= V_{RMS} I_{RMS} e^{j\theta_L}$

③ $\boxed{S = I_{RMS}^{2} Z_L}$ Use this one because it is easy.

(Since $S = I_{RMS} I_{RMS} Z_L = I_{RMS} \underbrace{I_{RMS} |Z_L|}_{V_{RMS}} e^{j \overbrace{\angle Z_L}^{\theta_L}}$

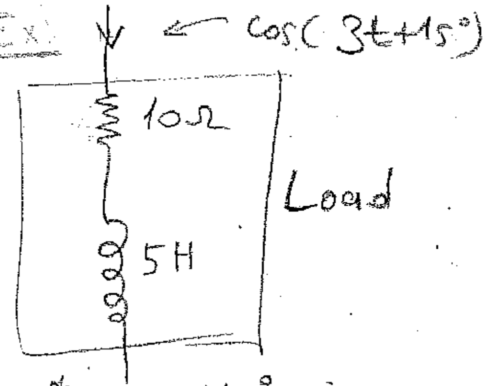
$= I_{RMS} V_{RMS} e^{j\theta_L}$

$|Z_{load}| = \frac{|V_{load}^{ph}|}{|I_{load}^{ph}|}$

(4) $\boxed{S=\frac{V_{RMS}^2}{(Z_L)^*}}$ - conjugate   (Since $S=V_{RMS}\cdot\frac{V_{RMS}}{(|Z_L|e^{j\angle Z_L})^*}$

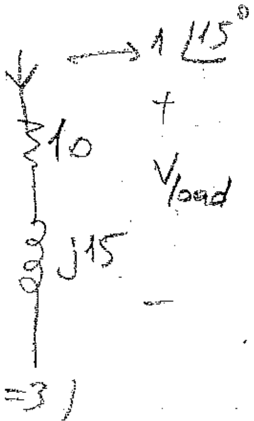
$= V_{RMS} I_{RMS} e^{j\angle Z_L}$

Ex:



Find S_{Load}.

Phasor:



$V^{ph}_{Load} = 1\angle 15^\circ \; 5(2+j3)$

$= 5\sqrt{13}\; \angle 15^\circ + \tan^{-1} 3/2$

$I^{ph} = 1\angle 15^\circ$

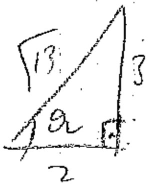
$Z^{load} = 10+j15$

$= 5\sqrt{13}\; \angle \tan^{-1} 3/2$

① $S = V_{RMS} I_{RMS} e^{j\theta_{Load}}$

$= \frac{5\sqrt{13}}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot e^{\underbrace{j\tan^{-1}(\frac{3}{2})}_{\theta_L}}$

$= \frac{5\sqrt{13}}{2}(\cos\theta_L + j\sin\theta_L)$



$= \frac{5\sqrt{13}}{2}\left(\frac{2}{\sqrt{13}} + j\frac{3}{\sqrt{13}}\right)$

$= \underset{\uparrow \atop P_{load}}{5} + j\underset{\uparrow \atop Q_{load}}{7.5}$ V.A

② $S = \frac{1}{2} V^{ph}(I^{ph})^*$

$= \frac{1}{2} 5\sqrt{13}\; \angle 15^\circ + \tan^{-1}\frac{3}{2} \cdot 1\angle -15^\circ$

$= \frac{5\sqrt{13}}{2}\; \angle \tan^{-1} 3/2 = 5+j7.5$

③ $S = I_{RMS}^2\, Z_L = \left(\frac{1}{\sqrt{2}}\right)^2 (10+j15) = 5+j7.5$ (VA)

④ $S = \frac{V_{RMS}^2}{Z_L^*} = \frac{(5\sqrt{13}/\sqrt{2})^2}{5\sqrt{13}\; \angle -\tan^{-1}(3/2)} = \frac{5\sqrt{13}}{2}\; \angle \tan^{-1}(\frac{3}{2}) = 5+j7.5$ (VA)

Important Note: Before finding S look at the phasor domain component and see whether
$Q>0$ or $Q<0$.

Remember $Q>0 \rightarrow$ Inductive components
$Q<0 \rightarrow$ Capacitive

Power Factor Definition:

$\rightarrow$ power factor $= \cos(\theta_{Load})$

Question: Can you uniquely find θ_{Load}, given that p.f. of the load is $\frac{\sqrt{2}}{2}$?

Answer: No:

$$\cos(\theta_{Load}) = \frac{\sqrt{2}}{2} \rightarrow \theta_{Load} = \{45°, -45°\}$$

For inductive loads, that is $Z_L = R + jX$ and $X > 0$ p.f. is said to be lagging ($\theta_{Load} > 0$).

For capacitive loads, that is $Z_L = R + jX$, $X < 0$, p.f. is said to be leading ($\theta_{Load} < 0$)

Q: Find θ_{load} for p.f. $\frac{1}{\sqrt{2}}$ lagging.

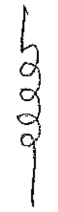
A. θ_{Load} is positive, since Load is inductive.

$$\theta_{Load} = 45°$$

$$\boxed{P = V_{RMS} I_{RMS} \cos(\theta_{Load})}$$

$$|H(j\omega)| = \frac{V_{out}(j\omega}{V_{in}(j\omega)}$$

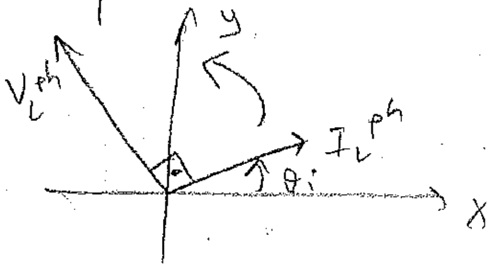
Lagging and leading Concept



$Z_L = j\omega L \leftarrow$ p.f. lagging

$I_{ph} = I_m \angle \theta_i$

$V_{ph} = Z_L I_{ph} = \omega L \angle \theta_i + 90°$



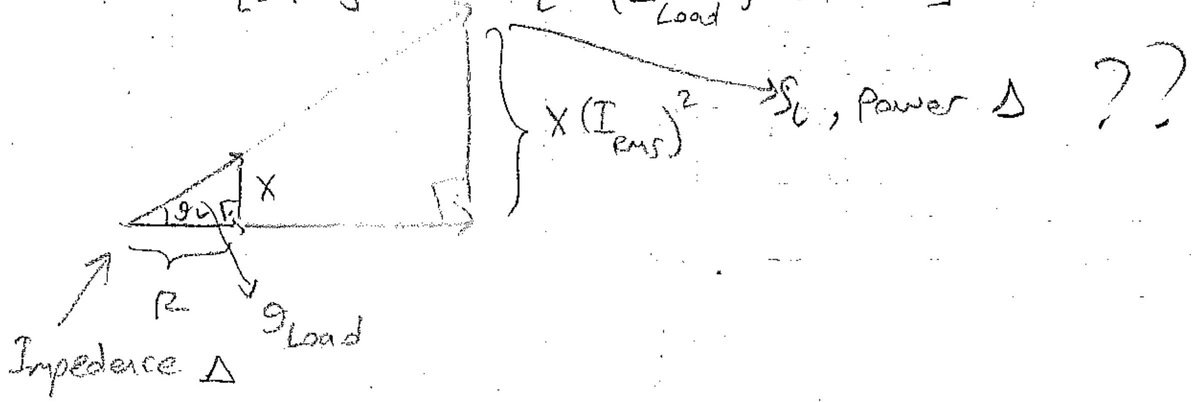
$\leftarrow$ phasor diagram

Current is lagging voltage by 90°.

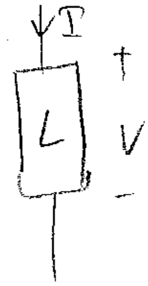
Impedance Triangle and Power Triangle

$$S_{Load} = (I_{Load}^{RMS})^2 Z_{load}$$

$$Z_L = R + jX \longrightarrow S_L = (I_{Load}^{RMS})^2 [R + jX]$$

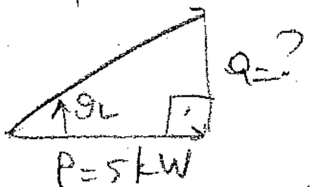


Ex:



$\vartheta = 100$ V (RMS)
$P = 5$ kW
p.f. $= 0.8$. lagging

a) Find VAR of load.
b) Find Appearent Power.

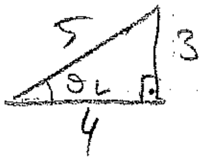


lagging info → inductive load

$$\downarrow$$

$$Q > 0$$

$$Q = (5000) \cdot \underbrace{\tan(\theta_L)}_{\frac{3}{4}} = 3750 \text{ VAR's}$$



b) Appearent Power $\triangleq V_{Rms} I_{Rms} \triangleq |S|$

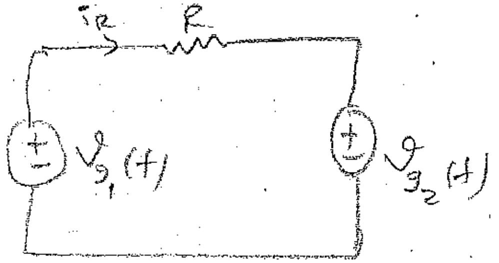
↳ since ① $S = V_{Rms} I_{Rms} e^{j\theta_L}$

$S = P + jQ$

$= 5000 + j\,3750$

Appearent power $= \sqrt{5000^2 + 3750^2} = \dfrac{5000}{\cos(\theta_L)} = \dfrac{5000}{4/5} = 6250$

Superposition in AC Power



$V_{g_1} = V_1 \cos(\omega_1 t + \theta_1)$ Volts

$V_{g_2} = V_2 \cos(\omega_2 t + \theta_2)$ Volts

$$i_R(t) = \frac{\vartheta_{g_1}}{R} - \frac{\vartheta_{g_2}}{R} = \frac{1}{R}\Big(V_1 \cos(\omega_1 t + \theta_1) - V_2 \cos(\omega_2 t + \theta_2)\Big)\, A$$

$$\langle p_R(t) \rangle = \langle R\, i_R^2(t) \rangle = R \langle i_R^2(t) \rangle = R \lim_{T_x \to \infty} \frac{1}{T_x} \int_{-\cancel{T}/2}^{T_{\cancel{x}}/2} i_R^2(t)\, dt$$

$$\lim_{T_x \to \infty} \frac{1}{T_x} \int_{-T_x/2}^{T_x/2} \frac{1}{R}\Big[V_1^2 \cos^2(\omega_1 t + \theta_1) - 2 V_1 V_2 \cos(\omega_1 t + \theta_1) \cos(\omega_2 + \theta_2) + V_2^2 \cos(\omega_2 t + \theta_2)\Big]\, dt = P_{AV}$$

$$= \underbrace{\frac{1}{R}\left(\lim_{T_x \to \infty} \frac{1}{T_x} \int_{-T_x/2}^{T_x/2} V_1^2 \cos^2(\omega_1 t + \theta_1)\, dt\right)}_{(V_{g_1}^{Rms})^2} + \underbrace{\frac{1}{R}\left(\lim_{T_x \to \infty} \frac{1}{T_x} \int_{-T_x/2}^{T_x/2} V_2^2 \cos^2(\omega_2 t + \theta_2)\, dt\right)}_{(V_{g_2}^{Rms})^2}$$

$$- \frac{2}{R}\left(\lim_{T_x \to \infty} \frac{1}{T_x} \int_{-T_x/2}^{T_x/2} V_1 V_2 \cos(\omega_1 t + \theta_1) \cos(\omega_2 t + \theta_2)\, dt\right)$$

$$= \frac{(\vartheta_{g_1}^{Rms})^2}{2} + \frac{(\vartheta_{g_2}^{Rms})^2}{2} - \frac{2}{R}(\text{Bracket})$$

$$\frac{\cos((\omega_1 + \omega_2)t + \theta_1 + \theta_2) + \cos((\omega_1 - \omega_2)t + \theta_1 - \theta_2))}{2}$$

Bracket $= 0$ if $\omega_1 \neq \omega_2$

Note: (Bracket) = 0 if $\omega_1 \neq \omega_2$ but
but (Bracket) $\neq 0$ if $\omega_1 = \omega_2$

$$\hookrightarrow (\text{Bracket}) = \frac{\cos(\theta_1 - \theta_2)}{2} V_1 V_2$$

Conclusion
If we have two sources with different frequencies, average power dissipated over a component is

$$P_{AVG} = P_{AVG}^{\text{source ①}} + P_{AVG}^{\text{source ②}}$$

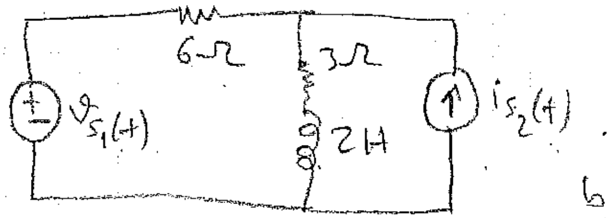
Avg. Power Consumption due to source ①

to source ②

So power can be superposed only if the sources have <u>different</u> frequencies!!!

A.C. Power Analysis (cont'd)

EX:



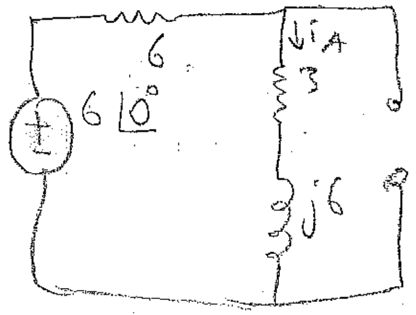
$v_{s_1}(t) = 6\cos(3t)$ Volts
$i_{s_2}(t) = 2\cos(4t + 30°)$ A

Find average power consumed by 3Ω resistor.

$$P_{AVG} = P_{AVG}^{\text{source with freq } w_1} + P_{AVG}^{\text{source with freq } w_2}$$, provided that $w_1 \neq w_2$

Superposition in A.C. s.s. power calculations (only valid when sources have different frequencies)

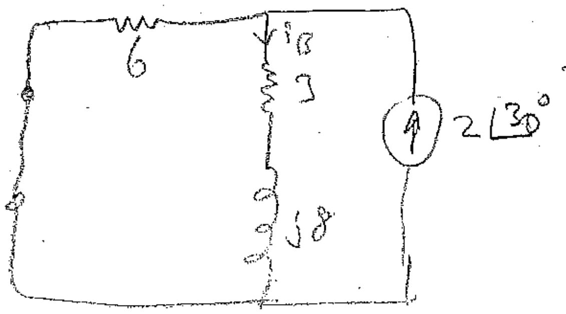
$w = 3$



$$\rightarrow i_A = \frac{\cancel{6}^{\,2}}{\cancel{9}_{3} + \cancel{6}_{2} j} = \frac{2(3-2j)}{13} = \frac{2}{13}\sqrt{13}\angle -\tan^{-1} 2/3$$

$$\rightarrow i_{Ⓐ}^{s.s.}(t) = \frac{2}{\sqrt{13}}\cos(3t - \tan^{-1} 2/3)\,A$$

$$\hookrightarrow i_{Ⓐ}^{RMS} = \frac{2}{\sqrt{2}\sqrt{13}} \rightarrow P_{3\Omega}^{Ⓐ} = (i_A^{RMS})^2 \cdot 3 = \frac{6}{13} \text{ watts}$$



$$\rightarrow i_{Ⓑ} = 2\angle 30° \frac{6}{9 + j8} = \frac{12\angle 30°}{\sqrt{145}\angle \tan^{-1} 8/9}$$

$$\hookrightarrow i_{Ⓑ}(t) = \frac{12}{\sqrt{145}}\cos(4t + 30° - \tan^{-1} 8/9)$$

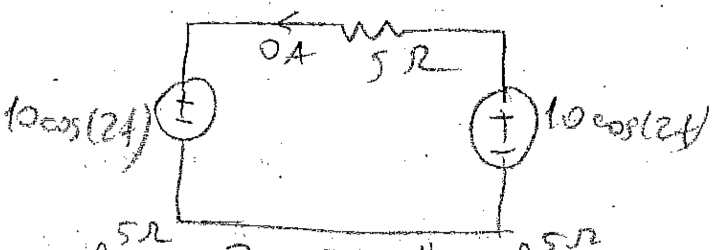
$$P_{3\Omega}^{Ⓑ} = \left(\frac{12}{\sqrt{2}\sqrt{145}}\right)^2 \cdot 3 \cong 1.5 \text{ Watts}$$

$$i_{3\Omega}^{s.s}(t) = i_A(t) + i_B(t) = \boxed{\sqrt{\qquad}\ \boxed{1}} + \boxed{\qquad\ \boxed{2}}\ A$$

$$\downarrow$$

$$P_{3\Omega}^{AVG} = P_{3\Omega}^{Ⓐ} + P_{3\Omega}^{Ⓑ} = \frac{6}{13} + 1.5 \approx 2 \text{watts}$$

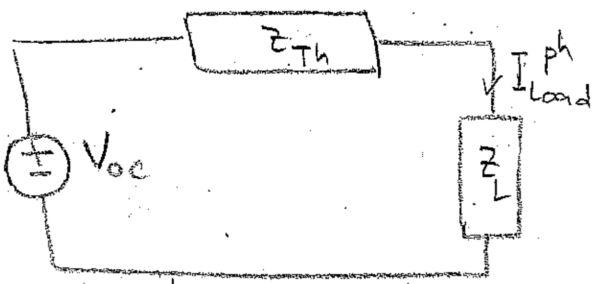
Ex: A silly but important example



$P_{AVG}^{5\Omega} = ?$ Trivially $P_{AVG}^{5\Omega} = 0$, since $i_{5\Omega}(t) = 0 \ \forall t$

→ You cannot apply superposition of A.C. powers when sources have the same frequency.

Maximum Power Transfer



V_{oc} and Z_{th} is fixed (we're not allowed to set their values) and we would like to adjust Z_L s.t. power delivered to Z_L is maximized.

$$I_{Load}^{ph} = \frac{V_{oc}^{ph}}{\underbrace{Z_{Th}}_{R_{Th}+jX_{Th}} + \underbrace{Z_L}_{R_L+jX_L}} = \frac{V_{oc}^{ph}}{(R_{Th}+R_L)+j(X_{Th}+X_L)}$$

$$P_{AVG}^{Load} = \left(\frac{|I_{Load}^{ph}|}{\sqrt{2}}\right)^2 \cdot R_L = \frac{|V_{oc}^{ph}|^2}{2} \underbrace{\frac{R_L}{(R_{Th}+R_L)^2+(X_{Th}+X_L)^2}}_{P_{AVG}^{Load}(R_L, X_L)}$$

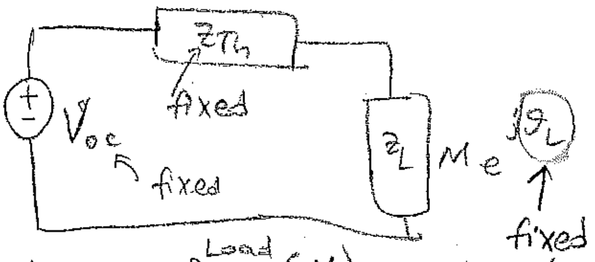
Q: Maximize $P_{AVG}^{Load}(R_L, X_L)$.

A. $\frac{\partial P_{AVG}^{Load}}{\partial R_L} = 0 \ \Big| \ \frac{\partial P_{AVG}^{L}}{\partial X_L} = 0$

$X_L = -X_{Th}$

$R_L = R_{Th}$ $\longrightarrow$ $\boxed{\begin{aligned} Z_L &= R_{Th} - jX_{Th} \\ &= (Z_{Th})^* \end{aligned}}$ optimum impedance for maximum power transfer

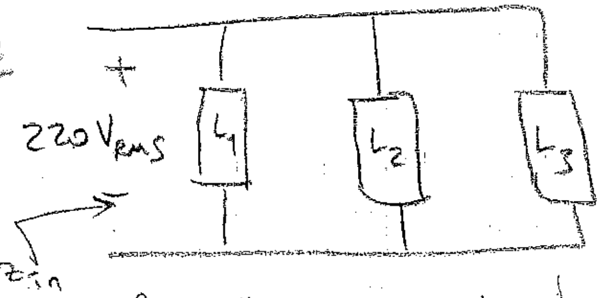
Some other maximum power transfer condition.



You can only change M but not the angle of the load.

$$\text{Maximize } P_{AVG}^{Load}(M) \longrightarrow M = |Z_{Th}|$$

EX:



inductor

L_1 : 16 kW, 18 kVAR

L_2 : 10 kVA at 0,6, p.f leading

L_3 : 8 kW at unity p.f.

capacitor

resistor

power factor

a) Find p.f. at the source side.

b) Find appearent power at the source side.

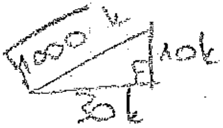
c) Find the impedence seen by the source.

$S_1 = 16 + j18$ kVA

$S_2 = 6 - j8$ kVA

$S_3 = 8$ kVA

a) $S_{Total} = S_1 + S_2 + S_3$
$= 30 + j10$ kVA



p.f. of source side $= \cos(\theta_s) = \frac{3}{\sqrt{10}}$ (lagging)

b) Appearent Power at Source side $= (220\, V_{RMS}) I_{RMS} = |S_{Total}|$
$= 10\sqrt{10}$ kVA

c) $S_{total} = (I_{RMS})^2 Z_{in}$

$$I_{RMS} = \frac{|S_{tot}|}{220 V_{RMS}} = \frac{10\sqrt{10} \times 10^3}{220} = \frac{10^3\sqrt{10}}{22}$$

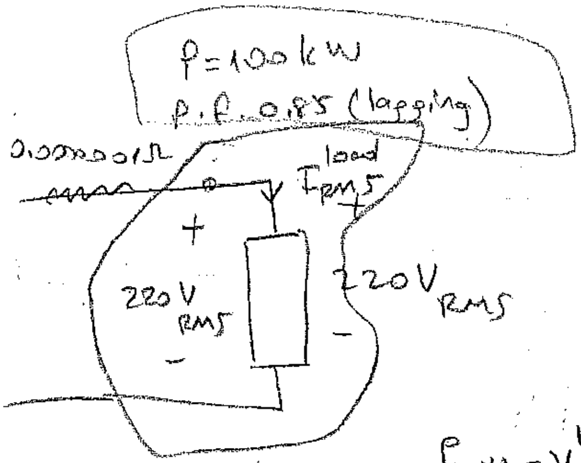
$0.484(3+j)$

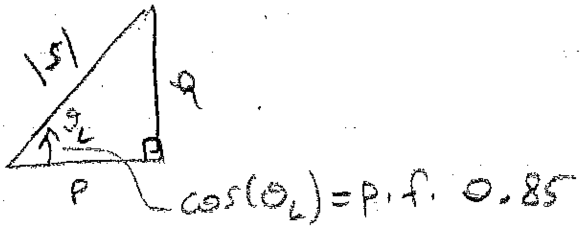
$$Z_{IN} = \frac{S_{total}}{(I_{RMS})^2} \text{ - - - - } = \frac{(30+j10)\,10^3}{\frac{10 \cdot 10^6}{(22)^2}}$$

Ex1 A mill consumes 100 kW, 220 V (RMS) at p.f. 0.85, lagging.

a) Find RMS current supplied by 220 V source to the mill.

b) Find the current in RMS, if p.f. were 0.95 lagging.





a) $S_{Load} = 100 + j100\tan(\cos^{-1}(0.85))$ kVA

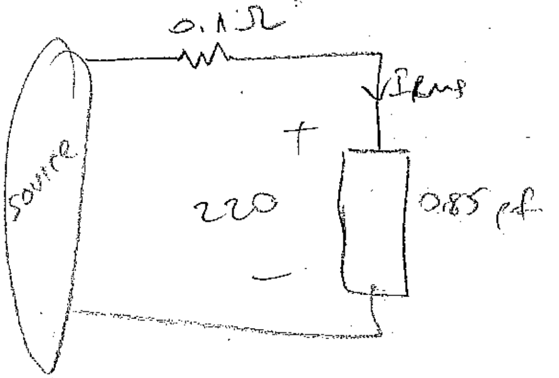
$= V^{Load}_{RMS} I^{Load}_{RMS} (\cos\theta_L + j\sin\theta_L)$

$$P_{mill} = V^{Load}_{RMS}\, I^{Load}_{RMS} \cos(\theta_L)$$

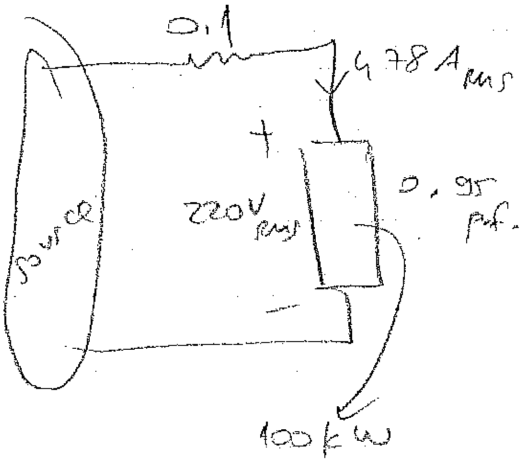
$$I^{Load}_{RMS} = \frac{\overset{100kW}{P_{mill}}}{\underset{220V_{RMS}}{V^{Load}_{RMS}}\,\underset{0.85}{\cos(\theta_L)}} = 534.8\,A$$

b) $$I^{Load}_{RMS} = \frac{100\text{ kW}}{220V_{RMS}\cdot(0.95)} = 478\,A$$

c) If there is a line (feeder) connecting load to the source and $R_{Line} = 0.1\,\Omega$. Find the power loss over the line for both p.f. conditions



$P = (I_{RMS})^2(0.1) = 28.6$ kW



$$P_{Line} = \underset{\underset{478}{\uparrow}}{(I_{RMS})}^2 \times 0.1 = 22.9\text{ kW}$$

d) Define efficiency as $\frac{\text{Real power delivered to the load (mill)}}{\text{Real Power supplied by the source}}$ $\frac{\text{Watt}}{\text{Watt}}$

η

Find efficiency for both cases,

Almost always asked in exams!!

i) p.f. 0.85 $P_{load} = 100kw$

$P_{supplied} = P_{Load} + P_{line} = 128.6 kw$

$$\eta = \frac{100 \text{ kWatt}}{128.6 \text{ Watt}} \cong 77\%$$

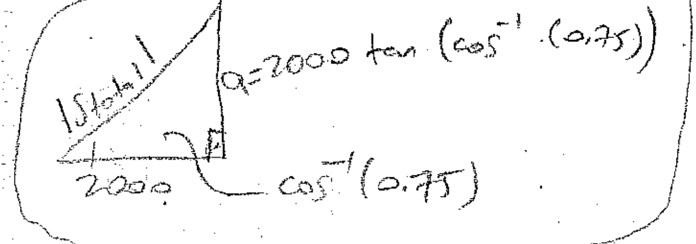
p.f. 0.95

$P_{Load} = 100 kw$

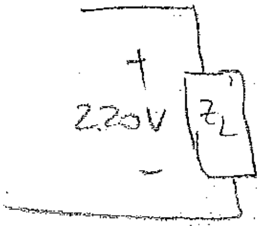
$P_{supplied} = 100 + 22.9 kw$ $\rightarrow \eta = \frac{100}{122.9} \cong 82\%$

Ex1 Load requires 2kW at 0.75 p.f. lagging at 220 V_{rms}. Calculate the reactive power supplied by the compensating capacitor to make p.f. 0.9 lagging. Find the impedence and the capacitance in Farads. (Assume 220 V_{rms}, f=50 Hz)

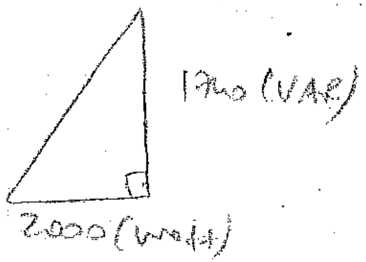
Soln: S_{load} =



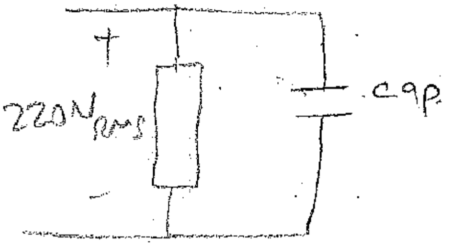
Before compensation



$S_{Load} = 2000 + j\,1740$

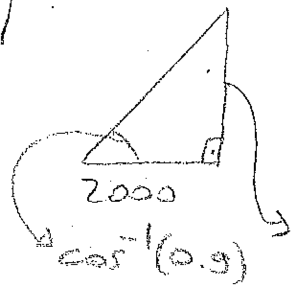


After compensation:



$S_{cap} = -jX$

$S_{Load}^{After} = 2000 + j(1740 - X)$



$S_{Total}^{desired, after} = 2000 + j$

$2000 \cdot \tan(\cos^{-1}(0.9)) = 968$

$$S_{cap} = \frac{(V_{rms})^2}{Z_{cap}^*}$$

$1740 - X = 968 \rightarrow X = 772$

$S_{cap} = -j772$

$Q_{cap} = -772$ VARs

$$\rightarrow -j772 = \frac{(220^2)}{Z_{cap}^*}$$

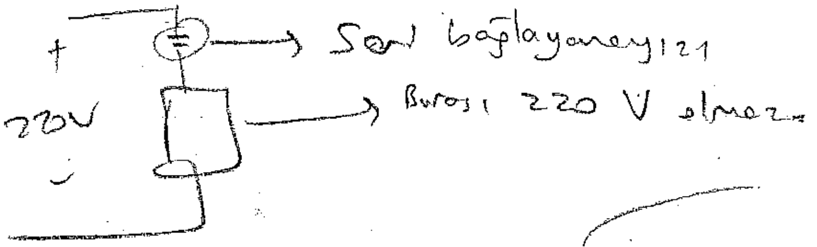
$$Z_{cap} = -j\frac{(220)^2}{772}$$ $\boxed{Z_{cap} = -j61.07\,\Omega}$

$Z_{cap} = \frac{1}{jwc} = -j61.07$

$\frac{1}{wc} = 61.07$

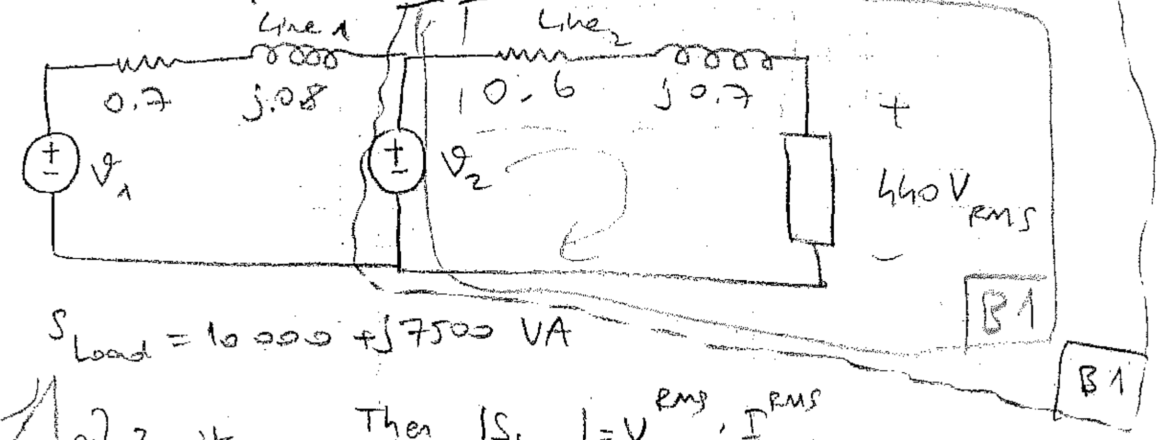
$\frac{1}{2\pi \times 50 \times C} = 61.07$

$C = \frac{1}{100\pi \times 61.07} F \cong 52\mu F$

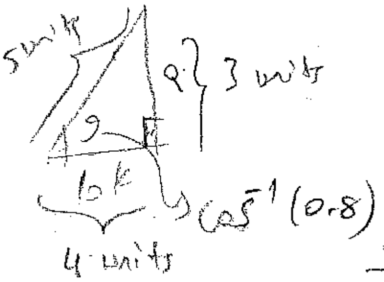


✱ check the example on the site.

Ex1 Load consumes 10KW at 0.8 p.f. lagging.
p.59 The generator 2 supplies 5KW at 0.6 p.f. lagging.
Find V_1 and V_2 (in RMS), the apparent power of generator 1 and p.f. of generator 1.



A: $S_{Load} = 10\,000 + j7500$ VA



Then $|S_{load}| = V^{RMS}_{Load} \cdot I^{RMS}_{Load}$

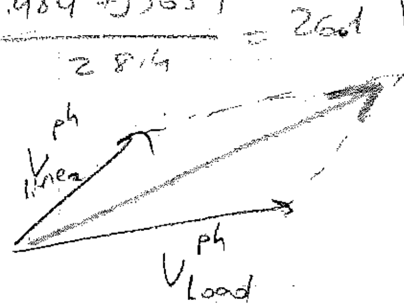
$\longrightarrow I^{RMS}_{Load} = \frac{10\,000/0.8}{440\,(V_{RMS})} = 28.4\,(A_{RMS})$

$$S_{Line_2} = \underset{(28.4)^2}{\underset{\uparrow}{I^2_{Line,RMS}}} \cdot \underset{0.6+j0.7}{\underset{\uparrow}{Z_{Line_2}}} = 484 + j565$$

$|S_{line_2}| = V^{RMS}_{Line_2} \cdot I^{RMS}_{line_2} \longrightarrow V^{RMS}_{Line_2} = \frac{|484 + j565|}{28.4} = 26.1\ V_{RMS}$

Let's find v_2 in RMS

Remember $V_2^{ph} = V^{ph}_{line_2} + V^{ph}_{Load}$



Method 1:

$$\underset{10\,000+j7500}{\underset{\nearrow}{S_{Load}}} = \frac{1}{2} \underset{440\sqrt{2}\angle 0^\circ}{\underset{\uparrow}{V^{ph}_{Load}}} \cdot (I^{ph}_{Load})^* \xrightarrow{Find} I^{ph}_{Load} = - - - -$$

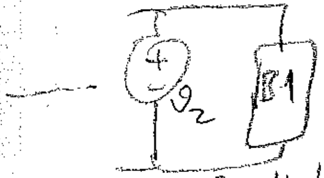
Reference Load phase angle

Find Z_{Load}

$V_2 = (Z_{Load} + Z_{Line_2})\, I^{ph}_{Load}$

Method 2 (Recommended)

$S_{B1} = S_{Line_2} + S_{Load} = 10484 + j8065$



$|S_{B1}| = V_{B1}^{RMS} \cdot I_{B1}^{RMS} \rightarrow \frac{|10484 + j8065|}{28.4} = V_{B1}^{RMS} = 466 \ V_{RMS}$

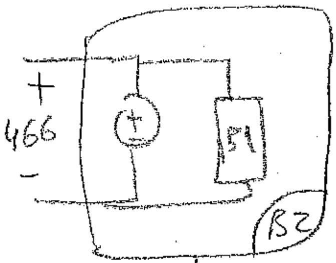
$\boxed{\vartheta_2 = 466 \ V_{RMS}}$

$S_2^{\text{supplied power}} = 5000 + j6666 \ VA$

2nd generator supplied power

$S_{B2} = S_{B1} - S_2^{\text{supplied}} = 5484 + j1400$

$|S_{B2}| = \underset{466}{V_{B2}^{RMS}} \cdot \underset{?}{I_{B2}^{RMS}} \longrightarrow I_{B2}^{RMS} = \frac{|5484 + j1400|}{466} = 12.14 \ A_{RMS}$



$S_{Line_1} = (I_{Line_1}^{RMS})^2 Z_{Line_1} = (12.14)^2 [0.7 + j0.8]$

$= 103 + j116$

$S_1^{\text{supplied}} = S_{Line_1} + S_{B2} = 5587 + j1516$

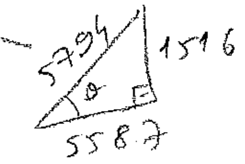
generator 1 supplied

$V_1^{RMS} = \frac{|S_1|}{12.14 \ A_{RMS}} = \boxed{477 \ V_{RMS}}$

Appearant power of Gen. 1 $= V_1^{RMS} I_1^{RMS} = |S_1| = 5794 \ VA$

p.f. of Gen. 1 $\Rightarrow \frac{5587}{5794}$ lagging

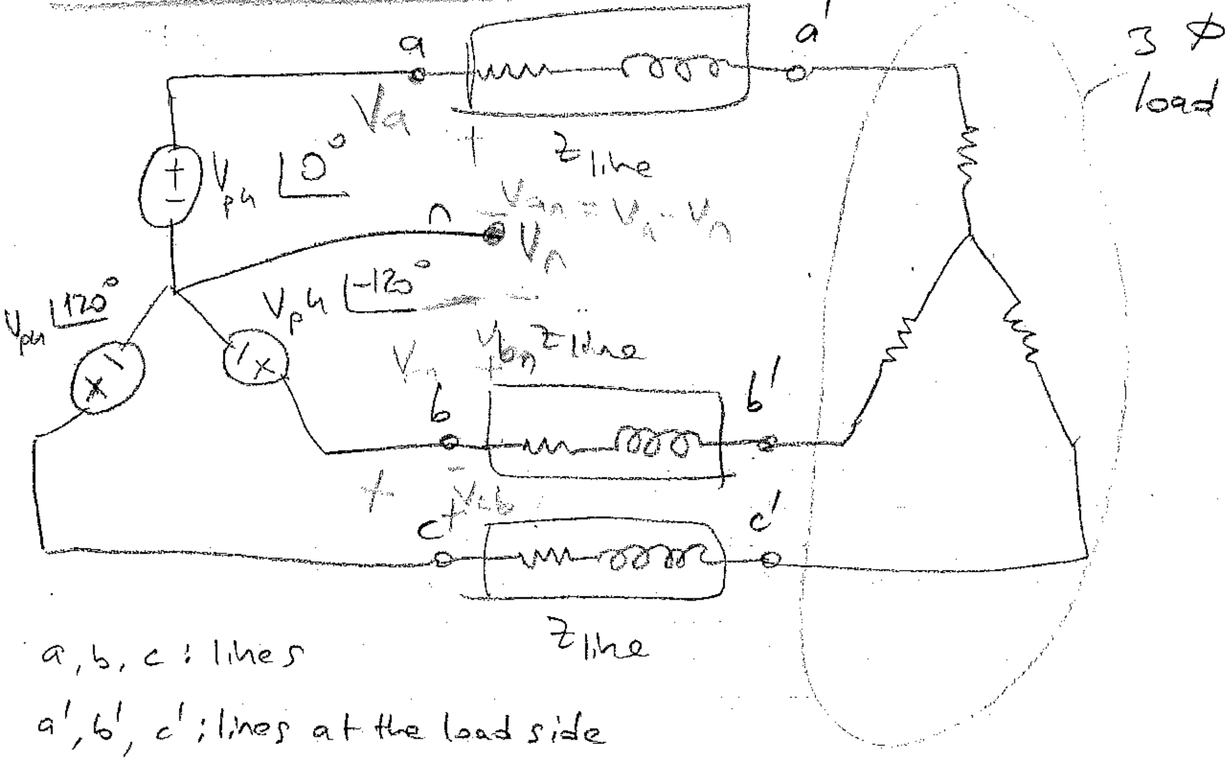
$\cos(\theta)$



$\boxed{S_{Load} = \frac{1}{2} V_{Load}^{ph} (I_{Load}^{ph})^*}$

→ we can use this if we need angle

3 phase Systems



a, b, c : lines

a', b', c' ; lines at the load side

a, b, c : 3 lines powering the system

n : neutral line (common line)

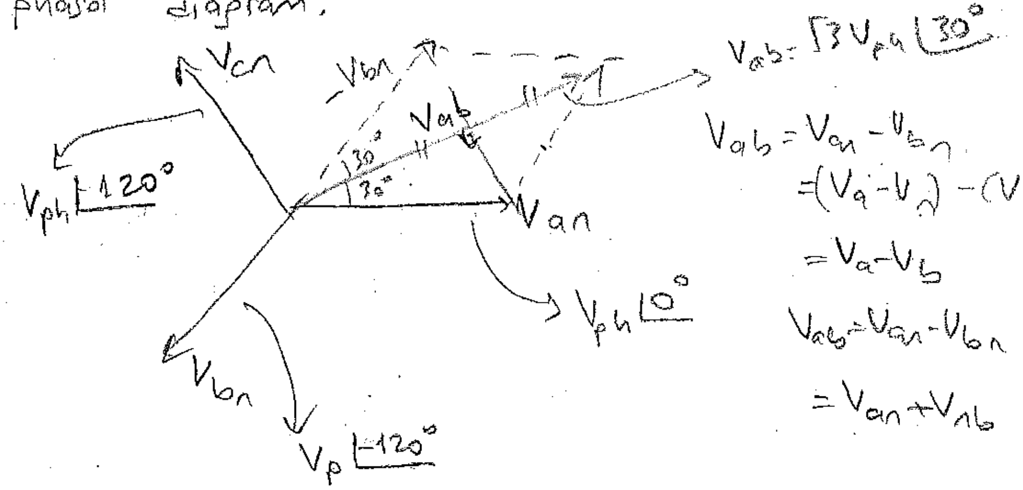
$$\begin{cases} V_{an} = V_a - V_n = V_{ph} \angle 0° \\ V_{bn} = V_b - V_n = V_{ph} \angle -120° \\ V_{cn} = V_c - V_n = V_{ph} \angle 120° \end{cases}$$

phase Voltages

line-to-line voltages

$$\begin{cases} V_{ab} = V_a - V_b = V_{an} - V_{bn} \\ V_{ac} = U_a - U_c = V_{an} - V_{cn} \\ V_{cb} = U_c - V_b = V_{cn} - V_{bn} \end{cases}$$

Let's find line-to-line voltages V_{ab}, V_{ac}, V_{cb} using phasor diagram.



$$V_{ab} = V_{an} - V_{bn}$$
$$= (V_a - V_n) - (V_b - V_n)$$
$$= V_a - V_b$$

$$V_{ab} = V_{an} - V_{bn}$$
$$= V_{an} + V_{nb}$$

$$V_{ab} = V_{an} - V_{bn} = V_{ph}\angle 0^\circ + \overset{\xrightarrow{} 1\angle 180^\circ}{(-1)}V_{ph}\angle -120^\circ$$

$$= V_{ph}\angle 0^\circ + V_{ph}\angle 60^\circ = V_{ph} + V_{ph}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= V_{ph}\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$$

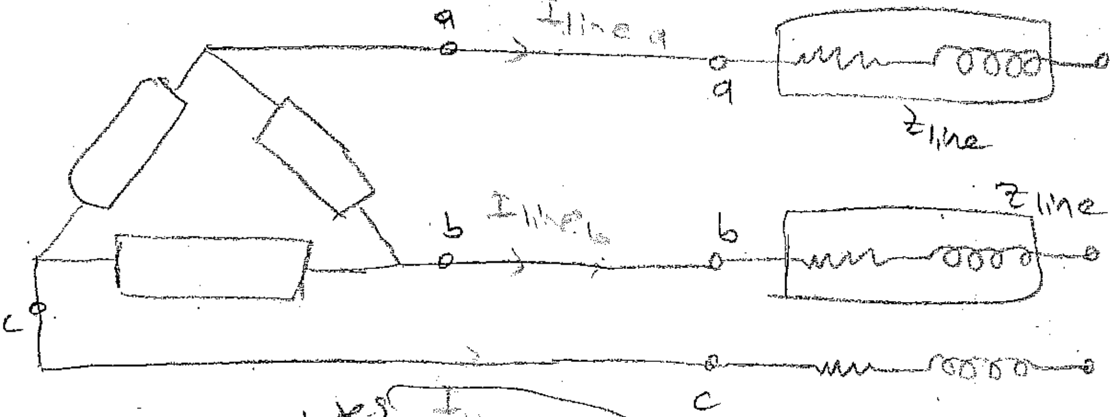
$$= \sqrt{3}\,V_{ph}\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right)$$

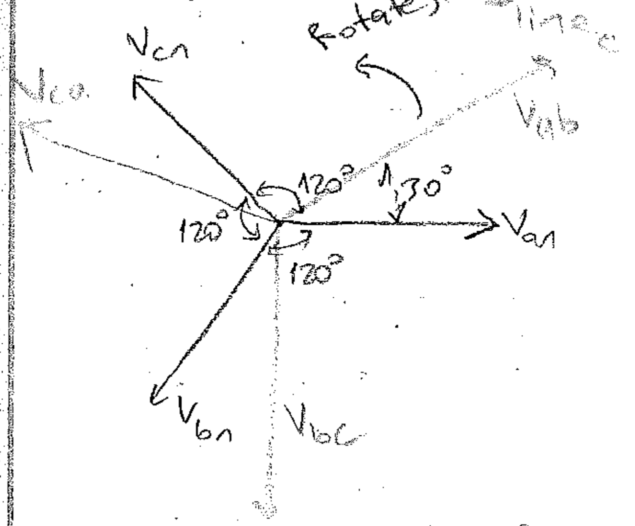
$$V_{ab} = \sqrt{3}\,V_{ph}\angle 30^\circ$$

$$V_{ac} = \sqrt{3}\,V_{ph}\angle -30^\circ$$

$$V_{cb} = \sqrt{3}\,V_{ph}\angle 90^\circ$$

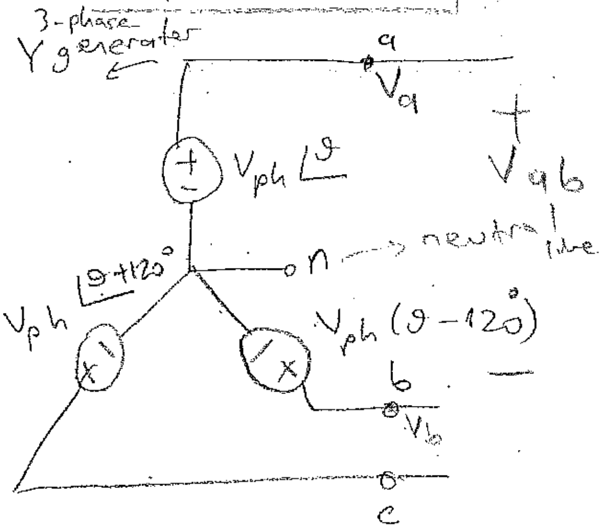
Line Currents , Phase Currents





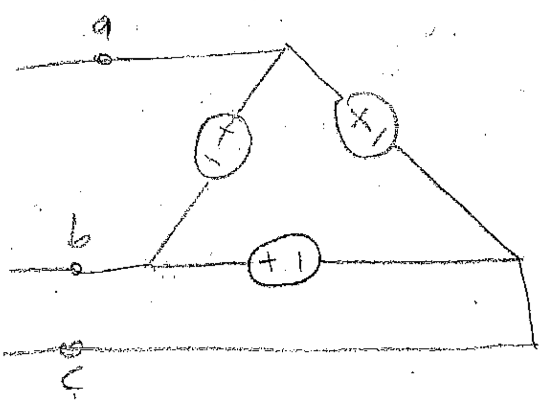
→ with angular velocity of ω = 2πf

3-phase System (Continued)
Line / Phase Voltages



There is no neutral line.

Δ 3-phase generator



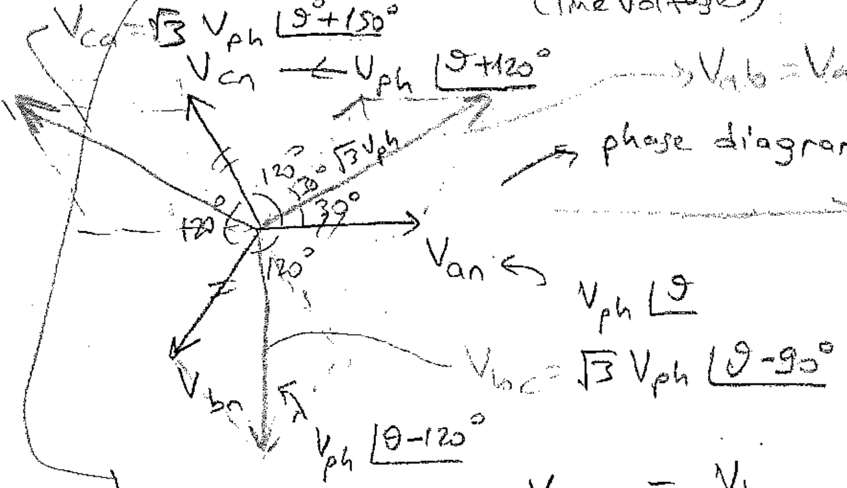
a, b, c : lines

n : neutral line

V_{an}, V_{bn}, V_{cn} : Phase voltages

$\{V_{ph}\angle\theta,\ V_{ph}\angle\theta-120^\circ,\ V_{ph}\angle\theta+120^\circ\}$

V_{ab}, V_{bc}, V_{ca} : line-line voltages (line voltages)



$V_{ca} = \sqrt{3}\, V_{ph} \angle \theta° + 150°$

$V_{cn} \leftarrow V_{ph} \angle \theta + 120°$

$\rightarrow V_{ab} = V_{an} - V_{bn} = \sqrt{3}\, V_{ph} \angle \theta + 30°$

→ phase diagram

→ Red ones are line voltages

$V_{an} \leftarrow V_{ph} \angle \theta$

$V_{bc} = \sqrt{3}\, V_{ph} \angle \theta - 90°$

$V_{ph} \angle \theta - 120°$

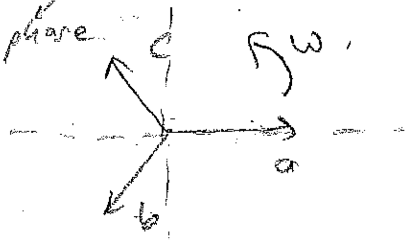
$$V_a - V_b = V_{ab}$$

$$V_{ab} = \underbrace{(V_a - V_n)}_{V_{an}} - \underbrace{(V_b - V_n)}_{V_{bn}}$$

$$= V_a - V_b$$

Positive Sequence

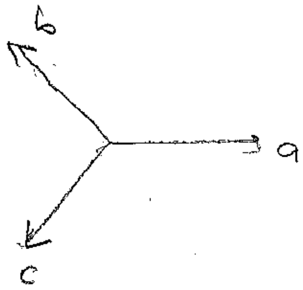
3ϕ systems can be configured in two ways:



"a" leads "b" leads "c"

Config. 1

↑ Positive sequence (abc sequence)



"a" leads "c" leads "b"

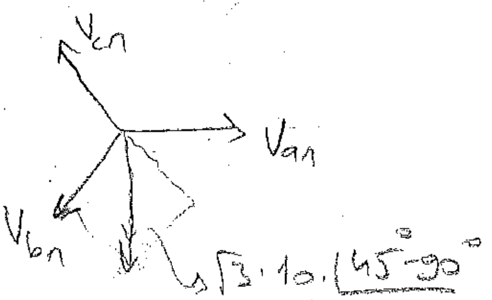
Config 2

Negative Sequence (acb sequence)

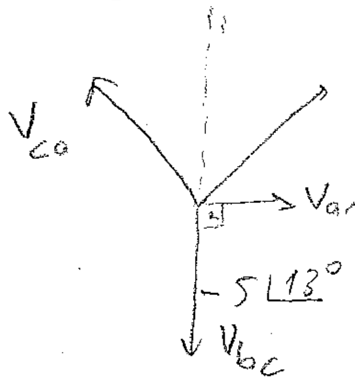
In EE202 we'll only use positive sequence for 3ϕ systems.

Ex: $V_{an} = 10 \angle 45°$ Volts (abc sequence)

Find V_{bc}.



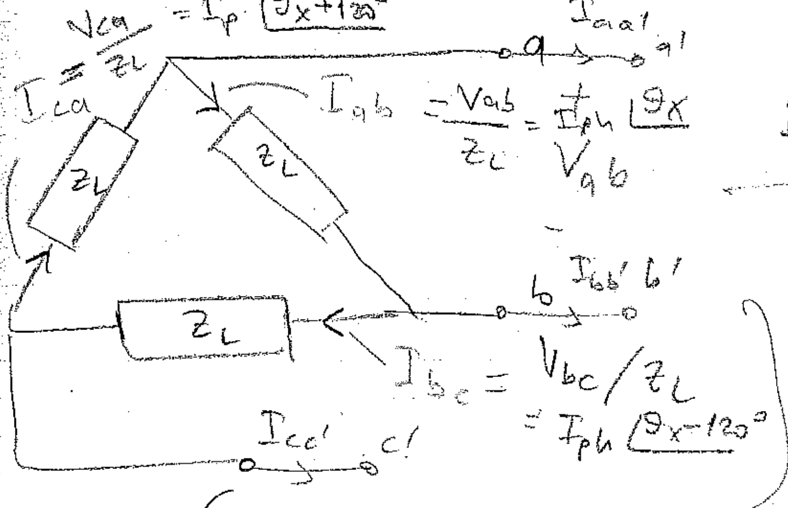
Ex: $V_{cb} = 5\angle 13^\circ$ Volts $V_{an} = ?$



$$V_{an} = \frac{-5}{\sqrt{3}}\angle 13^\circ + 90^\circ = \frac{5}{\sqrt{3}}\angle 13^\circ + 90^\circ - 180^\circ$$

$$= \frac{5}{\sqrt{13}}\angle -77^\circ \text{ Volts.}$$

Line Current / phase Current



$$I_{ca} = \frac{V_{ca}}{Z_L} = I_p\angle\theta_x + 120^\circ$$

$$I_{ab} = \frac{V_{ab}}{Z_L} = I_{ph}\angle\theta_x \quad V_{ab}$$

$$I_{bc} = V_{bc}/Z_L = I_{ph}\angle\theta_x - 120^\circ$$

I_{ab}, I_{bc}, I_{ca} : phase currents

$I_{aa'}, I_{bb'}, I_{cc'}$: Line current

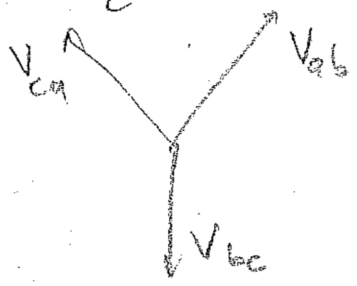
$I_{aa'} = ?$

$$I_{aa'} = I_{ca} - I_{ab}$$

$$= \frac{V_{ca}}{Z_L} - \frac{V_{ab}}{Z_L}$$

$$= I_{ph}\angle\theta_x + 120^\circ - I_{ph}\angle\theta_x$$

$$= \sqrt{3}\, I_{ph}\angle\theta_x + 150^\circ$$



$$I_{bb'} = I_{ab} - I_{bc}$$

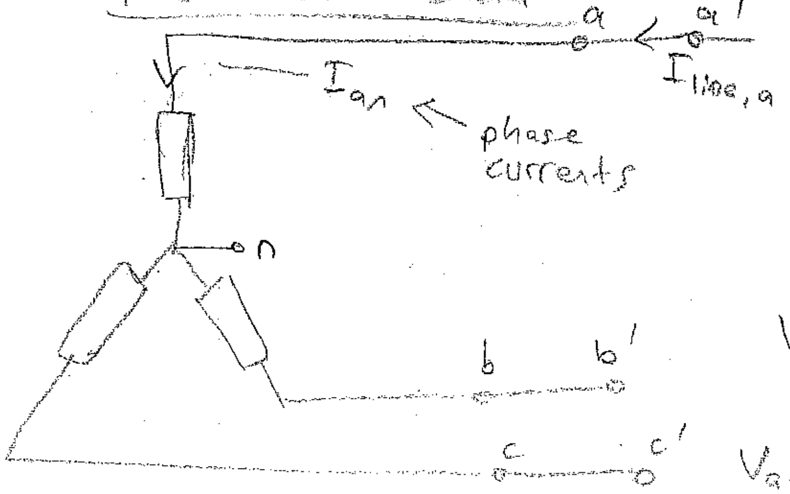
$$= I_{ph}\angle\theta_x - I_{ph}\angle\theta_x - 120^\circ$$

$$= \sqrt{3}\, I_p\angle\theta_x + 30^\circ$$

$$I_{cc'} = \sqrt{3} I_{ph}\angle\theta_x - 90^\circ$$

Note: The magnitude of line current is bigger by a power factor of $\sqrt{3}$ for a Δ connected 3ϕ Load/Systems.

Y-Connected Load



$I_{line,a} = I_{an}$

$I_{line,b} = I_{bn}$

$I_{line,c} = I_{cn}$

Y-connected Load phase Current = Line current

V_{an}, V_{bn}, V_{cn} ← Phase Voltages

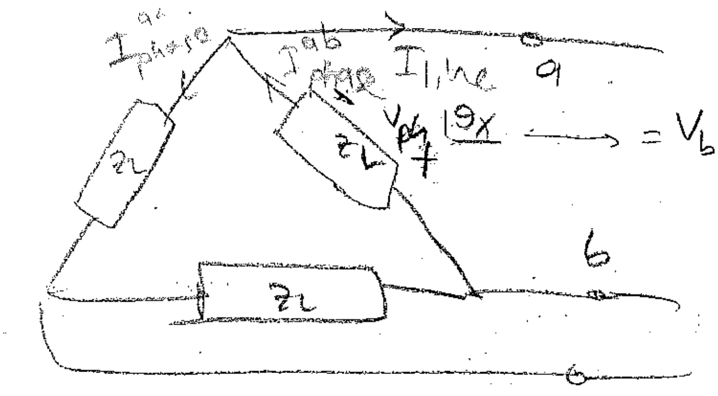
V_{ab}, V_{bc}, V_{ca} ← Line Voltages

Y-connected Load Rms line voltages are $\sqrt{3}$ times Rms phase voltages (Their angles also differ)

For a Δ connected Load:

$V_{phase\ voltage} = V_{line-line\ Voltage}$

$\sqrt{3}\, I^{RMS}_{phase} = I^{RMS}_{line}$

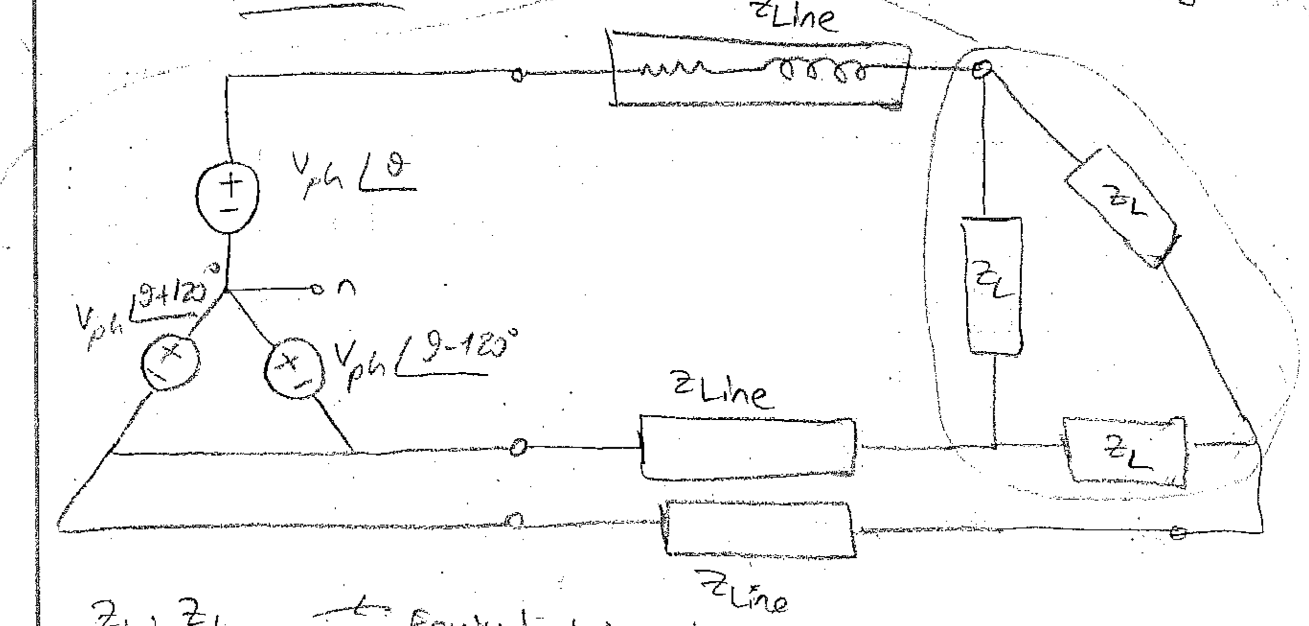


$V_{ph}\angle\theta x \longrightarrow = V_{ba} \rightarrow$ line-to-line voltage

Analysis of Balanced 3φ Systems

Balanced

We are always using abc connected systems.

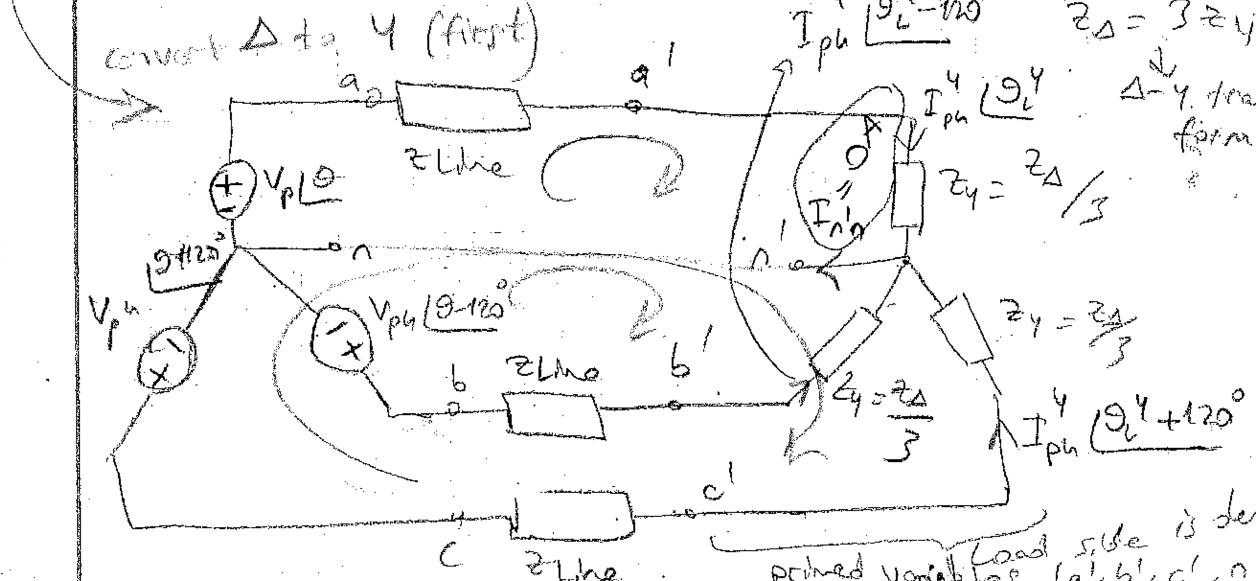


Z_L: Z_{Load} ← Equivalent Impedance of a 3φ motor

In balanced 3φ systems, the circuit in each phase is identical except the voltage source phase difference of ±120°.

convert Δ to Y (first)

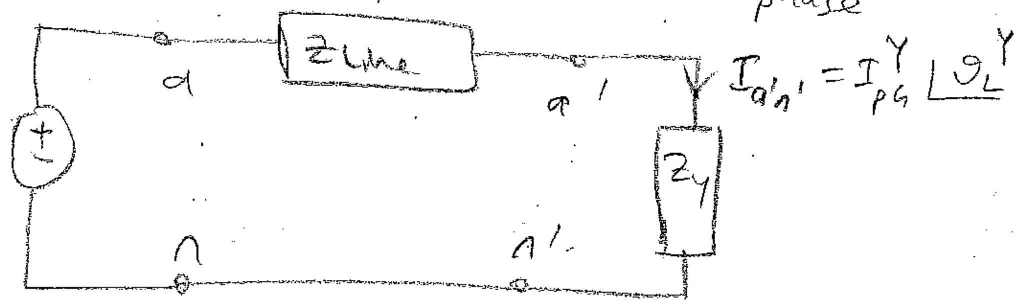
$Z_\Delta = 3 Z_Y$ ↓ Δ-Y transformation formula



Load side is denoted by primed variables. (a', b', c', n')

Single Phase Equivalent System

We only sketch a single phase (say "a"–"n" phase)

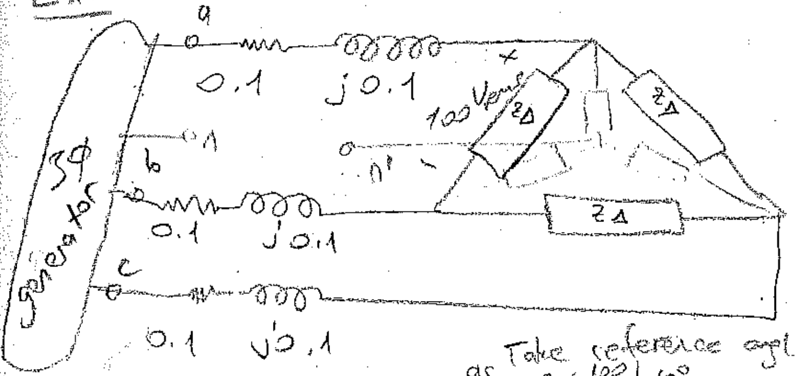


Note that $I_{n'n} = 0$ A since we have a balanced three-phase system and currents through each Y-connected load impedance are identical in RMS, but differ with ± 120°.

Therefore, the line from n to n' can be disconnected and totally removed for three phase balanced systems.

Also look at Sadiku's Textbook!!!

Ex:



From these we understand it is P (real power)

A 3φ Load consumes 1200 Watts at p.f. of 0.8 lagging.

a) Find complex power generated by generator

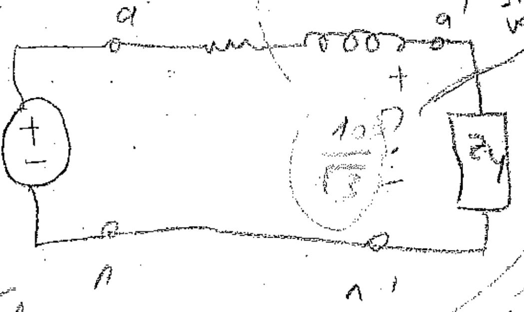
b) Find V_{ph} (RMS) of generator

i) if generator is Δ connected

ii) " " " Y " .

Soln.

a)



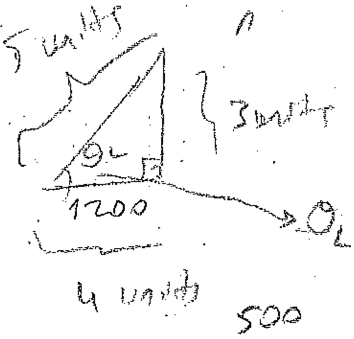
Take reference angle as 0° $\left(\frac{100}{\sqrt{3}}\right) \angle 0^\circ$ → RMS value

$Z_Y = \frac{Z_\Delta}{3}$

$$\frac{V^{RMS}_{a'b'}}{\sqrt{3}} = \frac{100}{\sqrt{3}} \text{ Volts}$$

per phase power consumption

3φ power consumption



$\theta_L = \cos^{-1}(0.8)$

$$S^{3\phi}_{Load} = 1200 + j\,900 \text{ VA}$$

Total (3φ) complex power

$$S^{\phi}_{load} = \frac{S^{3\phi}_{Load}}{3} = 400 + j300 \text{ VA}$$

$$\overbrace{|S^{\phi}_{Load}|}^{500} = \overset{100/\sqrt{3}}{\uparrow} V^{\phi,RMS}_{Load} \cdot I^{\phi,RMS}_{Load} \Longrightarrow I^{RMS,\phi}_{Load} = 5\sqrt{3} \text{ A}_{(RMS)}$$

$$S^{\phi}_{Line} = (I^{RMS}_{Line})^2 Z_{Line} = (5\sqrt{3})^2 [0.1 + j0.1]$$

$$= 7.5 + j\,7.5 \text{ VA}$$

$$S^{\phi}_{\text{supplied}} = S^{\phi}_{\text{Load}} + S^{\phi}_{\text{line}} = 407.5 + j\,307.5 \ \text{VA}$$

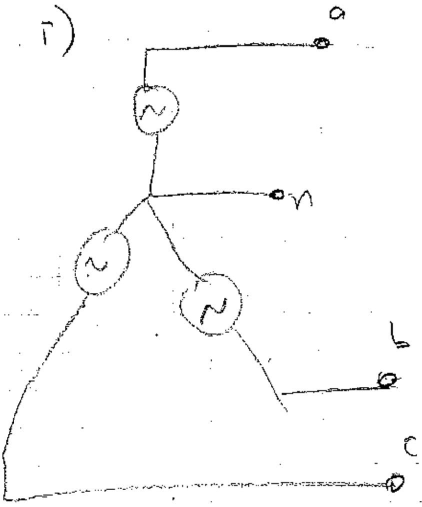
$$\boxed{S^{3\phi}_{\text{supplied}} = 3 S^{\phi}_{\text{Supplied}} = 1222.5 + j\,922.5 \ \text{VA}}$$

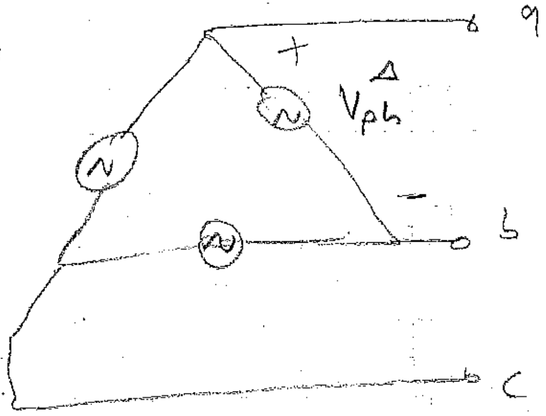
b) From single phase equivalent, let's find V_x (RMS).

$$|S^{\phi}_{\text{supplied}}| = V_x^{\text{rms}} \underbrace{I_{\text{line}}^{\text{rms}}}_{5\sqrt{3}\ A_{\text{rms}}} \longrightarrow V_x^{\text{rms}} = \frac{|407.5 + j\,307.5|}{5\sqrt{3}} = 58.94 \ V_{\text{rms}}$$

$$\sqrt{(407.5)^2 + (307.5)^2}$$

c)

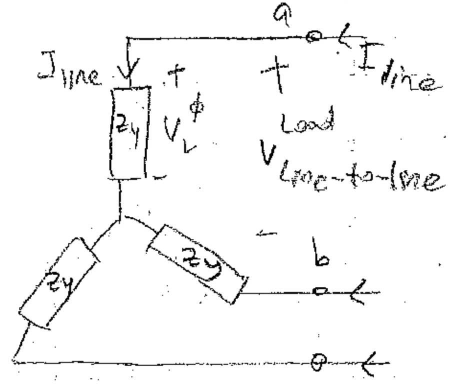


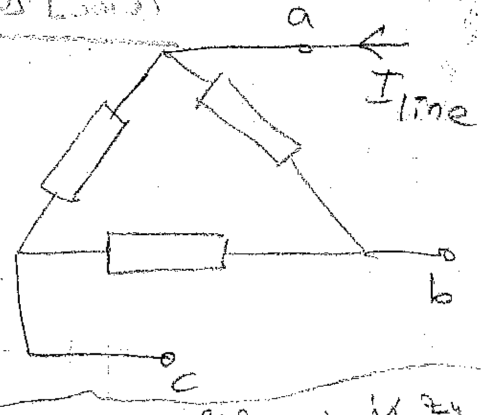


$$V^{Y\text{-Generator}} = 58.14 \ V_{\text{rms}}$$

$$V_{\text{phase}}^{\Delta\text{-Generator}} = \sqrt{3}\, V_X = 102.1 \ V_{\text{rms}}$$

3φ Power Calculations for Y-Δ Loads





$$S_L^{3\phi} = 3 S_L^{\phi} = 3 (I_{\text{Line}}^{\text{rms}})^2 \cdot Z_Y$$

$$= 3 I_{\text{Line}}^{\text{rms}} \cdot I_{\text{Line}}^{\text{rms}} \cdot Z_Y$$

$$Z_Y \rightarrow |Z_Y| e^{j\angle Z_y}$$

$$= 3 I_{\text{Line}}^{\text{rms}} (I_{\text{Line}}^{\text{rms}} |Z_Y|) e^{j\angle Z_y}$$

$$= 3 I_{\text{Line}}^{\text{rms}} (V_{\text{Load}}^{\phi,\text{rms}}) \cdot e^{j\angle Z_y}$$

$$= 3 I_{\text{Line}}^{\text{rms}} \frac{V_{\text{Line-to-line}}^{\text{rms,Load}}}{\sqrt{3}} \cdot e^{j\angle Z_y}$$

$$\boxed{S_L^{3\phi} = \sqrt{3}\, I_{\text{Line}}^{\text{rms}} V_{\text{Line-to-line}}^{\text{rms}} e^{j\angle Z_y}}$$

For Δ connection:

$$S_l^{3\phi} = 3\,S_L^{\phi} = 3\,(I_{Load}^{\phi,RMS})^2\, Z_\Delta$$

$$= 3\, I_{Load}^{\phi,RMS}\, I_{Load}^{\phi,RMS}\, \underbrace{Z_\Delta}_{|Z_\Delta|\, e^{j\measuredangle Z_\Delta}}$$

$$= 3\, I_{load}^{\phi,RMS} \cdot V_{Load}^{\phi,RMS} \cdot e^{j\measuredangle Z_\Delta}$$

$$= 3\, \underbrace{I_{Load}^{RMS}}_{\frac{I_{line}^{RMS}}{\sqrt{3}}} \cdot \underbrace{V_{Load}^{\phi,RMS}}_{V_{Line}^{RMS}} \cdot e^{j\measuredangle Z_\Delta}$$

$$\boxed{S_L^{3\phi} = \sqrt{3}\, I_{Line}^{RMS} \cdot V_{Line\text{-}to\text{-}line}^{RMS}\, e^{j\measuredangle Z_\Delta}}$$

Note that, we have the same total 3ϕ power formulas for both Δ and Y connected loads. That is

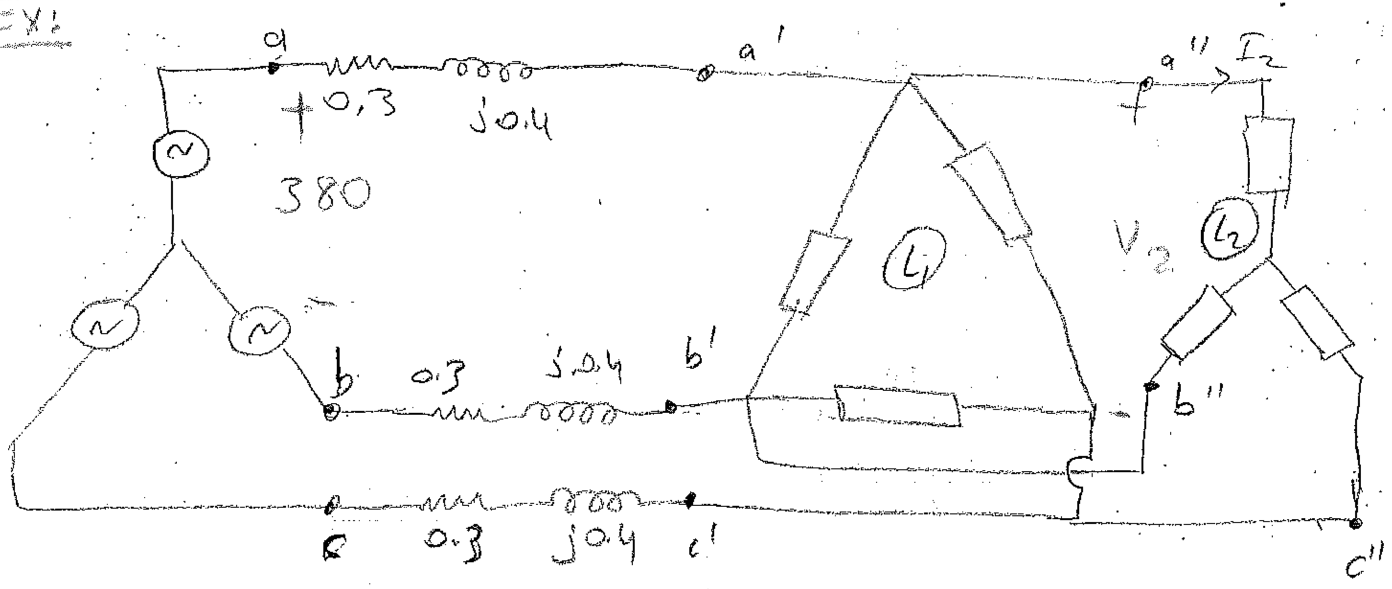
$$S_{Load}^{3\phi} = \sqrt{3}\, V_{Line\text{-}to\text{-}line}^{Load\text{-}RMS} \cdot I_{Line}^{Load,RMS} \cdot e^{j\overbrace{(\measuredangle Z_L)}^{= \theta_{Z_L}}}$$

$$\text{Re}\{\cdot\cdot\} \rightarrow P_{Load}^{3\phi} = \sqrt{3}\, V_{Line\text{-}to\text{-}line}^{Load\text{-}RMS}\, I_{Line}^{Load,RMS}\, \underbrace{\cos(\theta_{Z_L})}_{\text{power factor}}$$

or

$$\text{Im}\{\ldots\} \rightarrow Q_{Load}^{3\phi} = \sqrt{3}\, V_{Line\text{-}to\text{-}line}^{Load,RMS}\, I_{Line}^{RMS}\, \sin(\theta_{Z_L})$$

Ex:

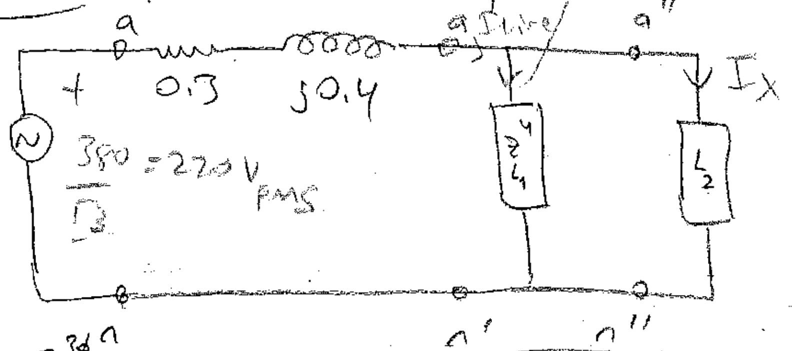


* Generator provides 380 V_{RMS} (line-to-line) at 9 kW and 9 kVAR

* L_1: 6 kW at 0.6 p.f. lagging

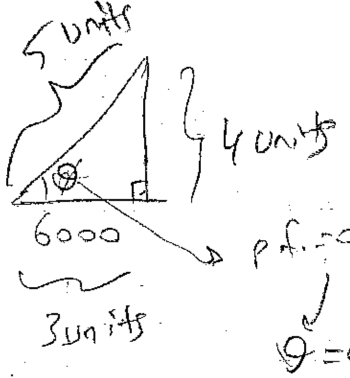
Find V_2^{RMS}, I_2^{RMS} and power absorbed by L_2.

Sol. ①: (Single phase Equivalent)



★★ This is not $I_{Line}^{RMS} - I_X^{RMS}$!!!

$$S_{L_1}^{3\phi} = 6000 + j8000\ VA \longrightarrow \boxed{S_{L_1}^{\phi} = 2000 + j2666\ VA}$$



$$\boxed{S_{Source}^{\phi} = 3000 + j3000}$$

p.f. = 0.6

$\theta = \cos^{-1}(0.6)$

$$\underset{3000\sqrt{2}}{|S_{Source}^{\phi}|} = 220\ V_{RMS} \cdot I_{Line}^{RMS} \longrightarrow I_{Line}^{RMS} = 19.3\ A_{RMS}$$

$$S_{Line}^{\phi} = (I_{Line}^{RMS})^2 Z_{Line} = (19.3)^2 \times (0.3 + j0.4) = 112 + j149$$

$$S_{L_1+L_2}^{\phi} = S_{supplied}^{\phi} - S_{Line}^{\phi} = (3000 + j3000) - (112 + j149)$$

$$= 2888 + j2851$$

$$S_{L_2}^{\phi} = S_{L_1+L_2}^{\phi} - S_{L_1}^{\phi} = \boxed{888 + j187\ VA}$$

$$|S_{L_1+L_2}^{\phi}| = V_X^{RMS} \cdot I_{Line}^{RMS} \longrightarrow |2888 + j2851| = V_X^{RMS} \times (19.3)$$

$$\longrightarrow \boxed{V_X^{RMS} = 210.3\ V_{RMS}}$$

$$|S_{L_2}| = V_X^{RMS} \cdot I_X^{RMS}$$

$$\sqrt{888^2 + 187^2} = (210.3) I_X^{RMS}$$

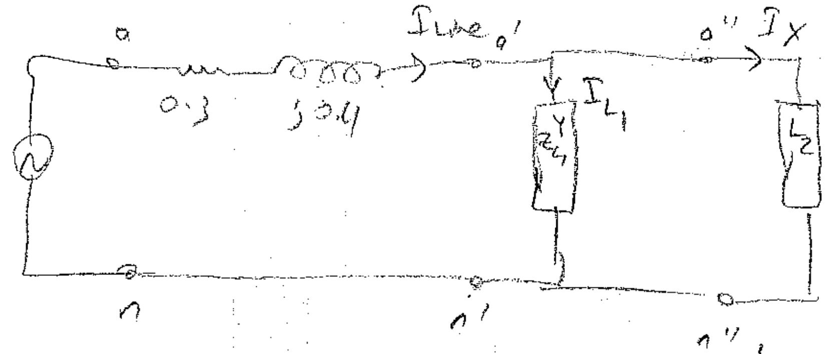
$$I_X^{RMS} = 4.3\ A_{RMS}$$

<u>Do not</u> write KCL and KVL over <u>RMS values</u>.

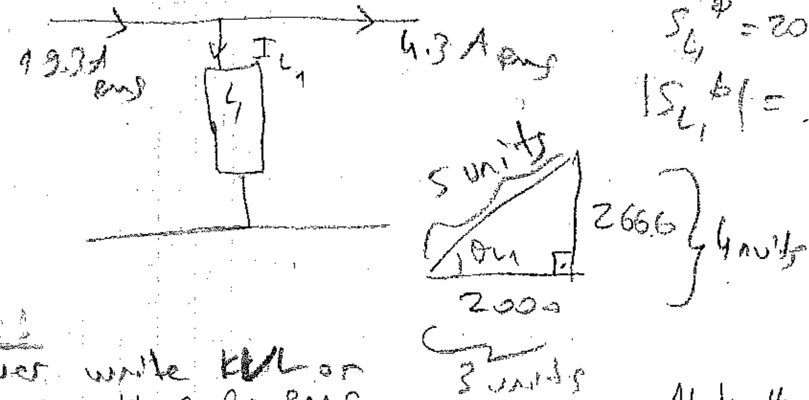
$$V_2^{RMS} = \sqrt{3}\, V_X = 364\ V_{RMS}$$

$$I_2^{RMS} = I_X = 4.3\ A_{RMS}$$

$$S_{L_2}^{3\phi} = 3 S_{L_2}^{\phi} = 3(888 + j187)\ VA$$



Note: In the previous problem, let's find I_{L_1}.



$$S_{L_1}^{\phi} = 2000 + j2666$$

$$|S_{L_1}^{\phi}| = \frac{2000}{0.6} = 3333 = \overset{210.3}{\boxed{V_{L_1}^{rms}}} \cdot I_{L_1}^{rms}$$

$$\cos(\theta_{L_1}) = 0.6$$

$$\boxed{I_{L_1}^{rms} = 15.84\ A_{rms}}$$

Note that: $I_{L_1}^{rms} \neq 19.3 - 4.3$

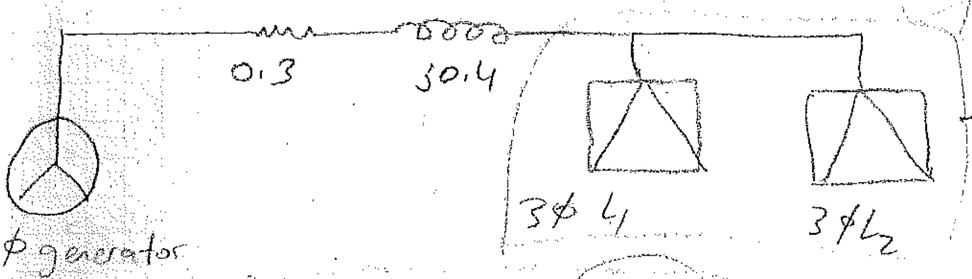
$15.84 \neq 15$

Note!

Never write KVL or KCL equations for RMS quantities, it should be clear that RMS quantities by definition >0, so $V_1^{rms} + V_2^{rms} + V_3^{rms} \neq 0$

can never be equal to zero.

Sol. ②: Schematic for a 3φ system



two loads are "hanging"!

→ we don't have any information about them being Y or Δ; because this formula is valid for both cases

$$\boxed{S_{gen}^{3\phi} = \sqrt{3}\, V_{Line-to-line}^{rms}\, I_{Line}^{rms}\, e^{j\boxed{\angle\theta_{gen}}}} \rightarrow \text{p. f. angle}$$

$$\underbrace{|S_{gen}^{3\phi}|}_{9000\sqrt{2}} = \sqrt{3}\, \underbrace{V_{Line-to-line}^{rms}}_{380}\, I_{Line}^{rms} \rightarrow I_{Line}^{rms} = 19.3\ A_{rms}$$

$$S_{Line}^{3\phi} = 3(I_{Line}^{rms})^2 \cdot Z_{Line} = 3(112 + j149) = \boxed{336 + j447}$$

$$S_{L_1+L_2}^{3\phi} = \underset{\overset{\uparrow}{9000 + j9000}}{S_{gen}^{3\phi}} - S_{Line}^{3\phi} = 8664 + j8553\ VA$$

$$|S_{L_1+L_2}^{3\phi}| = \sqrt{3}\, V_{L_1+L_2\,line-to-line}^{rms}\, I_{L_1+L_2}^{rms} \longrightarrow V_{(L_1+L_2)}^{line-to-line,\, rms} = \underset{\overset{\uparrow}{V_2}}{364}\ V_{rms}$$

$$S_{L_2}^{3\phi} = S_{L_1+L_2}^{3\phi} - S_{L_1}^{3\phi} = 2664 + j561$$

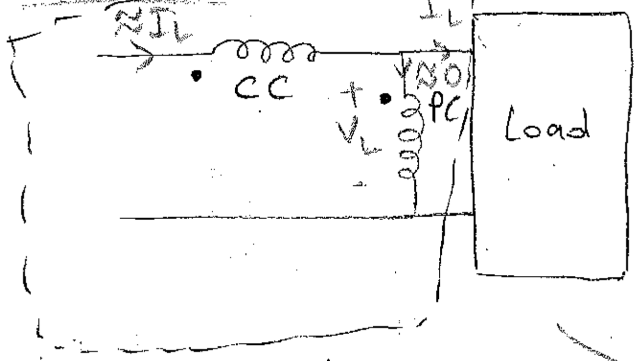
Then $\rightarrow |S_{L_2}^{3\phi}| = \sqrt{3}\, V_{L_2,\text{line-to-line}}^{RMS}\, I_{L_2}^{\text{line, RMS}} \longrightarrow I_{L_2}^{\text{line RMS}} = 4.3$ Amps

HW. Pls. also check notes p.69 for solution of a 3φ power compensation problem (A ZPS problem).

Check Sadiku's book for 3φ definition and examples.

3φ Power Measurement with 2 Wattmeters:

Wattmeter: An equipment built for power measurement.

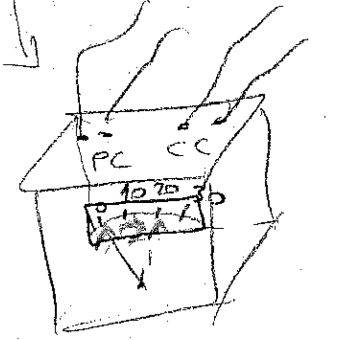


Wattmeter

PC: Potential coil → High impedence

CC: current coil → Low impedence

→ 1-φ Power Measurement



$$\boxed{S = \frac{1}{2} V_L^{ph} I_L^{ph\,*}}$$

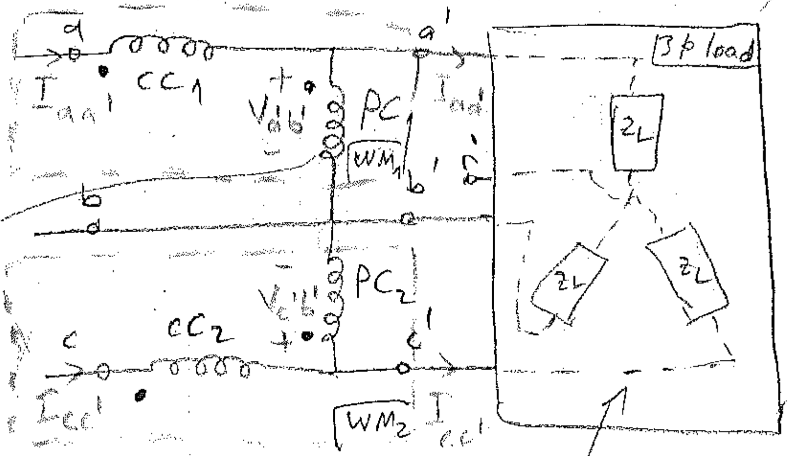
→ Complex power of the load

Wattmeter gives the reading of:

$$P_{\text{wattmeter}} = Re\{S\} = Re\left\{\frac{1}{2} V_L^{ph} (I_L^{ph})^*\right\}$$

$$\left(= \frac{1}{2} |V_L^{ph}| \cdot |I_L^{ph}| \cos(\theta_{Load}) = V_L^{RMS} \cdot I_L^{RMS} \cos(\theta_{Load}) \right)$$

3φ Power Measurement (with 2 wattmeters)



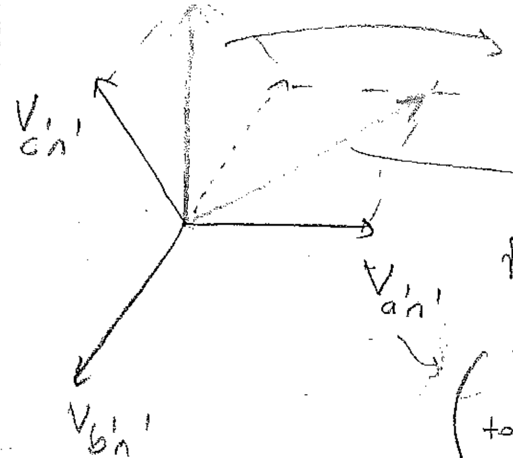
Power measured by WM's:

$$WM_1 \Rightarrow \frac{1}{2} Re\{V_{a'b'} I_{aa'}^*\}$$

$$WM_2 \Rightarrow \frac{1}{2} Re\{V_{c'b'} I_{cc'}^*\}$$

line-to-line voltage or line voltage V_{Line}^{ph}

balanced 3φ Load



$V_{c'b'} = |V_{Line}^{ph}| \angle 90^\circ$ ③

$V_{a'b'}$ ① $= \sqrt{3}|V_{a'n'}| \angle 30^\circ = |V_{Line}^{ph}| \angle 30^\circ$

($V_{a'n'}$ is assumed to be at 0° of reference phase)

$$I_{aa'} = \frac{V_{a'n'}}{Z_L} = \frac{V_{a'n'} \angle 0^\circ}{|Z_L| \angle \theta_{Z_L}} = |I_{Line}^{ph}| \angle -\theta_{Z_L} \quad ②$$

$$I_{cc'} = |I_{Line}^{ph}| \angle 120^\circ - \theta_{Z_L} \quad ④$$

positive sequence

WM$_1$:

$$P_{WM_1} = \frac{1}{2} Re\{\underbrace{V_{a'b'}}_{①} \underbrace{I_{aa'}^*}_{②}\} = \frac{1}{2} Re\{|V_{Line}^{ph}||I_{Line}^{ph}| \angle 30^\circ + \theta_{Z_L}\}$$

WM$_2$:

$$P_{WM_2} = \frac{1}{2} Re\{\underbrace{V_{c'b'}}_{③} \underbrace{I_{cc'}^*}_{④}\} = \frac{1}{2} Re\{|V_{Line}^{ph}||I_{Line}^{ph}| \angle -30^\circ + \theta_{Z_L}\}$$

$$P_{WM_1} = \frac{1}{2}|V_{Line}^{ph}||I_{Line}^{ph}| \cos(30^\circ + \theta_{Z_L})$$

$$P_{WM_2} = \frac{1}{2}|V_{Line}^{ph}||I_{Line}^{ph}| \cos(-30^\circ + \theta_{Z_L})$$

$$P_{WM_1} + P_{WM_2} = \frac{1}{2}|V_{Line}^{ph}||I_{Line}^{ph}| \cdot (2 \cdot \cos(\theta_{Z_L}) \cos(30^\circ))$$

$$= \frac{\sqrt{3}}{2}|V_{Line}^{ph}||I_{Line}^{ph}| \cdot \cos(\theta_{Z_L})$$

$$\boxed{= \sqrt{3}\, V_{Line}^{RMS} I_{Line}^{RMS} \cos(\theta_{Z_L})}$$

Remember that $P_{Load}^{3\phi} = \sqrt{3}\, V_{Line}^{RMS} I_{Line}^{RMS} \cos(\theta_{Load})$

Hence for the given wattmeter set-up, sum of watt-meter reading gives

$$\boxed{P_{WM_1} + P_{WM_2} = P_{Load}^{3\phi}}$$ → Sınavda çıkabilir

Similarly:

$$\boxed{\begin{aligned} P_{W_1} - P_{W_2} &= V_{Line}^{RMS} I_{Line}^{RMS} \sin(\theta_{Load}) \\ &= Q_{Load}^{3\phi}/\sqrt{3} \end{aligned}}$$

→ total reactive power of 3φ Load

S-Domain (Laplace Domain) Circuit Analysis

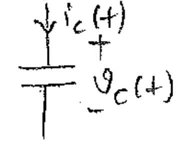
Previously we've studied application of Laplace transform in the analysis of N^{th} order LTI circuits.

Now, we'll directly transform the circuit to Laplace domain and give the solution in the Laplace domain.

laplace transform

Component | Time - Domain | Laplace Domain (S - domain)

C:



$$i_c(t) = C\frac{d}{dt}v_c(t) \xrightarrow{L.T.} I_c(s) = C\left[sV_c(s) - v_c(0^-)\right]$$

Upper case letter

lower case letter

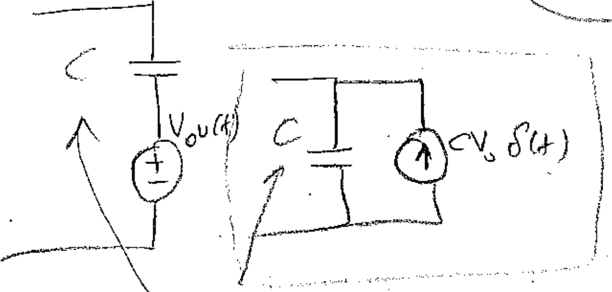
Initial condition

$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_0^t i_c(\tau)d\tau$$

$u(t)$

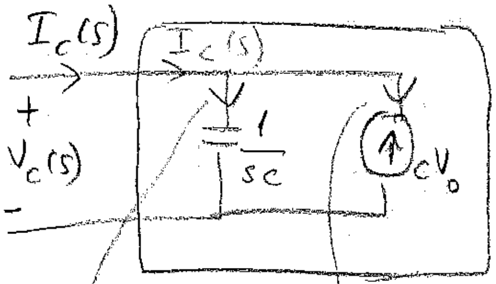
$t>0$

$$\xrightarrow{L.T.} V_c(s) = \frac{v_c(0^-)}{s} + \frac{1}{C}\frac{I_c(s)}{s}$$



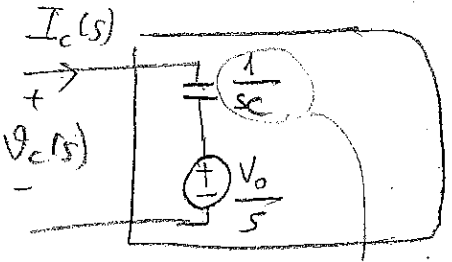
empty capacitor

Initial condition models for capacitor



$-CV_0$

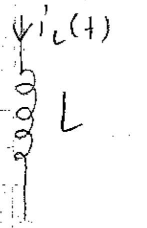
$$\frac{V_c(s)}{\frac{1}{sC}} = sCV_c(s)$$



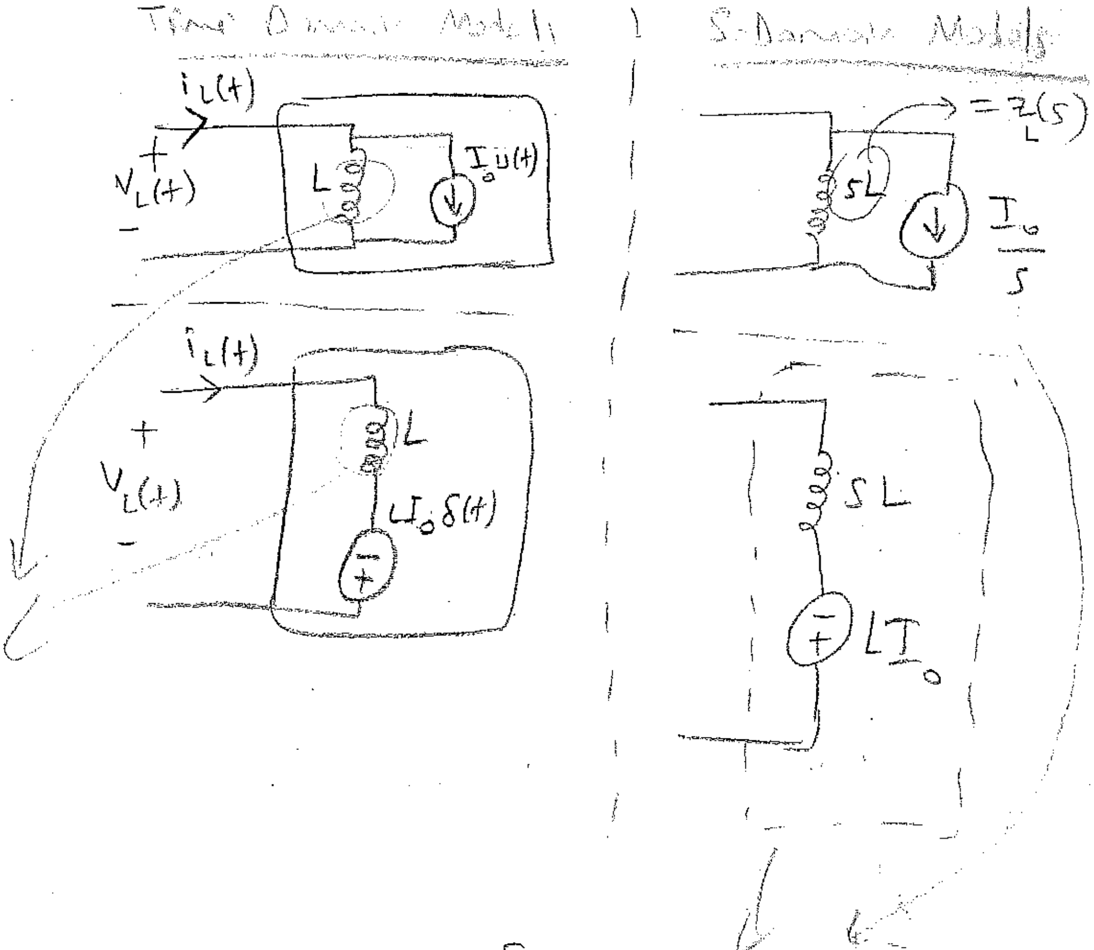
empty capacitor in Laplace domain

?

Inductor:



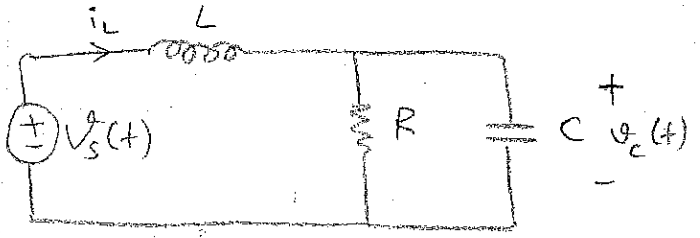
$i_L(0^-) = I_0$



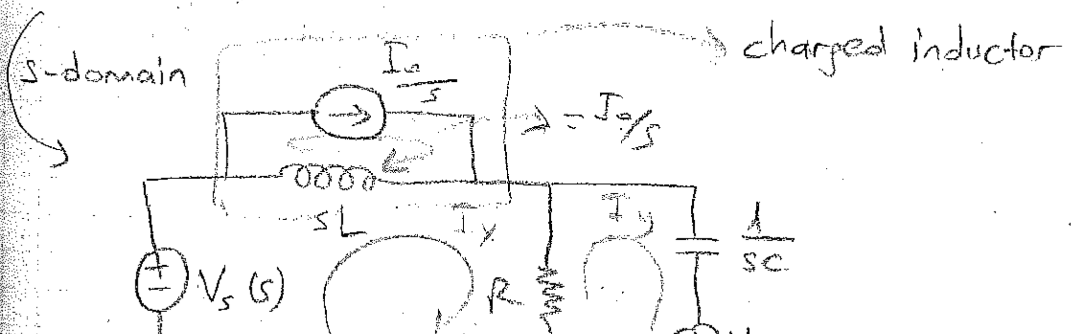
empty inductors (unfluxed

Source transformation of each other if sL is treated as an impedance with Z(s) definition.

Mesh Analysis in S-domain



$i_L(0^-) = I_0$
$v_c(0^-) = V_0$



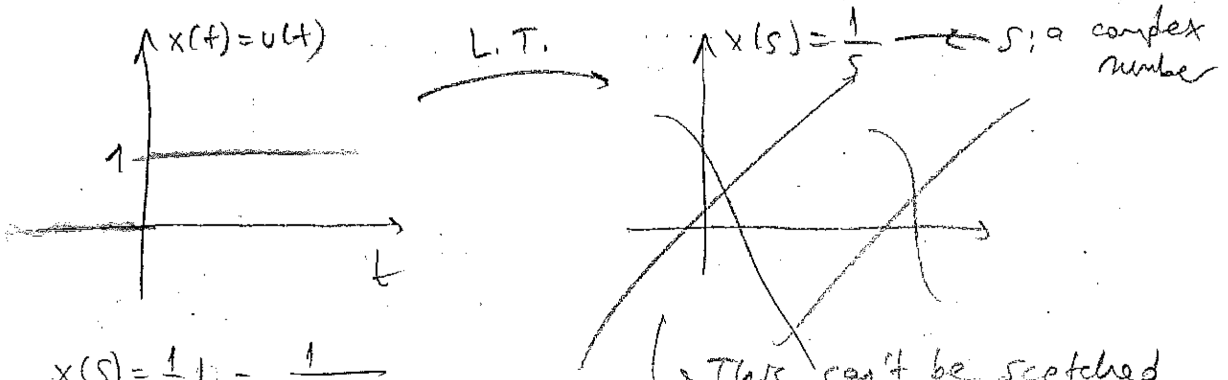
$$\begin{bmatrix} sL+R & -R \\ -R & R+\frac{1}{sC} \end{bmatrix} \begin{bmatrix} I_x(s) \\ I_y(s) \end{bmatrix} = \begin{bmatrix} v_s(s) + I_0 L \\ -\frac{V_0}{s} \end{bmatrix}$$

By calculating the inverse of matrix on LHS and applying from left to this equation, I can get

$$\underbrace{I_x(s)}_{\text{Complete solution in Laplace domain}} = \underbrace{\frac{\left(\frac{1}{RCL}+\frac{s}{L}\right)}{s^2+\frac{1}{RC}s+\frac{1}{LC}} v_S(s)}_{\text{Solution in Laplace domain for zero-state solution}} + \underbrace{\frac{\frac{I_0}{RC}-\frac{V_0}{L}+sI_0}{s^2+\frac{1}{RC}s+\frac{1}{LC}}}_{\text{zero-input solution}}$$

Note: $I_L(s)=I_x(s)$, that is changed inductor current is $I_x(s)$ in Laplace Domain.

Review: S-domain (Laplace Domain) Circuit Analysis:



$X(s)=\frac{1}{s}\Big\downarrow_{s=\sigma+j\omega} = \frac{1}{\sigma+j\omega}$ Upper-case "X"

lower case sigma

$\mathrm{Re}\{X(s)\} = \mathrm{Re}\left\{\frac{1}{\sigma+j\omega}\right\} = \frac{\sigma}{\sigma^2+\omega^2}$ ← function of two real variables

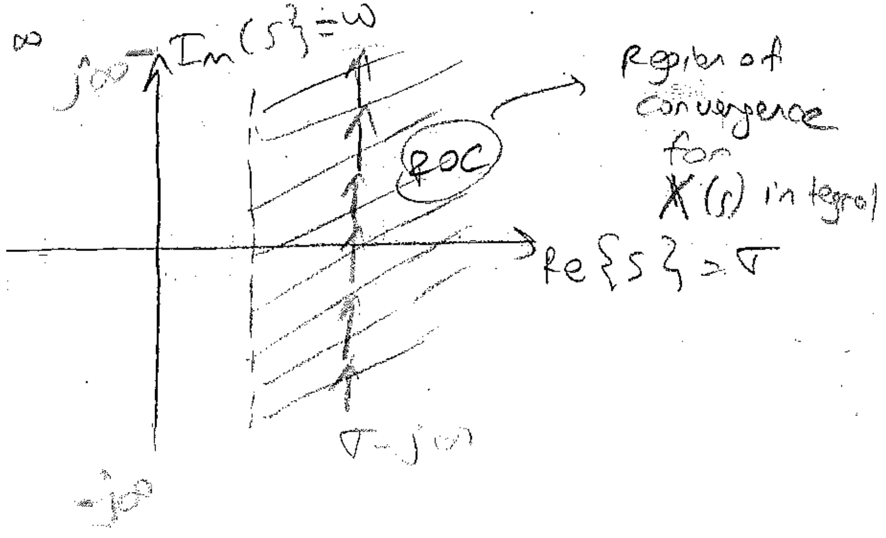
Laplace Transform

$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$ ← One-sided Laplace Transform (not two-sided)

There's a two-sided definition also.

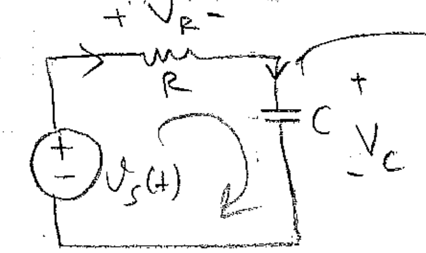
$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$

↑ inverse Laplace Transform



$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s) - \underbrace{x(0^-)}_{\text{Initial Condition}}$$

Circuit Analysis with Laplace Transform / in s-domain



$C\dot{V}_c(t)$

$(v_c(0^-) = V_0)$

KVL: $-v_s(t) + V_R + V_c = 0$

$$-v_s(t) + R\underbrace{C\dot{v}_c(t)}_{i_c = i_R} + v_c(t) = 0$$

Lower case lette

$$\boxed{\left(D + \frac{1}{RC}\right) v_c(t) = \frac{v_s(t)}{RC}\ ;\ v_c(0^-) = V_0}$$

L.T.

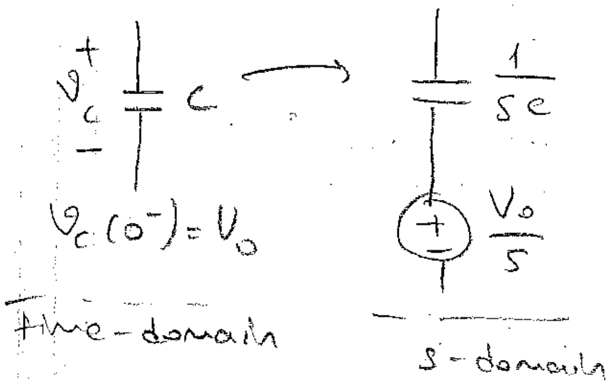
$$\mathcal{L}\left\{\dot{v}_c(t) + \frac{v_c(t)}{RC}\right\} = \mathcal{L}\left\{\frac{v_s(t)}{RC}\right\}$$

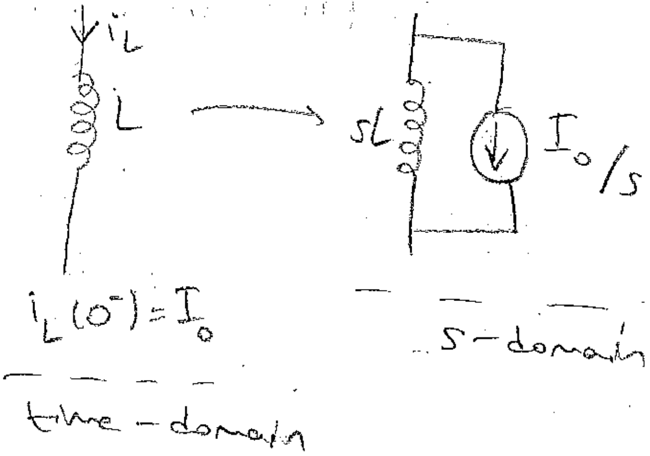
$$(sV_c(s) - V_0) + \frac{V_c(s)}{RC} = \frac{V_s(s)}{RC}$$

$$\boxed{V_c(s) = \frac{V_0}{s + \frac{1}{RC}} + \frac{1/RC}{s + 1/RC} \cdot V_s(s)}$$

$$v_c^{z.i.}(s) = \frac{V_0}{s + \frac{1}{RC}} \qquad v_c^{z.s.}(s) = \frac{1/RC}{s + 1/RC} V_s(s).$$

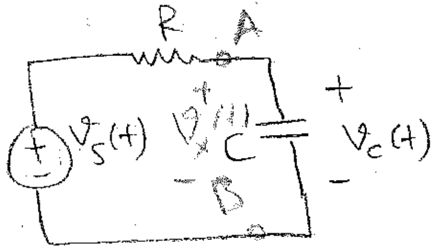
We'll further simplify the solution by transforming the circuit into Laplace domain.



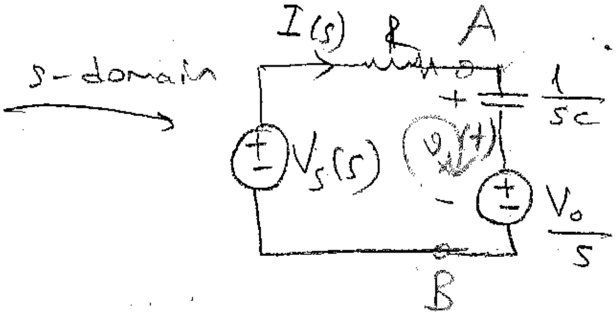


Note: when $s = j\omega$, we have the phasor domain equivalents except the sources which are related with I.C.s.

(Remember, phasor domain gives s.s. (steady-state) solution; hence I.C.'s are not important in phasor domain.)



$v_c(0^-) = V_0$



$$I(s) = \frac{v_s(s) - V_0/s}{R + \frac{1}{sC}}$$

$$V_x(s) = \frac{1}{sC} I(s) + V_0/s$$

$$= \frac{V_s(s) - V_0/s}{sCR + 1} + \frac{V_0}{s}$$

$$= \frac{V_s(s)}{sCR+1} + V_0/s \left(1 - \frac{1}{sCR+1}\right)$$

$$= \frac{V_s(s)/RC}{s + 1/RC} + \frac{V_0}{\cancel{s}} \left(\frac{\cancel{s}CR}{sCR+1}\right)$$

$$\downarrow$$

$$\frac{V_0}{s + \frac{1}{RC}}$$

Same with the result we have found.

Some Important Laplace Transform Related Properties

① Initial Value Theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Initial value

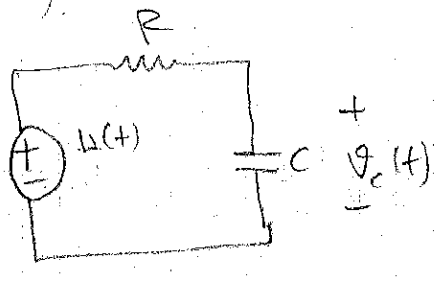
② Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Final value $f(\infty)$

Remark
① and ② is valid if all limits exist.

EX1



$v_c(0^-) = 10\,V$

$$v_c(t) = 1 + 9e^{-\frac{t}{\tau}}, \ t > 0 \ (\tau = RC)$$

$$\left.\begin{array}{l} v_c(0^+) = 10V \\ v_c(\infty) = 1V \end{array}\right\}$$

$$V_s(s) = \frac{V_s(s)/RC}{s + 1/RC} + \frac{V_0}{s + \frac{1}{RC}} \quad \rightarrow 10$$

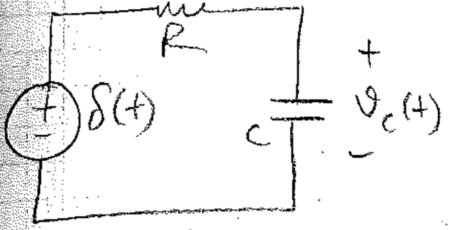
$\frac{1}{s}$

$\lim_{s\to\infty} sV_c(s) = 10 = \vartheta_c(0^+)$ $\lim_{s\to 0} sV_c(s) = 1 = \vartheta_c(\infty)$

$\vartheta_c(\infty) = 0$

$\vartheta_c(0^+) = 10 + \frac{1}{RC}$ V

Ex:



$$\downarrow$$

$$\boxed{\vartheta_c(t) = \left(10+\frac{1}{RC}\right)e^{-\frac{t}{\tau}}, t \geq 0}$$

$\vartheta_c(0^-) = 10$

$$V_c(s) = \frac{V_s(s)/RC}{s+\frac{1}{RC}} + \frac{\overset{\to 10}{V_0}}{s+\frac{1}{RC}} = \frac{10+\frac{1}{RC}}{s+\frac{1}{RC}}$$

$1 = \mathcal{L}\{\delta(t)\}$

Initial value $= \lim_{s\to\infty} sV_c(s) = 10 + \frac{1}{RC}$

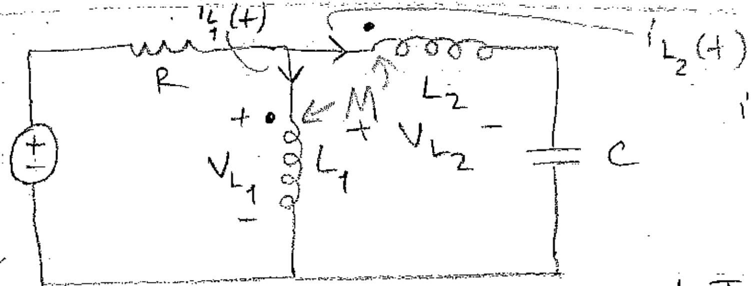
Final value $= \lim_{s\to 0} sV_c(s) = 0$

Second MT → It starts from A.C. analysis. HW's are on website.
Check Sadiku for A.C. analysis, 3ϕ systems, s-domain.

Additional Hours → Friday 17.40–19.40 (in usual lecture rooms)

You can also check ZPS problems

Ex:



$i_{L_1}(0^-) = I_1$

$i_{L_2}(0^-) = I_2$

$\vartheta_c(0^-) = V_0$

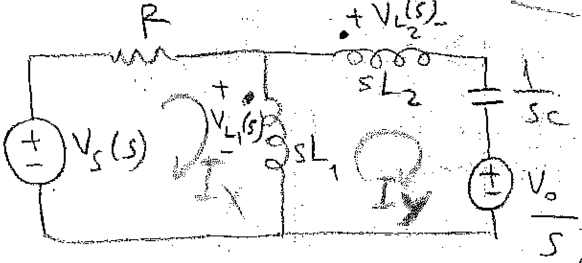
$i_{L_1}(0^-)$

$$\begin{bmatrix}\vartheta_{L_1}(t)\\ \vartheta_{L_2}(t)\end{bmatrix} = \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}\frac{d}{dt}i_{L_1}(t)\\ \frac{d}{dt}i_{L_2}(t)\end{bmatrix} \xrightarrow{\text{L.T.}} \begin{bmatrix}V_{L_1}(s)\\ V_{L_2}(s)\end{bmatrix} = \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}sI_{L_1}(s) - i_{L_1}(0)\\ sI_{L_2}(s) - i_{L_2}(0)\end{bmatrix}$$

$$= \begin{bmatrix}sL_1 & sM\\ sM & sL_2\end{bmatrix}\begin{bmatrix}I_{L_1}(s)\\ I_{L_2}(s)\end{bmatrix} - \begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}I_1\\ I_2\end{bmatrix}$$

$I_{L_1}(s) \to I_x(s) - I_y(s)$

$I_{L_2}(s) \to I_y(s)$



KVL (I_x):

$-V_s(s) + RI_x + V_{L_1}(s) = 0$

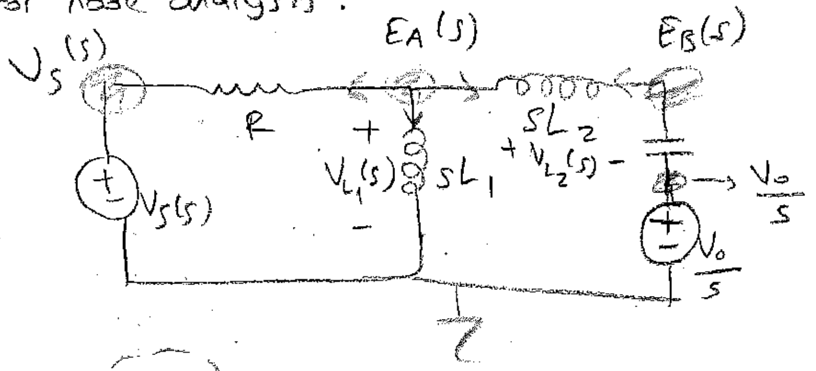
KVL (I_y mesh):

$-V_{L_1}(s) + V_{L_2}(s) + I_y\frac{1}{sC} + \frac{V_0}{s} = 0$

2 equations with 2 mesh currents as unknows

$$V_{L_1}(s) = sL_1(I_X - I_Y) + sMI_Y - L_1I_1 - MI_2$$
$$V_{L_2}(s) = sM(I_X - I_Y) + sL_2I_Y - MI_1 - L_2I_2$$

For node analysis:



$$\begin{bmatrix} V_{L_1}(s) \\ V_{L_2}(s) \end{bmatrix} = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix} \begin{bmatrix} I_{L_1}(s) \\ I_{L_2}(s) \end{bmatrix} - \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$E_A(s)$ $\quad$ $E_A(s) - E_B(s)$

Zero-state Responses in s-domain

Let's focus on zero-state response, that is the response when all I.C.'s are zero.

A typical solution is in the form

$$V_k^{z.s.}(s) = \frac{num(s)}{denum(s)} \cdot V_s(s)$$

External input in Laplace domain

For 1^{st} order RC circuit:

$$\boxed{v_c^{z.s.}(s) = \frac{1/RC}{s + 1/RC} V_s(s)}$$

Note that: for different inputs the zero-state solution is simply the product of $\frac{num(s)}{denum(s)} = H(s)$ and $V_s(s)$ in Laplace domain. That is, for any input $H(s)$ is fixed for a given system.

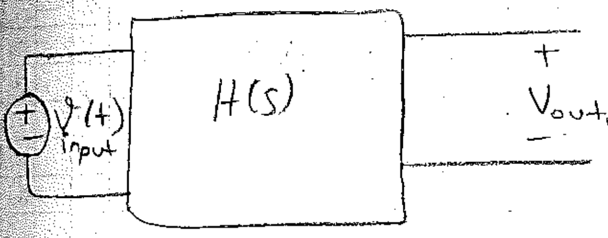
For example: $H(s) = \frac{1/RC}{s + 1/RC}$ ← 1st order R.C.

Case ① $v_s(t) = u(t) \rightarrow V_c^{z.s,\,step} = H(s) \cdot \mathcal{L}\{u(t)\} = \frac{1/RC}{s+1/RC} \cdot \frac{1}{s}$

Case ② $v_s(t) = \delta(t) \longrightarrow V_c^{z.s,\,impulse}(s) = \underbrace{\frac{\frac{1}{RC}}{s+\frac{1}{RC}}}_{H(s)} \cdot 1$

Case ③ $v_s(t) = 5\cos(2t) \longrightarrow V_c^{z,s,\,cosine}(s) = \underbrace{\frac{1/RC}{s+1/RC}}_{H(s)} \cdot \frac{5s}{s^2+4}$

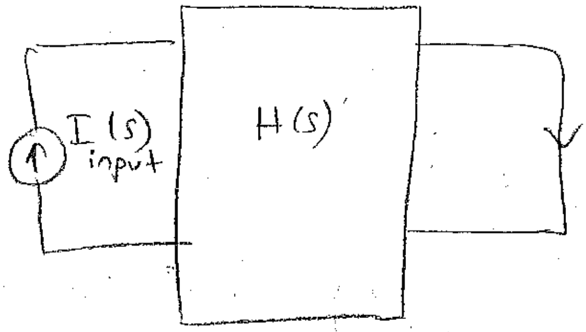
H(s) is called a transfer function and shown as:



$$\boxed{\overset{\text{Unitless}}{H(s)} = \frac{V_{output}(s)}{V_{input}(s)}}$$

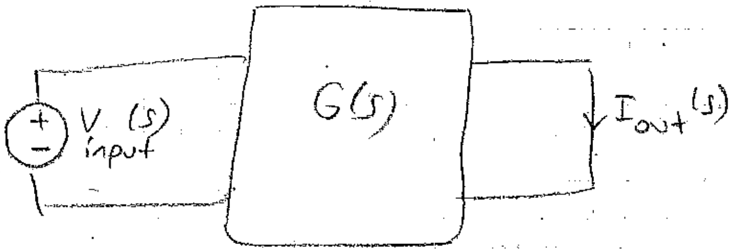
Voltage transfer function
input: voltage
output: voltage

Current transfer function:

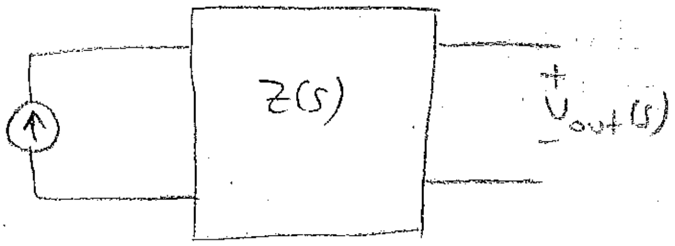


$\frac{I_{out}(s)}{I_{input}(s)} = H(s)$ → Unitless

Transfer admittance



$G(s) = \frac{I_{out}(s)}{V_{input}(s)}$ $\mho$



$Z(s) = \frac{V_{out}(s)}{I_{input}(s)}$ Ω → transfer impedence

Note: All transfer functions are only valid for z.s. circuits.

$$v_c(t) = 2\cos(2t+30^\circ) \text{ Volts}$$

$$\downarrow$$

$$v_c^{ph} = 2\angle 30^\circ$$

Phasor (Complex number)

$$V_c^{RMS} = \frac{2}{\sqrt{2}} = \sqrt{2}\ V_{RMS} \xrightarrow{V_{RMS}} v_c = \sqrt{2}\angle 30^\circ \text{ V (RMS)}$$

Magnitude of this phasor is in RMS!

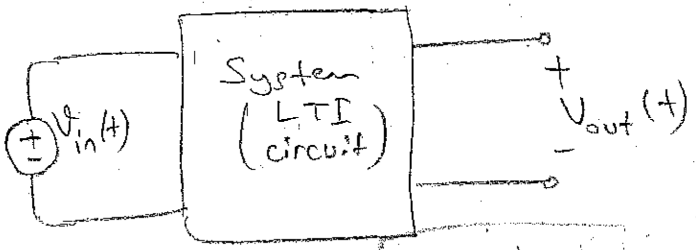
$$S_{cap} = \frac{1}{2}\frac{(v_{cap}^{ph})^2}{Z_{cap}^*} = \frac{(v_{cap}^{ph,RMS})^2}{Z_{cap}^*}$$

We can get rid of $\frac{1}{2}$ if we use $v_{cap}^{ph,RMS}$

$$X_{RMS} = \sqrt{\langle x^2(t)\rangle}$$

$$\langle x^2(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

Transfer Function:



$$H(s) = \frac{V_{out}(s)}{V_{input}(s)}$$

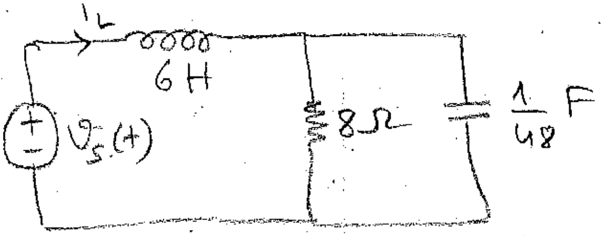
transfer function

$$V_{out}(s) = H(s)\,V_{input}(s)$$

Important Fact

Note that All transfer function related calculations assume that the initial conditions are all zero, that is they assume zero state conditions and transfer function is related with zero state input.

Ex:

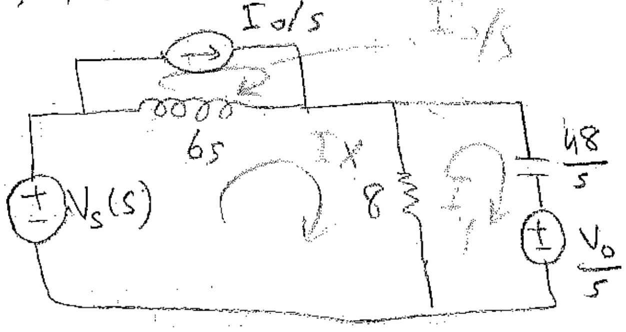


$v_c(0^-) = V_0$

$I_L(0^-) = I_0$

a) Find $i_L(t)$, $t>0$.

Solution:



$\underline{\text{KVL}}/I_x$: $$-V_s(s)+6s\left(I_x-\frac{I_0}{s}\right)+8(I_x-I_y)=0$$

KVL/I_y: $$8(I_y-I_x)+\frac{48}{s}\cdot I_y+\frac{V_0}{s}=0$$

$$\left[\begin{array}{c|c} 6s+8 & -8 \\ \hline -8 & \frac{48}{s}+8 \end{array}\right]\begin{bmatrix} I_x(s) \\ I_y(s) \end{bmatrix}=\left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right]V_s(s)+\left[\begin{array}{c} 6 \\ \hline 0 \end{array}\right]I_0+\left[\begin{array}{c} 0 \\ \hline \frac{-1}{s} \end{array}\right]V_0$$

$$\begin{bmatrix} I_x(s) \\ I_y(s) \end{bmatrix}=\underbrace{\left[\begin{array}{c|c} 6s+8 & -8 \\ \hline -8 & \frac{48}{s}+8 \end{array}\right]}_{M}^{-1}\begin{bmatrix} V_s(s)+6I_0 \\ -\frac{V_0}{s} \end{bmatrix}=\frac{1}{\Delta}\left[\begin{array}{c|c} \frac{48}{s}+8 & 8 \\ \hline 8 & 6s+8 \end{array}\right]\begin{bmatrix} V_s(s)+6I_0 \\ -\frac{V_0}{s} \end{bmatrix}$$

$$\Delta=\det(\underline{M})=(6s+8)\left(\frac{48}{s}+8\right)-64$$
$$=16\left[(3s+4)\left(\frac{6}{s}+1\right)-4\right]$$
$$=16\left[18+3s+\frac{24}{s}\right]$$
$$=\frac{16}{s}\left(3s^2+18s+24\right)$$
$$=\frac{48}{s}\left(s^2+6s+8\right)$$
$$=\frac{48}{s}(s+4)(s+2)$$

$$\begin{bmatrix} I_x(s) \\ I_y(s) \end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix} 8\left(\left(\frac{6}{s}+1\right)\left(V_s(s)+6I_0\right)-\frac{V_0}{s}\right) \\ 8\cdot 1\cdot\left(V_s(s)+6I_0\right)-(6s+8)\frac{V_0}{s} \end{bmatrix}$$

$$\frac{1}{\Delta}=\frac{s}{48(s+2)(s+4)}$$

$$I_x(s)=\frac{s}{6(s+2)(s+4)}\left[\left(\frac{6}{s}+1\right)V_s(s)+\left(\frac{6}{s}+1\right)6I_0-\frac{V_0}{s}\right]$$
$$=\frac{1}{6(s+2)(s+4)}\left[(6+s)V_s(s)+(6+s)6I_0-V_0\right]$$

$$\hookrightarrow \underbrace{i_x(t)}_{\text{complete solution}}=L^{-1}\{I_x(s)\}$$

Let $v_s(t) = u(t) \longrightarrow V_s(s) = \frac{1}{s}$

$$I_x^{\text{step, complete}}(s) = \frac{(s+6)}{6(s+2)(s+4)} \cdot \frac{1}{s} + \frac{(6+s)\,6I_o - V_o}{6(s+2)(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4} \qquad A = \frac{6}{6\cdot 2\cdot 4}$$ (Multiplied everyside with $s=0$)

$A = \frac{1}{8}$

$B = \frac{4}{6\cdot 2(-2)} + \frac{24I_o - V_o}{12}$ (Multiplied every side with $s=-2$)

$= \frac{24I_o - V_o - 2}{12}$

$C = \frac{+2}{6\cdot(-2)(-4)} + \frac{12I_o - V_o}{-12}$ (Multiplied both sides with $s=-4$)

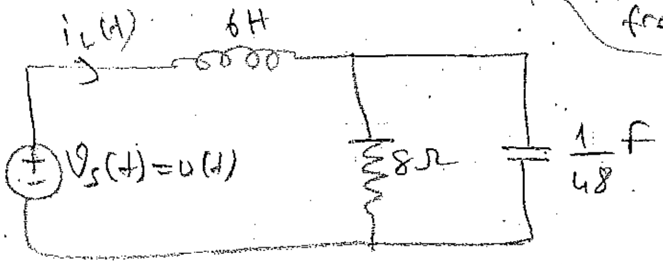
$= \frac{-24I_o + 2V_o + 1}{24}$

$i_x^{\text{complete, step}}(t) = Au(t) + Be^{-2t}u(t) + Ce^{-4t}u(t)$

Note: ① Natural frequencies of the system = $\{-2, -4\}$

↳ zero (0) isn't a natural frequency; it comes from input ($v_s(t) = u(t)$)

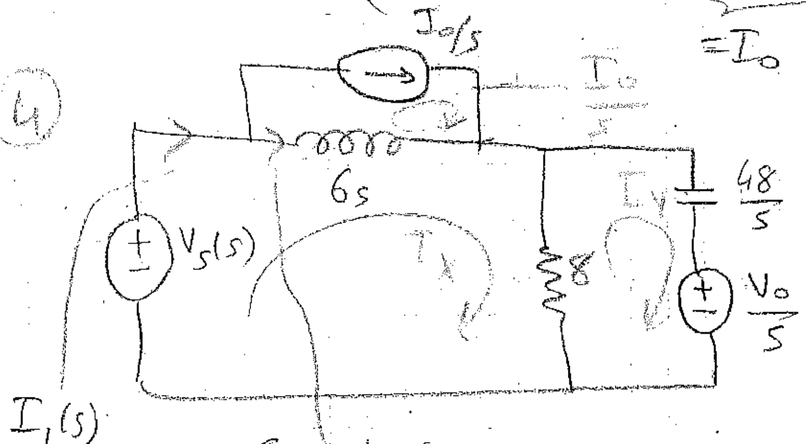
②



$i_L(\infty) = \frac{1}{8}$ (Also $i_L(\infty) = \lim_{s\to 0} sI_x(s)$) Final Value Theorem

③ $i_L(0^+) = I_o$ (Also $i_L(0^+) = \underbrace{\lim_{s\to\infty} sI_x(s)}_{= I_o}$ ✓) Initial Value Theorem

④



Current of the empty inductor that is the inductor without any initial charge.

⑤ The transfer function between $V_s(s)$ and $I_L(s)$ $\left(= I_x(s)\right)$.

Then taken initial conditions as zero and look at the relation between input and output.

$$\boxed{I_L(s)} = \frac{s+6}{6(s+2)(s+4)} V_s(s) \qquad \boxed{I_x^{\text{complete}}(s) = \frac{(s+6)\,V_s(s)}{6(s+2)(s+4)} + \frac{(6+s)\,6I_0 - V_0}{6(s+2)(s+4)}}$$

Output in s-domain

Input in s-domain

$$G(s) = \frac{I_L(s)}{V_s(s)} \overset{?}{=} \frac{s+6}{6(s+2)(s+4)} \;℧ \rightarrow \text{transfer function}$$

5.a) Find step response

$$v_s(t) = u(t) \longrightarrow V_s(s) = \frac{1}{s}$$

$$I_L^{step}(s) = G(s) \cdot V_s(s) = G(s) \cdot \frac{1}{s}$$

$$I_L^{step}(s) = \frac{s+6}{6(s+2)(s+4)s}$$

$$\boxed{i_L^{step}(t) = \frac{1}{8} - \frac{4}{24} e^{-2t} + \frac{1}{24} e^{-4t} \; A} \;,\quad t > 0$$

5.b) Find impulse response

$$v_s(t) = \delta(t) \longrightarrow V_s(s) = 1$$

$$I_L^{impulse}(s) = G(s) \cdot V_s(s) = \frac{s+6}{6(s+2)(s+4)} = \frac{1/3}{s+2} + \frac{-1/6}{s+4}$$

$$\boxed{i_L^{impulse}(t) = \left(\frac{1}{3} e^{-2t} - \frac{1}{6} e^{-4t}\right) u(t)}$$

Remember $\boxed{i_L^{impulse}(t) = \frac{d}{dt}\, i_L^{step}(t)}$ ← derivative of step response is the impulse response

or in Laplace domain

☆ $$\boxed{I_L^{\text{impulse resp.}}(s) = s\, I_L^{\text{step resp}}(s)} - \cancel{i_L^{step}(0^-)}$$

$= 0$ because step response calculation is done at zero I.C.'s or at zero-state.

☆ ← From Important Fact

5.c) Find the zero-state response to the input

$$V_s(t) = \frac{d}{dt}\delta(t) = \dot{\delta}(t) \leftarrow \text{doublet function}$$

$$I_L^{\text{Doublet response}}(s) = G(s)\,V_s(s) \leftarrow \mathcal{L}\{\dot{\delta}(t)\} = \mathcal{L}\left\{\frac{d}{dt}\delta(t)\right\}$$

$$= s\,\mathcal{L}\{\delta(t)\} - \overset{\to 0}{\cancel{\delta(0^-)}}$$

$$= s$$

$$= G(s)\cdot s$$

$$= \frac{(s+6)\cdot s}{6(s+2)(s+4)} = \frac{1}{6} + \underbrace{\frac{-8}{6(s+2)(s+4)}}$$

$$\neq \cancel{= \frac{-8/12}{s+2} + \frac{-8/-12}{s+4}}$$

They are not equal (wrong Method)

Remember to do partial fraction expansion as usual

$$\underbrace{\deg(\text{num}(s))}_{0} < \underbrace{\deg(\text{denum}(s))}_{2}$$

$$0 < 2$$

$$= \frac{1}{6} + \frac{-8/12}{s+2} + \frac{8/12}{s+4}$$

$$i_L^{\text{doublet}}(t) = \left(\frac{1}{6}\delta(t) - \frac{2}{3}e^{-2t} + \frac{2}{3}e^{-4t}\right) \quad A,\ t>0$$

5.d) Find zero-state response for $v_s(t) = 6\cos(2t)$

$$I_L(s) = G(s)\cdot\mathcal{L}\{v_s(t)\}$$

$$= \frac{s+6}{6(s+2)(s+4)}\cdot\frac{6s}{s^2+4}$$

$$= \frac{(s+6)\,s}{(s+2)(s+4)(s^2+4)} = \frac{\overset{1/2}{4\cdot\frac{(-2)}{2\cdot 8}}}{\underset{(s^2+4)(s+4)}{s+2}} + \frac{\overset{1/5}{2\cdot\frac{(-4)}{(-2)\cdot 20}}}{\underset{(s^2+4)(s+2)}{s+4}} + \frac{As+B}{\underset{(s+4)(s+2)}{s^2+4}}$$

$$\underbrace{-\frac{1}{2} + \frac{1}{5} + A = 0}_{\text{coefficients of } s^3} \Rightarrow \boxed{A = \frac{3}{10}}$$

$$\underbrace{-\cancel{8} + \frac{\cancel{8}}{5} + \cancel{8}B = 0}_{\text{Constant terms}} \quad \boxed{B = \frac{4}{5}}$$

$$i_L^{\text{cosine, z.s.}}(t) = -\frac{1}{2}e^{-2t} + \frac{1}{5}e^{-4t} + \underbrace{\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+4}\right\}}$$

$$A\,\underset{\cos(2t)u(t)}{\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}} + \overset{\sin(2t)u(t)}{\frac{B}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}}$$

$$= \left(-\frac{1}{2}e^{-2t} + \frac{1}{5}e^{-4t} + \frac{3}{10}\cos(2t) + \frac{4}{10}\sin(2t)\right) u(t) \text{ A}.$$

⑥ Let's find the differential equation between input $\vartheta_s(t)$ and output $i_L(t)$.

output

$$\frac{I_L(s)}{V_s(s)} \quad G(s) = \frac{s+6}{6(s+2)(s+4)}$$

input

$$I_L(s)(s+2)(s+4) = \frac{(s+6)}{6} \vartheta_s(s)$$

$$-6s\cdot 0 = -6si_L(0^-)$$

$$s^2 I_L(s) + 6s\, I_L(s) + 8 I_L(s) = \frac{s+6}{6}\vartheta_s(s)$$

$$-si_L(0^-) - \frac{di_L}{dt}(0^-)$$

$$\mathcal{L}^{-1}\{\ \}$$

$$\frac{d^2}{dt^2} i_L(t) + 6\left[\frac{d}{dt} i_L(t)\right] + 8\, i_L(t) = \frac{1}{6}\left[\frac{d}{dt}\vartheta_s(t) + 6\vartheta_s(t)\right]$$

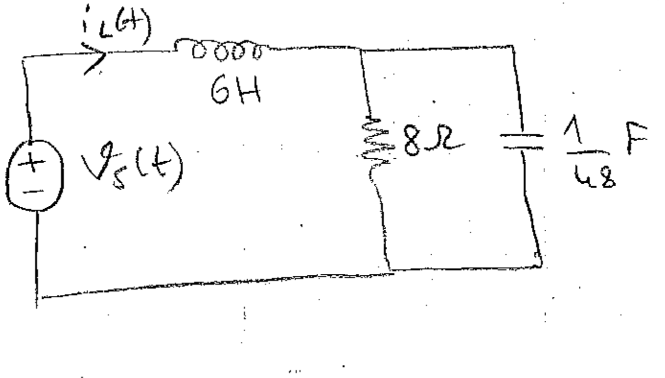
$$s^2 I_L(s) + 6s\cdot I_L(s) + 8I_L(s) = \frac{1}{6}\left(sV_s(s) + 6\vartheta_s(t)\right)$$

$\mathcal{L}^{-1}\{\ \}$ (Using the knowledge of the relation above is valid at zero-state, that is all I.C. at $t=0^-$ are zero!)

$$(D^2 + 6D + 8)\, i_L(t) = \frac{1}{6}(D+6)\,\vartheta_s(t)$$

Complete Solution via Zero-state, Zero-input Decomposition

Last lecture, we have studied the zero-state solution for LTI circuits in s-domain. We have studied transfer functions and established connection between transfer functions and zero-state responses such as step response, impulse response etc.



$\vartheta_c(0^-) = 0$
$I_L(0^-) = 0$ } Zero I.C.

zero-state

$$H(s) = \frac{s+6}{6(s+2)(s+4)}$$

$$\frac{\mathcal{L}\{i_L(t)\}}{\mathcal{L}\{v_s(t)\}}$$

Then for any input $\mathcal{L}\{v_s(t)\}$

$$I_L^{zero-state}(s) = H(s)\cdot V_s(s)$$

Case ① $v_s(t) = \delta(t)$

$$I_L^{zs, impulse}(s) = H(s)\cdot 1 \quad \mathcal{L}\{\delta(t)\}$$

$$\mathcal{L}^{-1}\{\ \}$$

$$i_L^{impulse, zs}(t) = \mathcal{L}^{-1}\{H(s)\} = \left(\frac{1}{3}e^{-2t} - \frac{1}{6}e^{-4t}\right)u(t)$$

$h(t)$

special notation for impulse response

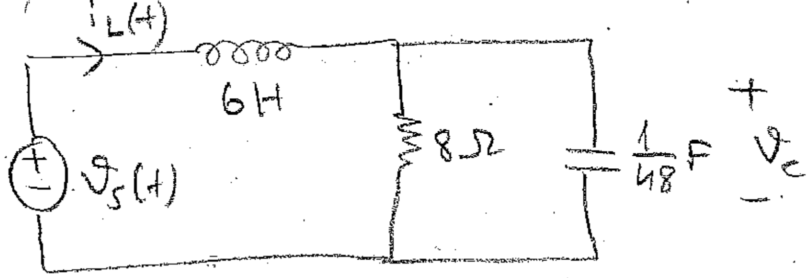
(Also discussed in EE201)

Case ②: $v_s(t) = u(t)$

$$I_L^{step, zs}(s) = H(s)\cdot\frac{1}{s}$$

$$\hookrightarrow i_L^{step}(t) = \left(\frac{1}{8} - \frac{6}{24}e^{-2t} + \frac{1}{4}e^{-4t}\right)u(t)$$

Complete Solution:

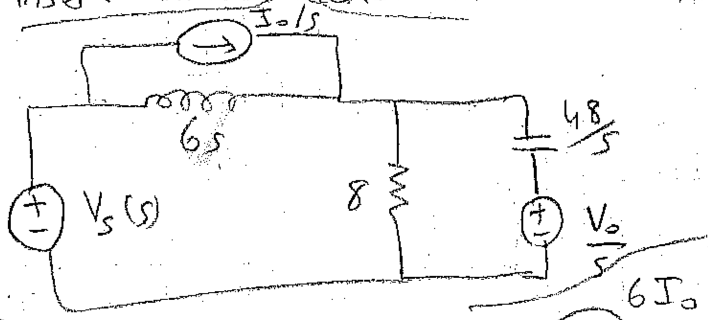


$v_c(0^-) = V_0$

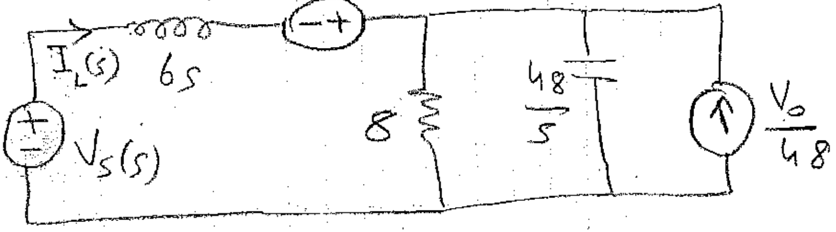
$I_L(0^-) = I_0$

Find the complete solution for $i_L(t)$ in s-domain.

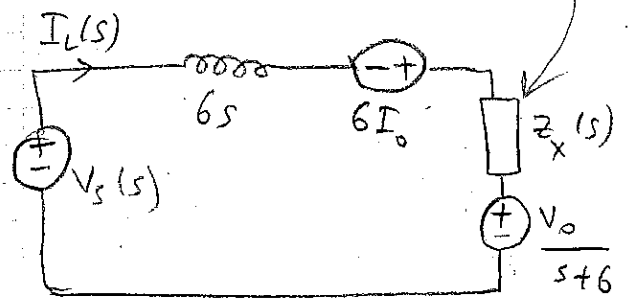
Insert initial condition modes for dynamic components.



Source Transformation



$$8 \,//\, \frac{48}{s} = \frac{\cancel{8}\cdot\frac{48}{s}}{\underset{1}{\cancel{8}}+\frac{\cancel{48}\,6}{s}} = \frac{48}{s+6} = Z_x(s)$$

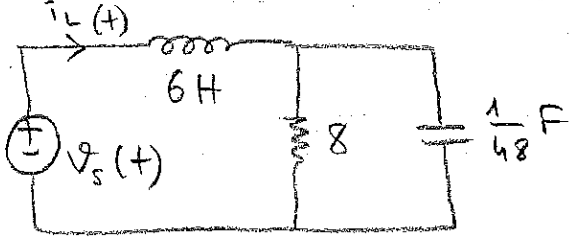


$$\frac{V_o}{s+6} = Z_x(s)\cdot\frac{V_o}{48}$$

$$H(s) = \frac{s+6}{6(s+2)(s+4)}$$

$$I_L(s) = \frac{V_s(s)+6I_o-V_o/(s+6)}{Z_x(s)+6s} = \underbrace{I_L(s)}_{\substack{\text{complete}\\\text{Solution}\\\text{in s-domain}}} = \underbrace{\boxed{\frac{1}{Z_x(s)+6s}}\,V_s(s)}_{\substack{\text{zero-state}\\\text{solution}\\\text{in}\\\text{s-domain}}} + \underbrace{\frac{6I_o-\frac{V_o}{s+6}}{Z_x(s)+6s}}_{\substack{\text{zero-input}\\\text{solution}\\\text{in}\\\text{s-domain}}}$$

Going back to zero-state responses one more time!



$v_C(0^-) = 0$

$I_L(0^-) = 0$

$$H(s) = \frac{s+6}{6(s+2)(s+4)}$$

$$\frac{I_L(s)}{V_s(s)}$$

Case ① $v_s(t) = u(t)$

$A = H(0)$

$$I_L^{step}(s) = H(s)\cdot\frac{1}{s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$i_L^{step}(t) \;\text{------}= (A + Be^{-2t} + Ce^{-4t})\,u(t)$$

Case 2 : $v_s(t) = e^{s_0 t}u(t)$, $s_0 \neq \{-2,-4\}$

$$I_L^{case ②, z.s.}(s) = H(s)\cdot\frac{1}{s-s_0}$$

$$= \frac{A}{s-s_0} + \frac{B}{s+2} + \frac{C}{s+4}$$

$\mathcal{L}^{-1}\{\}$

$$i_L^{case 2, z.s.}(t) = (A e^{s_0 t} + Be^{-2t} + Ce^{-4t})\,u(t)$$

$$\boxed{A = H(s_0)}$$

Case 3! $v_s(t) = e^{j\omega t} u(t)$

(Note: this is a special case ②, $s_o = j\omega$)

$$I_L^{\text{case ③, zs.}}(s) = H(s)\frac{1}{s-j\omega}$$

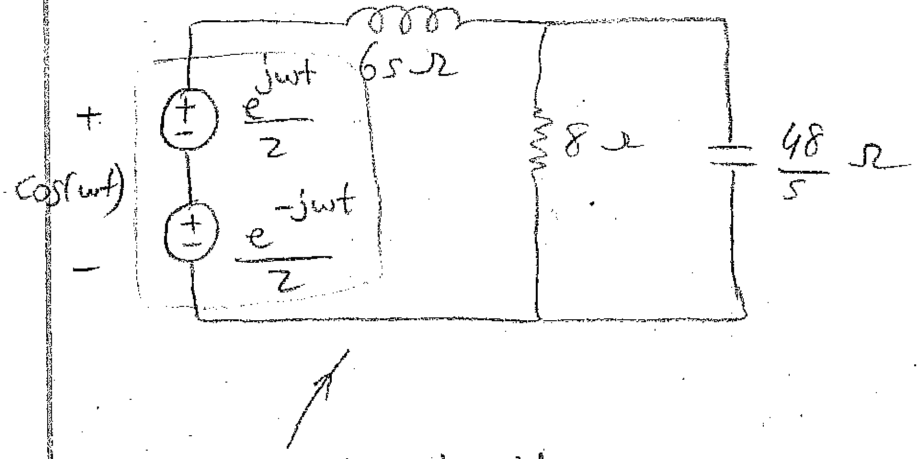
$$= \frac{A}{s-j\omega} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$i_L^{\text{case ③, zs.}}(t) = Ae^{j\omega t} + \_\_\_ e^{-2t} + \_\_\_ e^{-4t}$$

$$\hookrightarrow \boxed{A = H(j\omega)}$$

Case ④ $v_s(t) = \cos(\omega t)\, u(t)$

$$= \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) u(t)$$



Zero-state circuit

When $\frac{e^{j\omega t}}{2}$ source is active (the other one is off.):

$$\rightarrow i_{L_{Ⓐ}}^{zs.}(t) = \underline{\frac{H(j\omega)}{2}} e^{j\omega t} + \_\_ e^{-2t} + \_\_ e^{-4t}$$

when $\frac{e^{-j\omega t}}{2}$ source is active (the other one is off):

$$\rightarrow i_{L_{Ⓑ}}^{zs}(t) = \frac{H(-j\omega)}{2} e^{-j\omega t} + \_\_\_ e^{-2t} + \_\_\_ e^{-4t}$$

When both $\frac{e^{j\omega t}}{2}$ and $\frac{e^{-j\omega t}}{2}$ sources are ON at the same time;

$$i_L^{z.s.}(t) = i_{L_{Ⓐ}}(t) + i_{L_{Ⓑ}}(t)$$

$$= \frac{H(j\omega)}{2} e^{j\omega t} + \frac{H(-j\omega)}{2} e^{-j\omega t} + \_\_\_ e^{-2t} + \_\_\_ e^{-4t}$$

Remember: $H(s) = \frac{s+6}{6(s+2)(s+4)}$. please note that $\boxed{H(j\omega) = H^*(-j\omega)}$

$H(s) = \frac{num(s)}{denum(s)}$ This equality is also valid for all LTI circuits, since and coefficients of num(s) polynomial and denum(s) polynomial are real valued.

$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{v_s(t)}{RC} \longrightarrow \frac{V_c(s)}{V_s(s)} = \frac{1/RC}{s+1/RC}$$

$$i_L^{z.s.}(t) = \frac{H(j\omega)}{2} e^{j\omega t} + \left(\frac{H(j\omega)\, e^{j\omega t}}{2}\right)^* + \text{---}\, e^{-2t} + \text{---}\, e^{-4t}$$

$$= 2\,\mathrm{Re}\left\{\frac{H(j\omega)}{2} e^{j\omega t}\right\} + \text{---}\, e^{-2t} + \text{---}\, e^{-4t}$$

$$= \mathrm{Re}\left\{H(j\omega) e^{j\omega t}\right\} + \text{---}\, e^{-2t} + \text{---}\, e^{-4t}$$

$\longrightarrow |H(j\omega)|\, e^{j\measuredangle H(j\omega)}$ } ; Polar coordinate representation of $H(j\omega)$

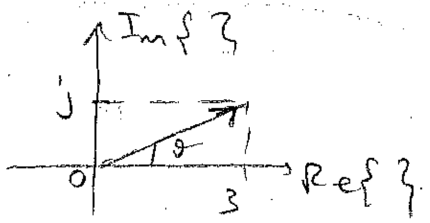
$$= |H(j\omega)| \cos(\omega t + \measuredangle H(j\omega)) + \text{---}\, e^{-2t} + \text{---}\, e^{-4t} \ A$$

$$i_L^{zs}(t) \xrightarrow[t\to\infty]{} \underbrace{|H(j\omega)| \cos(\omega t + \measuredangle H(j\omega))}_{\text{steady-state } (t\to\infty) \text{ of } i_L(t)} \ A$$

Polar coordinates:

$$z = 3+j = \sqrt{10}\, e^{j\underbrace{\tan^{-1}\left(\frac{1}{3}\right)}_{\theta}} = \sqrt{10}\left[\cos\theta + j\sin\theta\right]$$

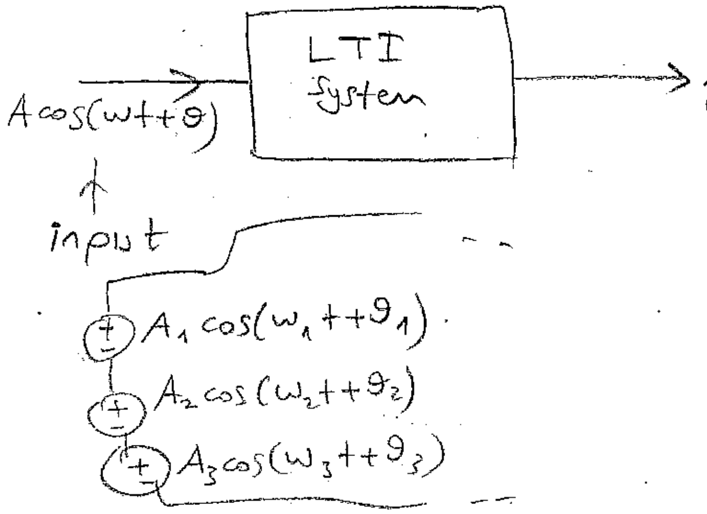
$= \sqrt{10} \angle \tan^{-1}\left(\frac{1}{3}\right)$ → Phasor domain notation



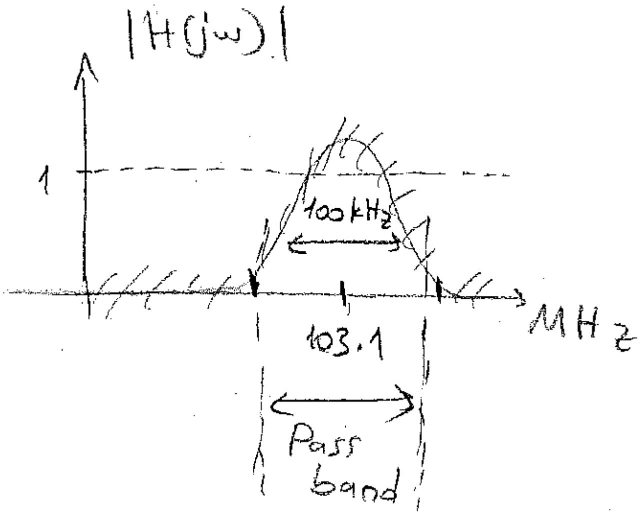
$\theta = \tan^{-1}\left(\frac{1}{3}\right)$

For $H(s)\Big|_{s=j\omega} = \frac{6+j\omega}{6(2+j\omega)(4+j\omega)}$, $H(j\omega)\Big|_{\omega=1} = \frac{6+j}{6(2+j)(4+j)}$

$$= \frac{\sqrt{37}}{6\sqrt{5}\sqrt{17}} \angle \tan^{-1}\frac{1}{6} - \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4}$$



$$\longrightarrow \underbrace{A\,|H(j\omega)| \cos(\omega t + \theta + \measuredangle H(j\omega))}_{\text{output of LTI system with impulse reponse } h(t) = \mathcal{L}^{-1}\{H(s)\}}$$



Pole-Zero Concept and H(s)

$$\prod_{k=1}^{3} k = 1\cdot 2\cdot 3 = 3!$$

Product symbol

$$H(s) = K \frac{\prod_{k=1}^{L}(s-z_k)}{\prod_{k=1}^{P}(s-p_k)}$$

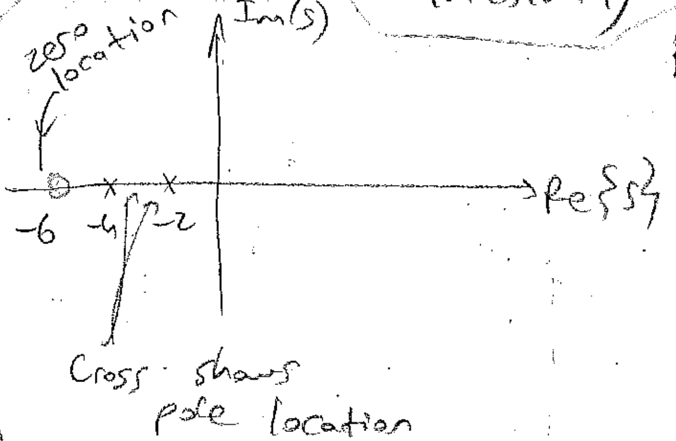
Since for LTI systems of interest,

H(s) is always ratio of two polynomials in "s" variable.

By fundamental theorem of algebra, → nom. and denum. polynomials can be factorized.

Then we may have a H(s) in the form given.

For Ex!
$$H(s) = \frac{1}{6}\,\frac{s+6}{(s+2)(s+4)}$$

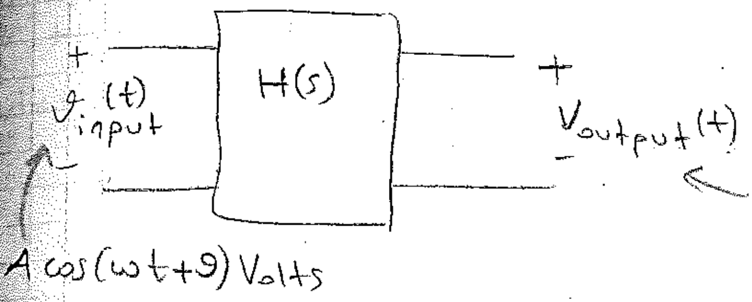


$K = \frac{1}{6}$

$z_1 = -6$ } zero

$p_1 = -2$
$p_2 = -4$ } poles

Frequency Response



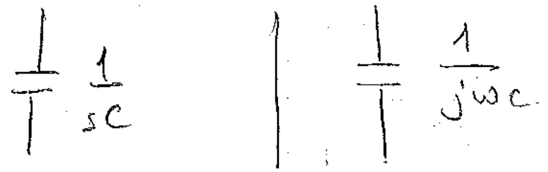
$$A|H(j\omega)|\cos(\omega t+\theta+\boxed{\measuredangle}H(j\omega))$$

= angle of

Response to cosine input at steady-state

H (s) : Transfer function

$V_{output}(s) = H(s)V_{input}(s)$



$$H(s)=\frac{s+\frac{1}{RC}}{s^2+\frac{R}{L}s+\frac{1}{LC}} \xrightarrow{} s=j2$$

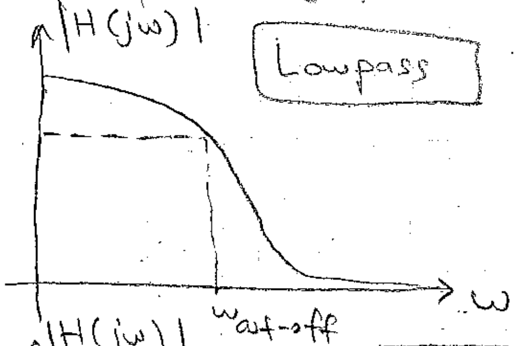
We consider:

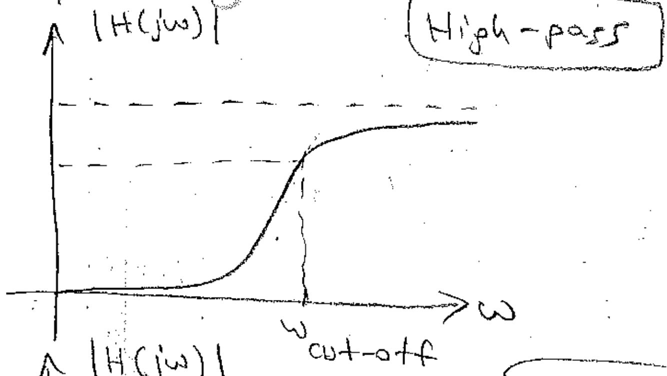
$|H(j\omega)|$: Gain

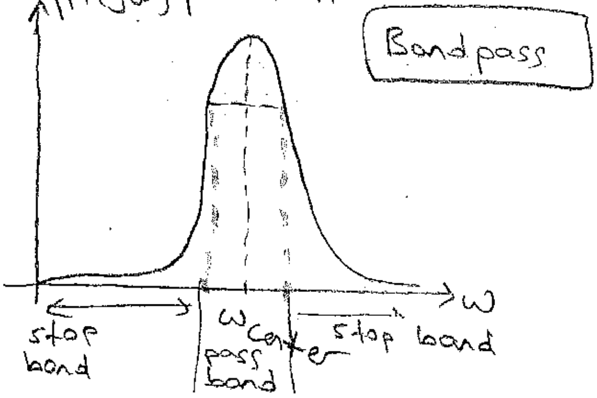
$\measuredangle H(j\omega)$: Added phase provided by the system denoted as H(s).

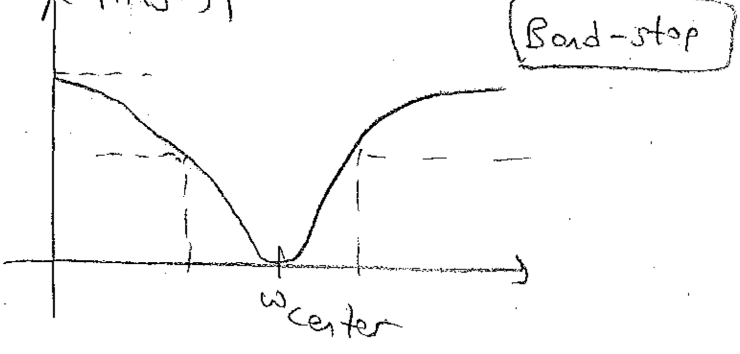
$|H(j\omega)|$: Magnitude response } of the system.
$\measuredangle H(j\omega)$: Phase response }

Lowpass, Highpass and Bandpass Systems

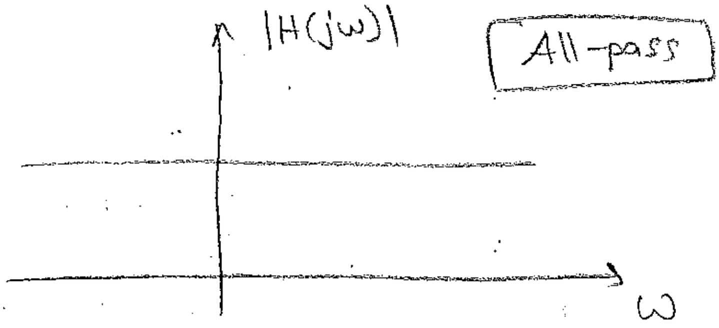






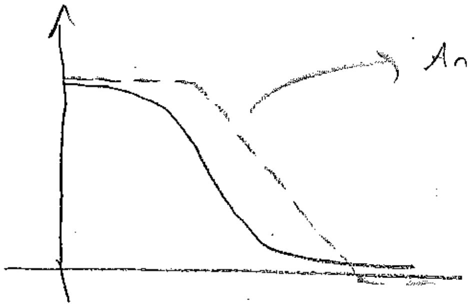


$\boxed{\text{All-pass}}$



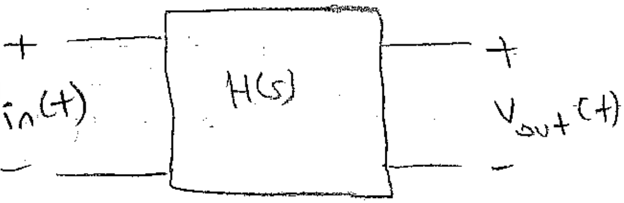
Used for phase corrections of the signal at the input.

(There is no amplification or atteneation).
→ ??

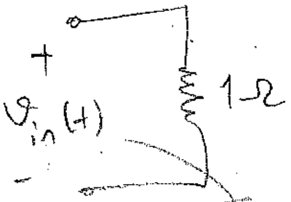


An approximate $[H(j\omega)]\,\underbrace{dB}_{\text{deciBell}}$

Decibell: Bell (Graham Bell)



Let's assume 1Ω resistance is connected to the $v_{input}(t)$ and $V_{output}(t)$;



$$P_{AV} = \frac{(V_{rms}^{Input})^2}{R \longrightarrow 1\Omega}$$

$A\cos(\omega t+\theta)$ Volts $= \dfrac{A^2}{2\boxed{R}} \leftarrow 1\Omega$

Decibell: $\boxed{10\log_{10}\dfrac{\text{Power at the output}}{\text{Power at the input}}}$ ← power ratio of two signals

$= 10\log_{10}|H(j\omega)|^2$ ← $A\cos(\omega t+\theta)$
$A|H(j\omega)|\cos(\omega t+\theta+\angle H(j\omega))$

$$\boxed{\begin{aligned}&\log(|H_1(j\omega)|\,|H_2(j\omega)|)\\&=\log(|H_1(j\omega)|)+\log(|H_2(j\omega)|)\end{aligned}}$$

$= 20\log_{10}\underbrace{|H(j\omega)|}$

$|H(j\omega)|$: Gain in amplitude for the voltage signal.

Decibell Table

$\lvert H(j\omega)\rvert^2$	$10\log_{10}\lvert H(j\omega)\rvert^2$
1	0
2	3 dB
3	4.77 dB
4	6 dB
5	7 dB
6	7.77 dB
7	8.5
8	9 dB
9	9.54 dB
10	10

$10\log_{10}(2) = ?$

$10\log_{10}(2^{10})$

$= 10\log_{10}(1024)$

$\approx 10\times 3$

$= 30$

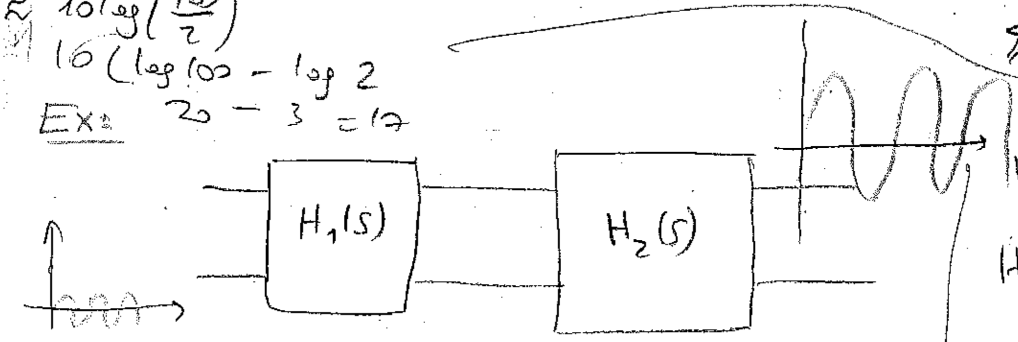
$10\log(7^2)$

$\approx 10\log\left(\frac{100}{2}\right)$

$10(\log 100 - \log 2$

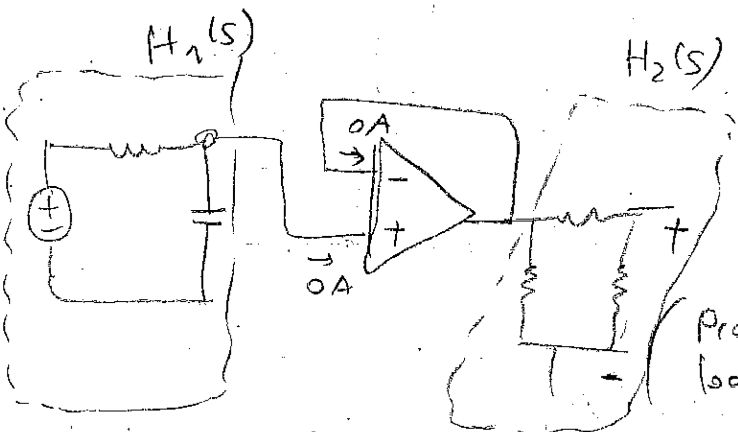
Ex: $20 - 3 = 17$

Remembering the values for 2 and 3 is helpful.



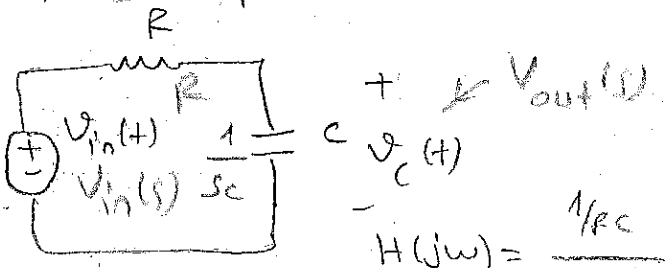
$H_1(s)$: Provides 2 dB gain

$H_2(s)$: Provides 3 dB gain



The overall system (the cascade application of $H_1(S)$ and $H_2(s)$) has a gain of 5 dB. Provided that $H_2(s)$ has no loading effect on $H_1(s)$.

Frequency Response Calculation for 1st order Circuits



$$H(s) = \frac{1/sC}{\frac{1}{sC} + R} = \frac{1/RC}{s + 1/RC}$$

$\left(\frac{s}{R}\right)$

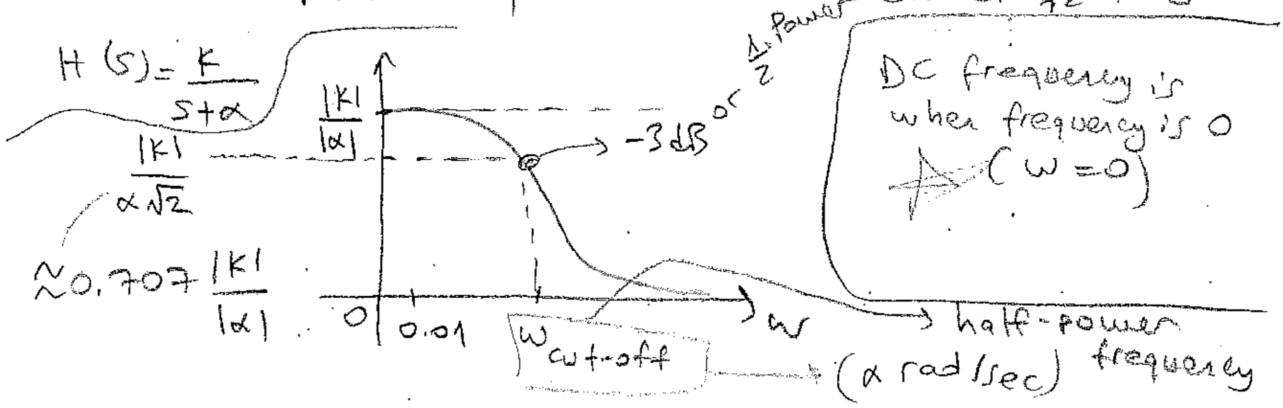
$$H(j\omega) = \frac{1/RC}{s + 1/RC}\Bigg\downarrow_{s=j\omega} = \frac{1/RC}{j\omega + \frac{1}{RC}}$$

K

$$|H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\measuredangle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{1/RC}\right) = -\tan^{-1}(\omega RC)$$

$|H(j\omega)| = \frac{|K|}{\sqrt{\omega^2+\alpha^2}}$ | $\measuredangle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$



DC frequency is when frequency is 0 (ω = 0)

Cut-off frequency is the frequency for which power gain is the half of the maximum value, also called as half-power frequency and also called as -3dB point.

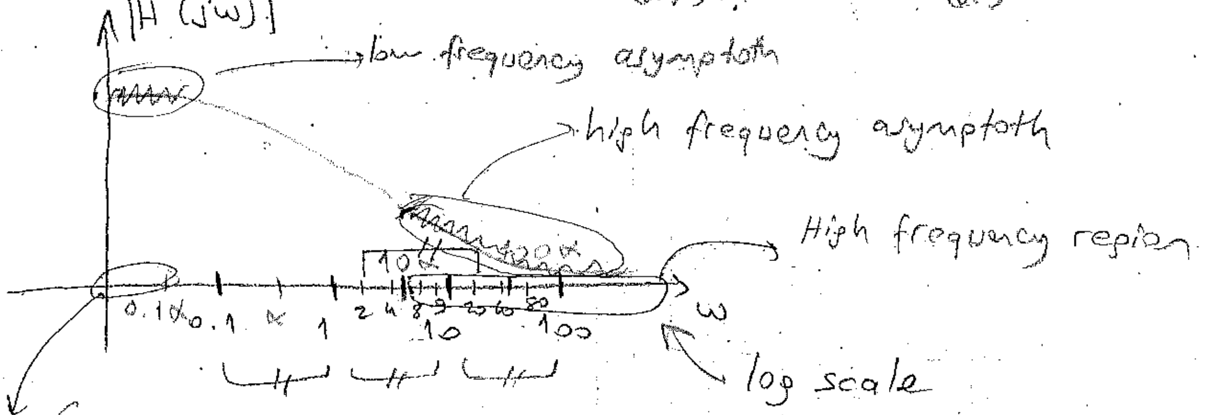
$$|H(j\omega_{cut-off})| = \frac{Max\,|H(j\omega)|}{\sqrt{2}} = \frac{|K|}{|\alpha|\sqrt{2}}$$

$\omega_{cut-off} = \alpha$ rad/sec

Asymptotic Approximation and Sketches:

$$H(s) = \frac{K}{s+\alpha} \qquad H(j\omega) = \frac{K}{j\omega+\alpha}$$

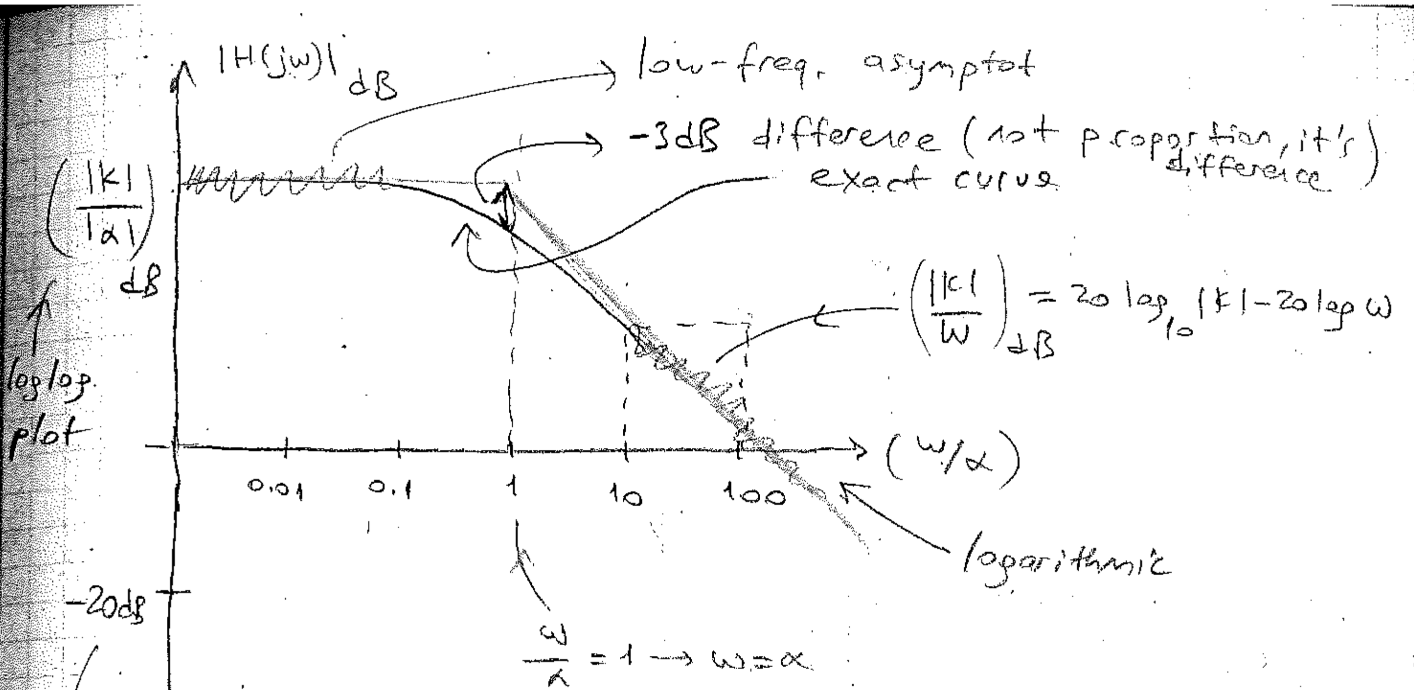
$$|H(j\omega)| = \frac{|K|}{\sqrt{\omega^2+\alpha^2}} = \frac{|K|}{\sqrt{\alpha^2\left(1+\left(\frac{\omega}{\alpha}\right)^2\right)}} = \frac{|K|}{|\alpha|\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}}$$



low frequency region

$$|H(j\omega)| \longrightarrow \omega \ll \alpha \left(1+\left(\frac{\omega}{\alpha}\right)^2 \approx 1\right) \rightarrow |H(j\omega)| \approx \frac{|K|}{|\alpha|}$$

$$\longrightarrow \omega \gg \alpha,\ \sqrt{\cancel{1}+\left(\frac{\omega}{\alpha}\right)^2} = \frac{\omega}{|\alpha|} \longrightarrow |H(j\omega)| = \frac{|K|}{\omega}$$

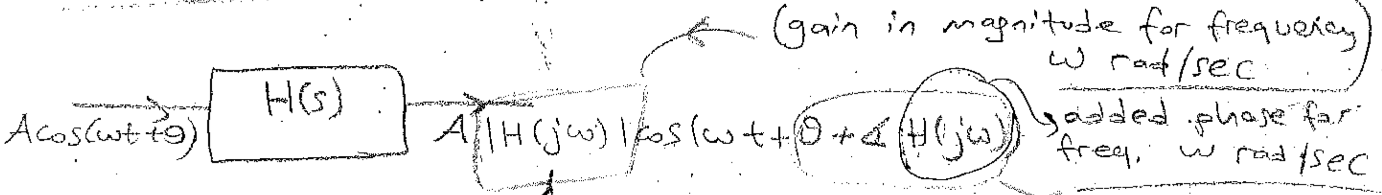


$|H(j\omega)| = \frac{1}{10}$

Note: We see straight lines as approximations to $|H(j\omega)|$ in log-log plot.

Let's find intersection point high/low freq. asymptots $\rightarrow \frac{|K|}{|\alpha|} = \frac{|K|}{\omega} \rightarrow \omega = \alpha$

Frequency Response (cont'd)



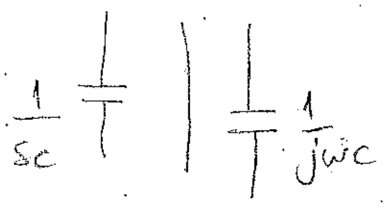
$A|H(j\omega)|\cos(\omega t + \theta + \measuredangle H(j\omega))$

(gain in magnitude for frequency ω rad/sec)

added phase for freq. ω rad/sec

output at steady state

phase of output



Low pass

$H(s) = \frac{100}{s+20}$ $\quad \underbrace{|H(j\omega)| = ?}_{\text{magnitude response}} \quad \underbrace{\measuredangle H(j\omega) = ?}_{\text{phase response}}$

$$H(s)\Big|_{s=j\omega} = \frac{100}{j\omega+20} \rightarrow |H(j\omega)| = \frac{100}{\sqrt{\omega^2+20^2}} \qquad \measuredangle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{20}\right)$$

$$H(s) = \frac{100}{20\left(1+\frac{s}{20}\right)} = \frac{5}{1+\frac{s}{20}}$$

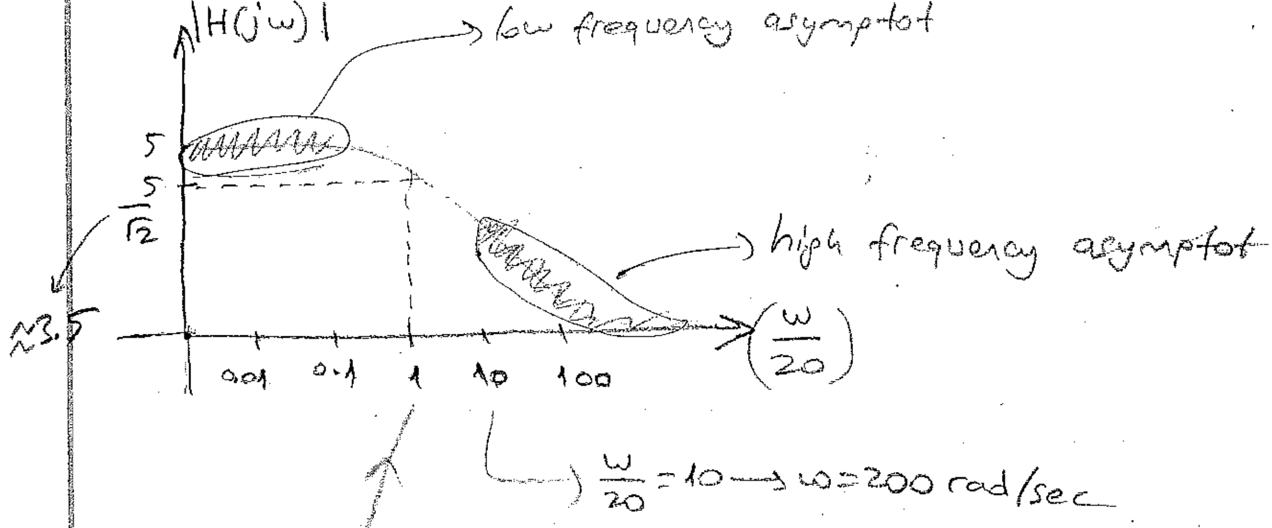
Standart form $= \frac{\boxed{K}\left(1+\frac{s}{\alpha_1}\right)\left(1+\frac{s}{\alpha_2}\right)\cdots}{\left(1+\frac{s}{\beta_1}\right)\left(1+\frac{s}{\beta_2}\right)\cdots}$

$K = 5$

$\beta_1 = 20$

$$H(j\omega) = \frac{5}{1+\frac{j\omega}{20}} \rightarrow \omega \ll 20 \rightarrow H(j\omega) \approx \frac{5}{1+0} = 5$$

$$\rightarrow \omega \gg 20 \rightarrow \frac{\omega}{20} \gg 1 \rightarrow H(j\omega) \approx \frac{5}{j\omega/20} = \frac{100}{j\omega}$$



$w_c = 20$ rad/sec is the frequency for which

$$|H(jw)| = \frac{\max(|H(jw)|)}{\sqrt{2}}$$

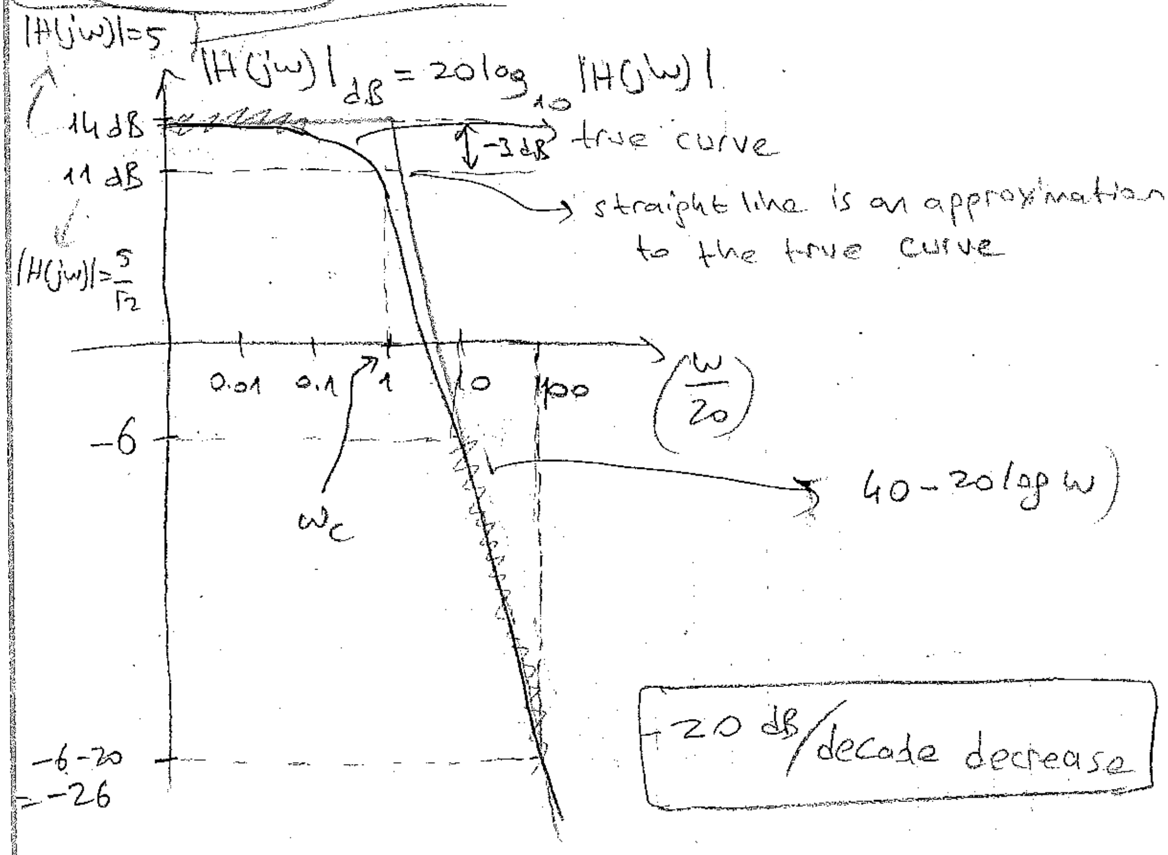
$\boxed{\text{Cut-off freq.}}$

||

$\boxed{-3\text{dB point}}$

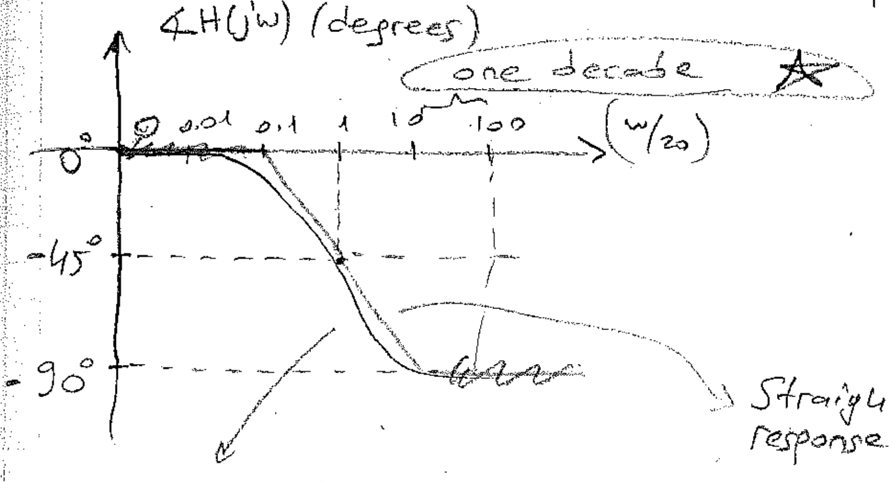
||

$\boxed{\text{Half power frequency}}$



Phase Response

$\measuredangle H(j\omega) = -\tan(\omega/20)$ $\qquad H(s) = \dfrac{5}{1 + \frac{j\omega}{20}}$



Straight line approximation to phase response

-45°/decade

High-pass

$H(s) = K \dfrac{s}{s+\alpha}$ $\qquad |H(j\omega)| = ?$ $\qquad \measuredangle H(j\omega) = ?$

$$H(s) = \frac{K}{\alpha} \cdot \frac{s}{\left(1+\frac{s}{\alpha}\right)} \longrightarrow H(j\omega) = \frac{K}{\alpha} \cdot \frac{j\omega}{1+\frac{j\omega}{\alpha}}$$

Low frequency

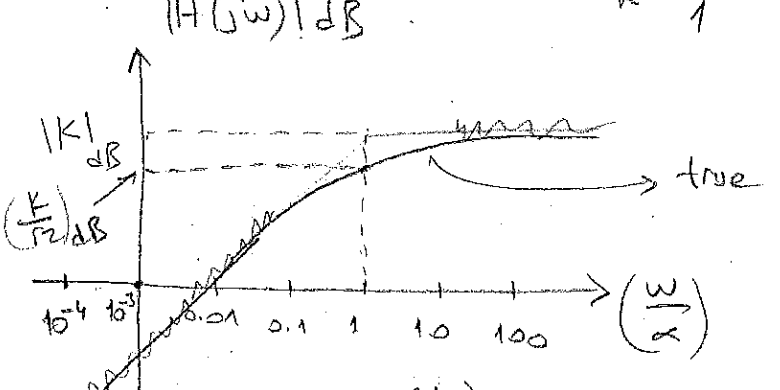
$\boxed{\omega << \alpha}$

$H(j\omega) \cong \dfrac{K}{\alpha} \cdot \dfrac{j\omega}{1}$

High frequency

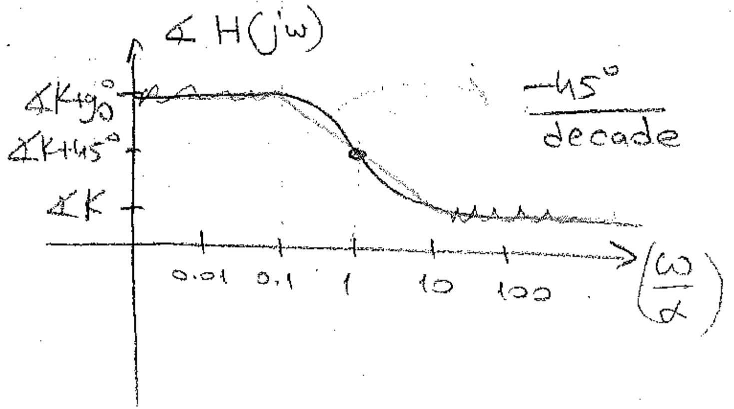
$\boxed{\frac{\omega}{\alpha} >> 1}$

$H(j\omega) \cong \cancel{\dfrac{K}{\alpha}} \cdot \dfrac{\cancel{j\omega}}{\frac{\cancel{j\omega}}{\cancel{\alpha}}} = K$

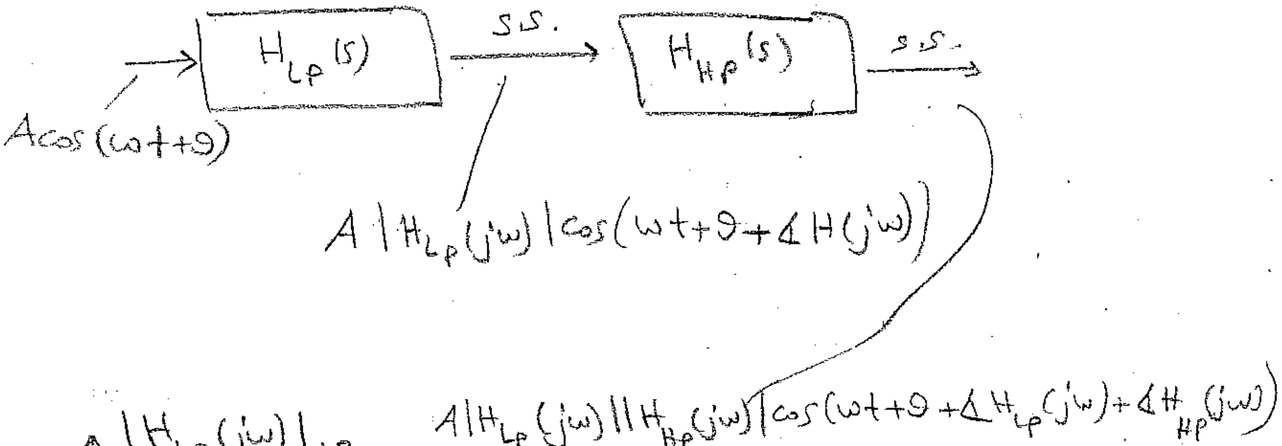


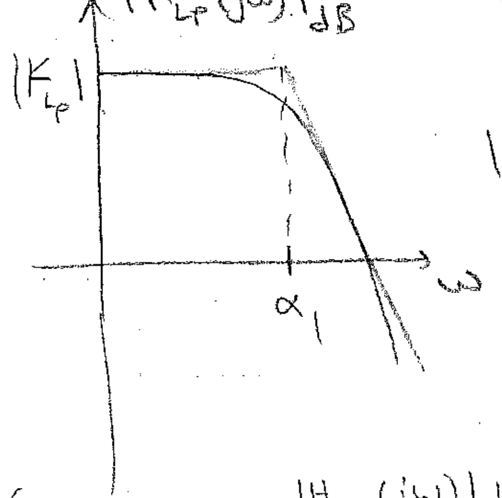
$K > 0$ (K=3) → $\measuredangle K = 0°$

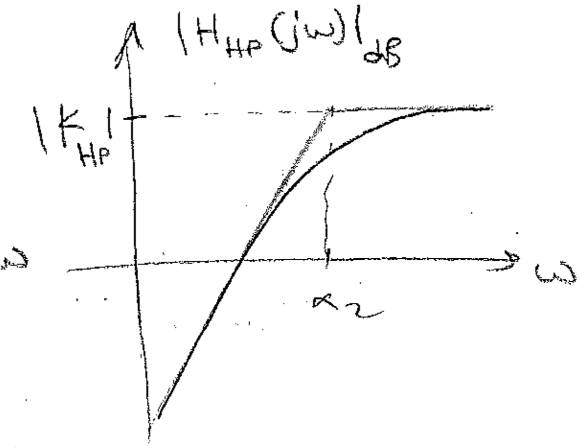
$K < 0$ (K=4) → $\measuredangle K = \mp 180°$

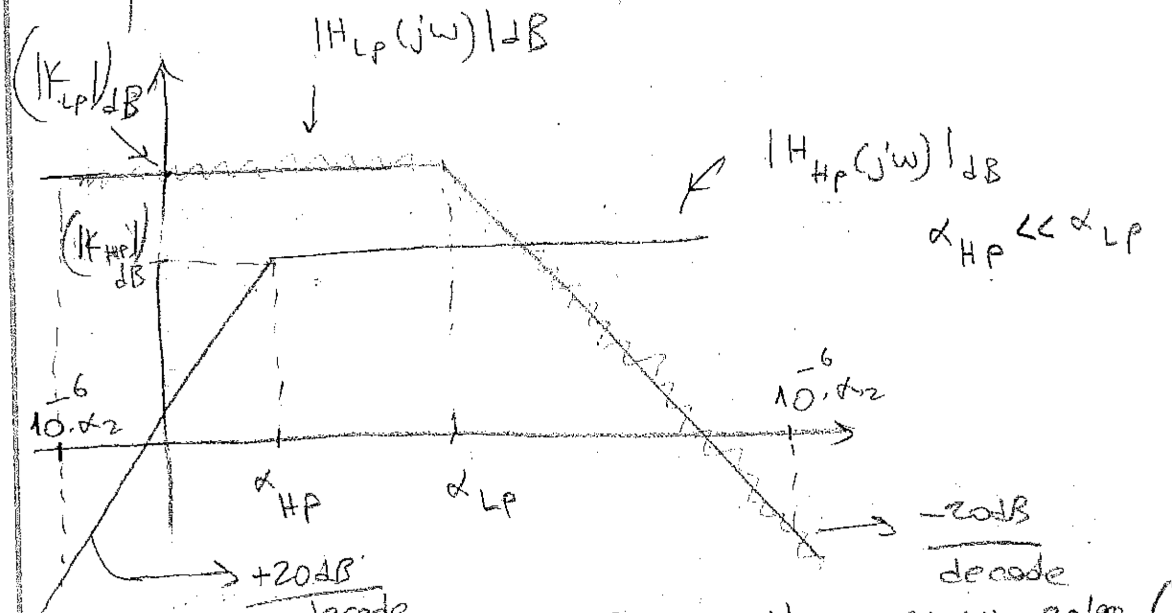


Band-pass Filter by Cascading Lowpass and High-pass Filter

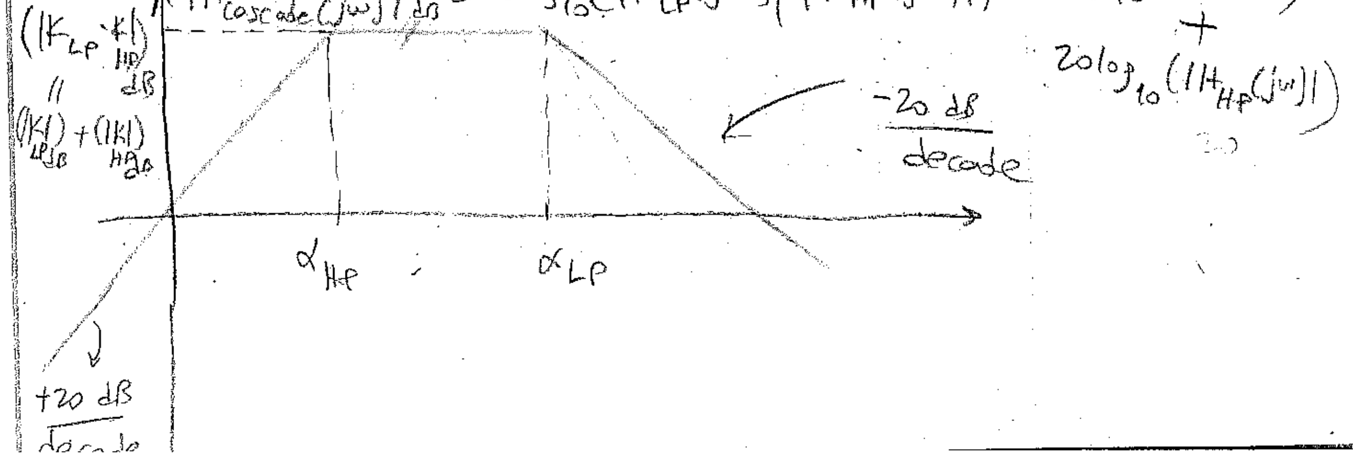




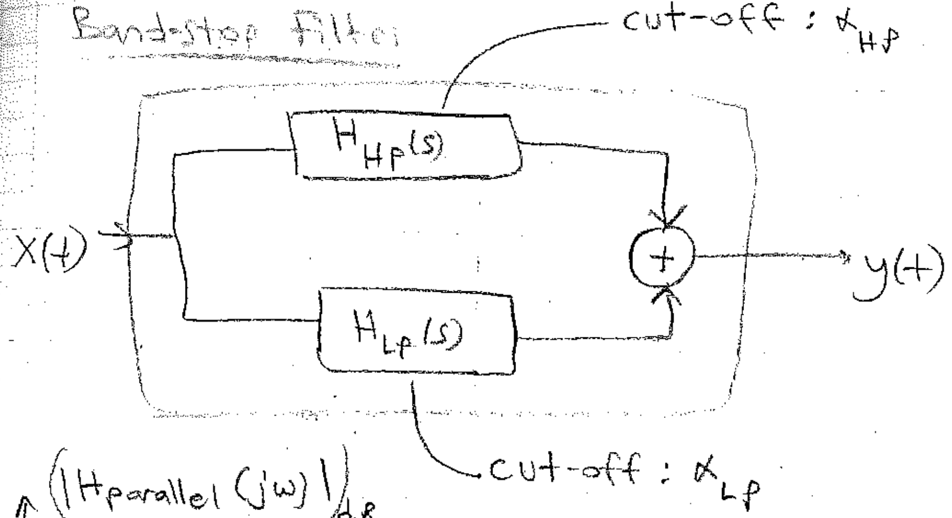


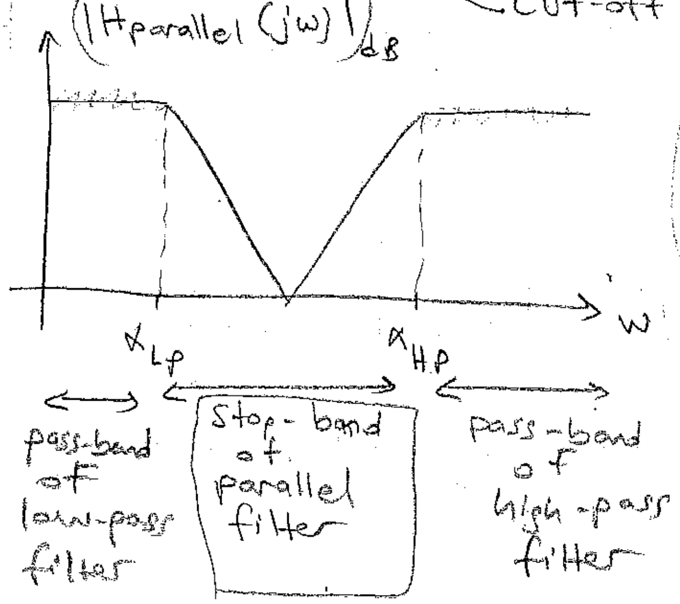


$$|H_{cascade}(j\omega)|_{dB} = 20\log_{10}(|H_{Lp}(j\omega)|\cdot|H_{Hp}(j\omega)|) = 20\log_{10}(|H_{Lp}(j\omega)|) + 20\log_{10}(|H_{Hp}(j\omega)|)$$



Band-stop filter

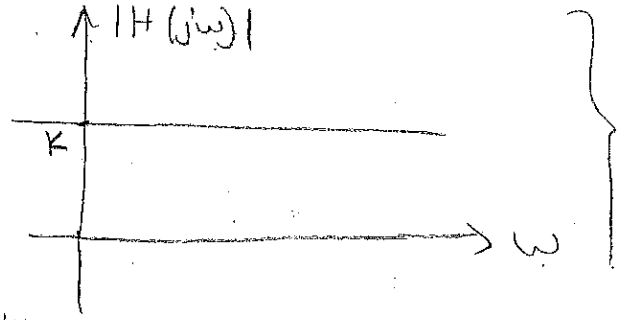




$$H_{parallel}(s) = H_{HP}(s) + H_{LP}(s)$$

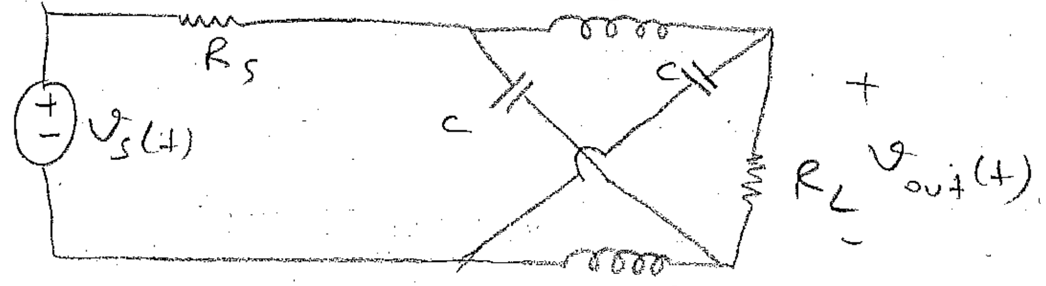
$$H_{parallel}(j\omega) = H_{HP}(j\omega) + H_{LP}(j\omega)$$

All-pass Filter



all-pass magnitude response

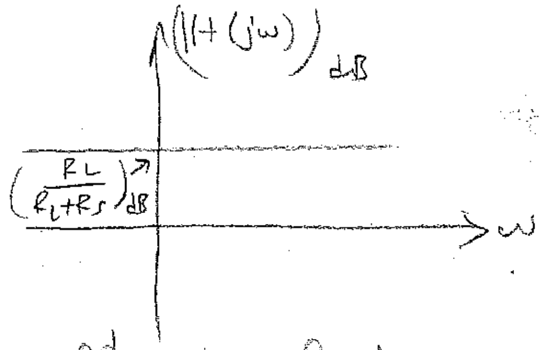
Ex

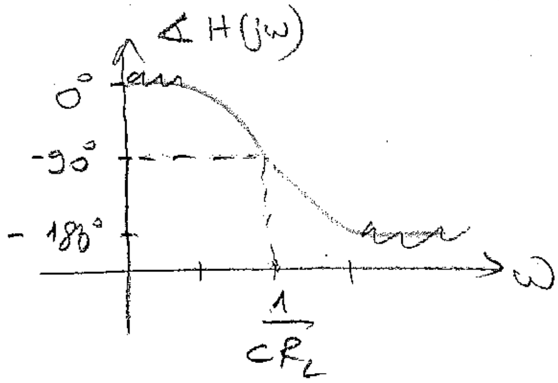


If $\frac{L}{C} = R_L^2 \rightarrow H(j\omega) = \frac{R_L}{R_L+R_S} \cdot \frac{1-j\omega C R_L}{1+j\omega C R_L}$

$$\frac{V_{out}(s)}{V_S(s)} \Big\downarrow_{s=j\omega}$$

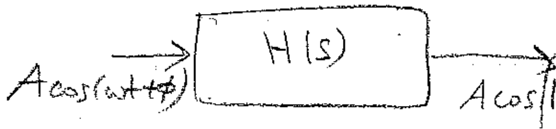
$$|H(j\omega)| = \frac{R_L}{R_L+R_S} \qquad \measuredangle H(j\omega) = -2\tan^{-1}(\omega C R_L)$$





2nd Order Band-pass System! → most important system we'll study.

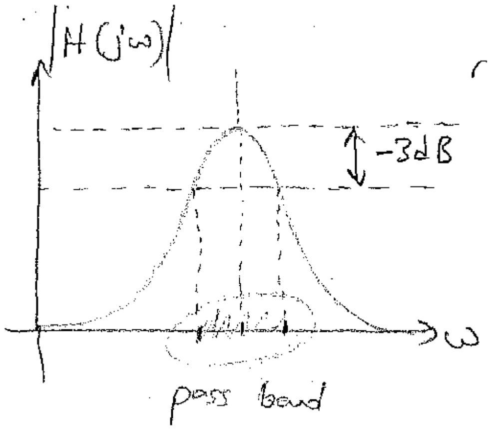
$$H(s) = \frac{ks}{s^2+2\gamma\omega_0 s+\omega_0^2}$$



$A\cos\boxed{|H(j\omega)|}\cos(\omega t+\phi+\boxed{\measuredangle H(j\omega)})$

↳ Gain: Magnitude response w.r.t. ω.

↳ phase response



$$H(s) = \frac{ks}{s^2+2\boxed{\gamma}\omega_0 s+\boxed{\omega_0}^2}$$

↳ γ: damping factor

↳ ω_0: resonance frequency

Poles/Zeros of H(s)

Zeros! $s=0 \rightarrow H(s)\big|_{s=0} = 0$

Poles! $s^2+2\gamma\omega_0 s+\omega_0^2 = 0$

$$s_{1,2} = \frac{-2\gamma\omega_0 \mp \sqrt{4\gamma^2\omega_0^2-4\omega_0^2}}{2}$$

$$\boxed{s_{1,2} = -\gamma\omega_0 \mp \omega_0\sqrt{\gamma^2-1}}$$

↳ pole locations!,

γ, in general, considered as : γ > 0.

observations!

① $\gamma = 1 \rightarrow s_{1,2} = -\overset{1}{\gamma}\omega_0 = -\omega_0$ → critically damped

$\left(s^2 + 2\gamma\omega_0 s + \omega_0^2 = 0\right) \downarrow \gamma = 1$

$\downarrow$

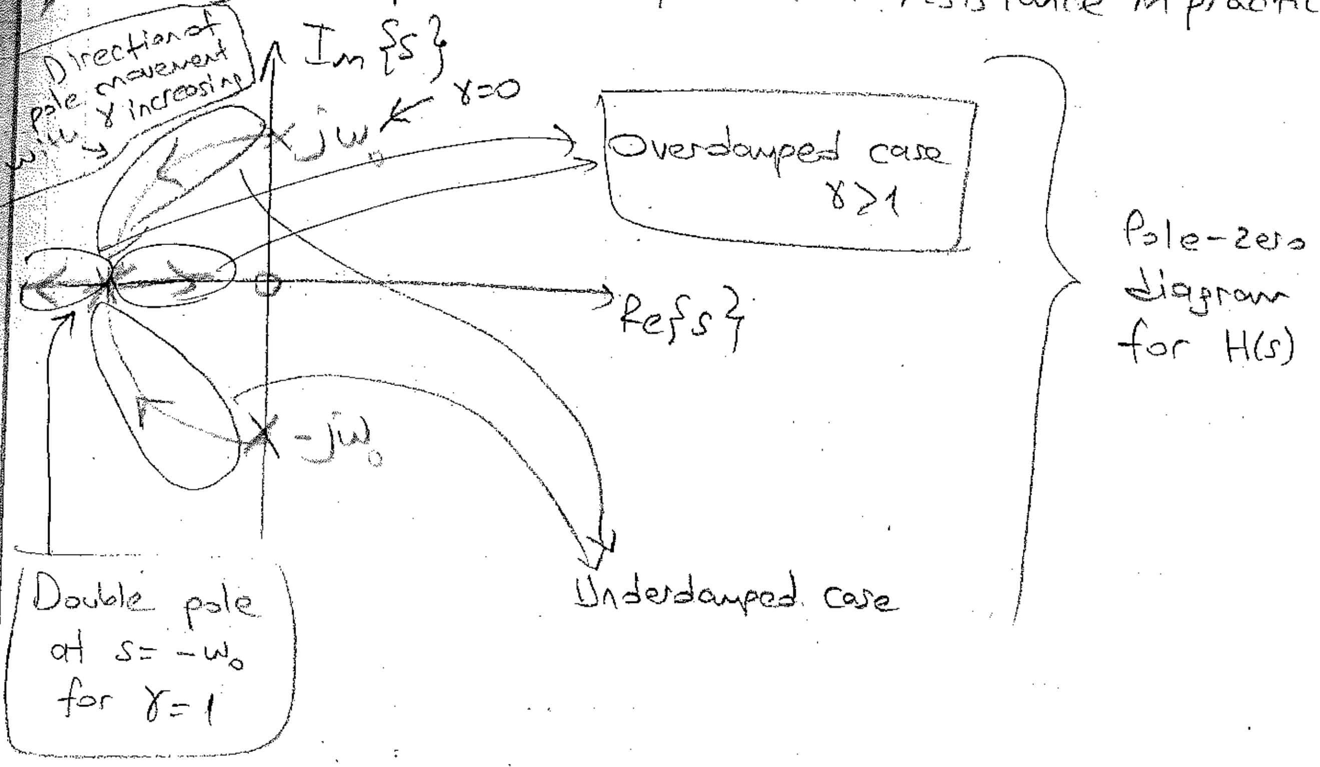
$\left(s^2 + 2\omega_0 s + \omega_0^2 = 0\right)$

↓ Double pole at $s = -\omega_0$

② $\gamma > 1 \rightarrow s_{1,2}$ are real valued and distinct $(s_1 \neq s_2)$ } overdamped

③ $\gamma < 1 \rightarrow s_{1,2}$ are complex valued and distinct } underdamped

☆ Damping corresponses to friction or resistance in practice.

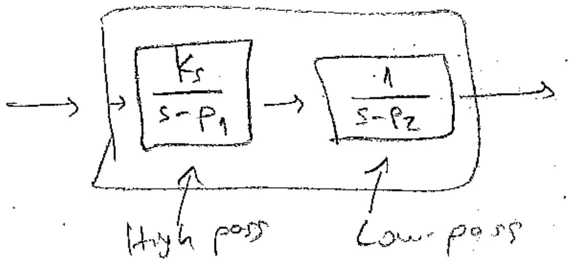


Note! The case of overdamped band-pass is not of much interest since that case can be written as:

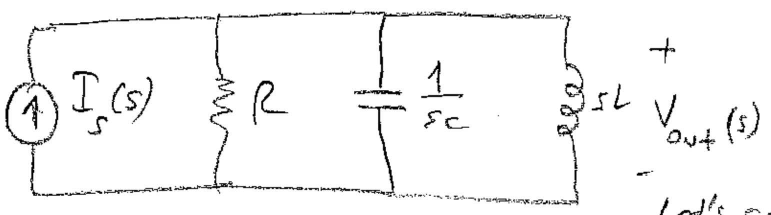
$$H(s) = \frac{ks}{(s-p_1)} \cdot \frac{1}{(s-p_2)}$$

→ This is simply the cascade of high-pass 1st order filter and low-pass 1st order filter.

↑ p_1, p_2 : real numbers



Parallel RLC:



$$H(s) = \frac{V_{out}(s)}{I_s(s)} = \frac{s/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Let's compare with

$$\frac{Ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$$H(s) = \frac{Ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2} \qquad K = \frac{1}{C} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \gamma = \frac{1}{2}\sqrt{\frac{L}{R}}$$

$$H(j\omega) = \frac{Kj\omega}{-\omega^2 + j2\gamma\omega_0\omega + \omega_0^2} \cdot \frac{\frac{1}{j\omega}}{\frac{1}{j\omega}}$$

$$= \frac{K}{2\gamma\omega_0 + \frac{\omega_0^2 - \omega^2}{j\omega}} = \frac{K}{\omega_0\left(2\gamma - j\left[\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right]\right)}$$

$$= \frac{K}{\omega_0\left(2\gamma + j\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]\right)}$$

$$|H(j\omega)| = \frac{|K|}{\omega_0\sqrt{4\gamma^2 + \left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]^2}}$$

no need for absolute value

magnitude response

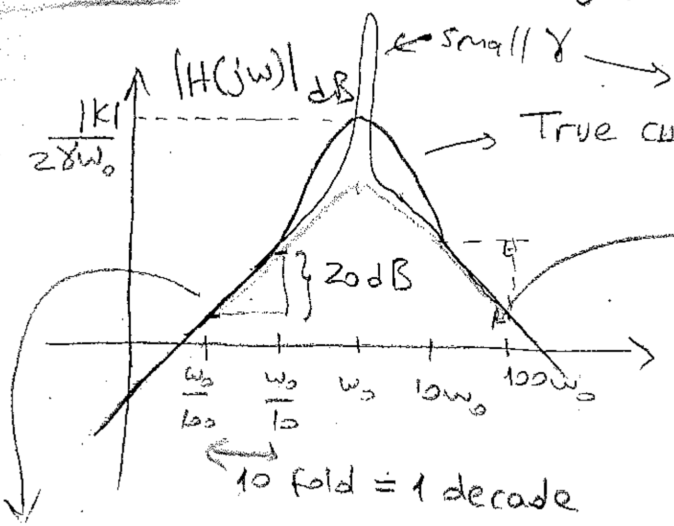
Case ①: $\omega << \omega_0$ (Low frequency asymptote) $\longrightarrow |H(j\omega)| \approx \frac{|K|}{\omega_0 \frac{\omega_0}{\omega}} \approx \frac{|K|}{\omega_0^2} \cdot \omega$

$$|H(j\omega)| = \frac{|K|}{\omega_0\sqrt{\cancel{4\gamma^2} + \left[\cancel{\frac{\omega}{\omega_0}} - \frac{\omega_0}{\omega}\right]^2}}$$

too small too small

case ② : $\omega >> \omega_0$ (High frequency asymptote) $|H(j\omega)| \cong \frac{K}{\omega_0 \frac{\omega}{\omega_0}} = \frac{K}{\omega}$

case ③ : $\omega = \omega_0 \longrightarrow |H(j\omega)| = \frac{K}{2\gamma\omega_0}$



small γ → (low ohmic loss, low damping factor)

True curve

$\frac{-20dB}{decade}$

the slope = $\frac{20dB}{decade}$

Let's find intersection point of asymptotes:

$$\underbrace{\frac{|K|}{\omega}}_{\text{High frequency}} = \underbrace{\frac{|K|}{\omega_0^2} \cdot \omega}_{\text{Low frequency}} \longrightarrow \boxed{\omega = \omega_0}$$

is the intersection point

Note! ① when $\gamma = \frac{1}{2}$ → true curve passe from the intersection point of asymptotes.

② When $\gamma << \frac{1}{2}$ → There is a high valued, much above the the intersection point of asymptotes

③ When $\gamma >> \frac{1}{2}$ → The peak location of true curve is below the intersection point of asymptotes.

Q: Does the true curve really have its peak location (maxima) at $\omega = \omega_0$?

A: $|H(j\omega)| = \frac{|K|}{\omega_0 \sqrt{4\gamma^2 + \left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]^2}}$

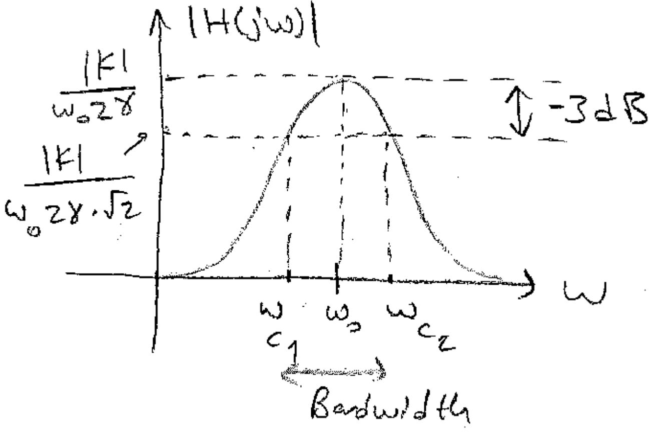
To maximize $|H(j\omega)|$
→ we need to minimize $\sqrt{\ldots\ldots}$

we need to minimize $\sqrt{\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$

Clearly $\omega = \omega_0$ minimizes

So indeed $\omega = \omega_0$ is the maxima of $|H(j\omega)|$.

Bandwidth:



Bandwidth = $\omega_{c_2} - \omega_{c_1}$

We can find ω_{c_1} and ω_{c_2} by

$$|H(j\omega_{c_1})| = \frac{|K|}{\omega_0 2\gamma} \cdot \frac{1}{\sqrt{2}}$$

$$\hookrightarrow \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 4\gamma^2$$

By solving the quadratic equation:

$$\omega_{c_2} = \omega_0 (\gamma + \sqrt{1+\gamma^2})$$

$$\omega_{c_1} = \omega_0 (-\gamma + \sqrt{1+\gamma^2})$$

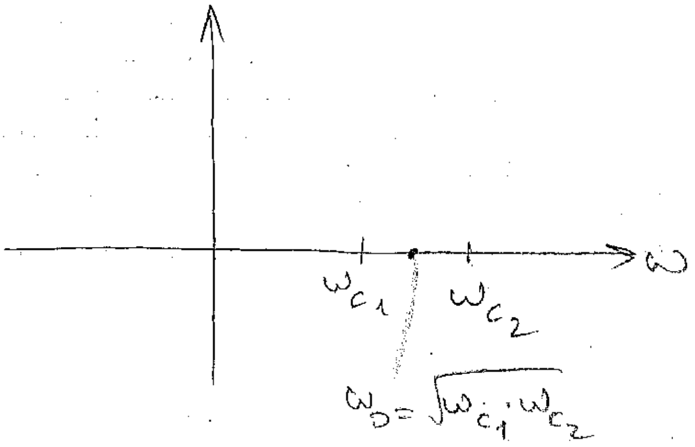
$$\boxed{B.W. = \omega_{c_2} - \omega_{c_1} = 2\gamma\omega_0}$$

Note: $H(s) = \frac{Ks}{s^2 + \boxed{2\gamma\omega_0} s + \omega_0^2}$ → resonant freq. (ω_0)

$\hookrightarrow$ B.W. $= 2\gamma\omega_0$

Note that, $\boxed{\omega_0 = \sqrt{\omega_{c_1} \cdot \omega_{c_2}}}$

Hence, ω_0 is at the geometric mean of ω_{c_1} and ω_{c_2}.



But, if ω axis is logaritmic

$$\omega_0 = \log\left((\omega_{c_1} \cdot \omega_{c_2})^{\frac{1}{2}}\right)$$

$$= \frac{\log(\omega_{c_1}) + \log(\omega_{c_2})}{2}$$

Quality Factor: $Q = \frac{\omega_0}{B.W.} \longrightarrow Q = \frac{1}{2\gamma}$ $\boxed{BW = 2\gamma\omega_0}$

Quality factor

So, a high Q filter has a high peak at $w = w_0$.

Quality factor indicates the bandwidth of the filter w.r.t. its expected operating frequency w_0.

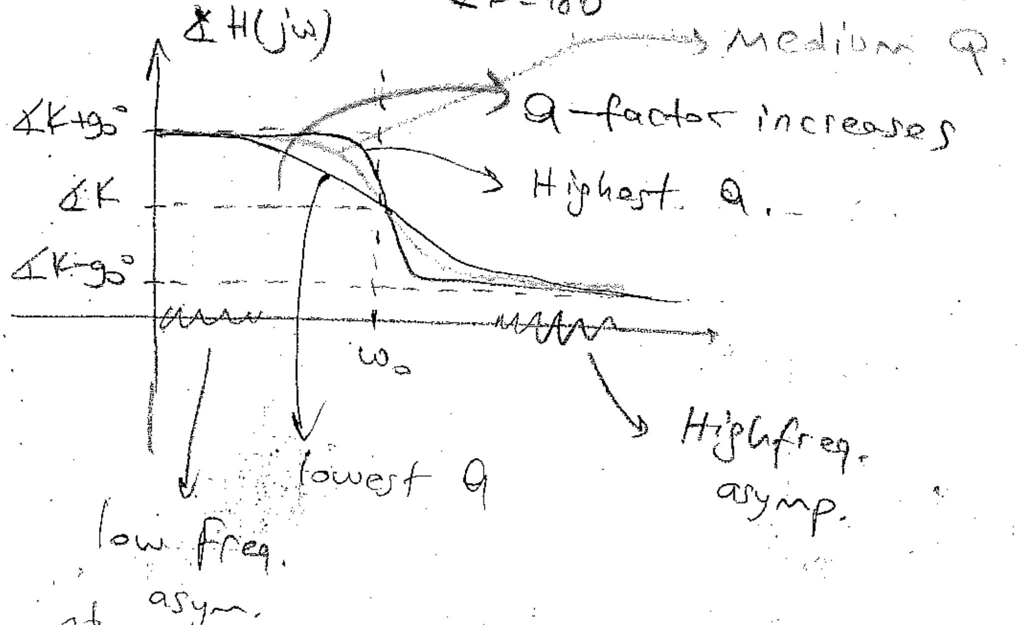
$$\boxed{Q = \frac{1}{2\gamma}} \quad \boxed{\gamma = \frac{1}{2Q}}$$

Phase Response: $H(j\omega) = \frac{K}{\omega_0\left(2\gamma + j\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]\right)}$

$$\measuredangle H(j\omega) = \measuredangle K - \tan^{-1}\left(\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \cdot \boxed{\frac{1}{2\gamma}}\right) \longrightarrow = Q$$

$K > 0 \rightarrow \measuredangle K = 0°$

$K < 0 \rightarrow \measuredangle K = 180°$



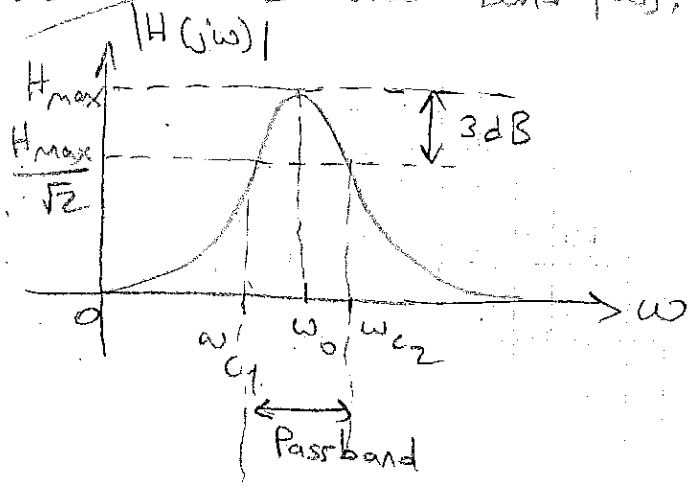
2nd Order Systems

23.05.2016

Last week: 2nd order Band-pass: $\frac{Ks}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$

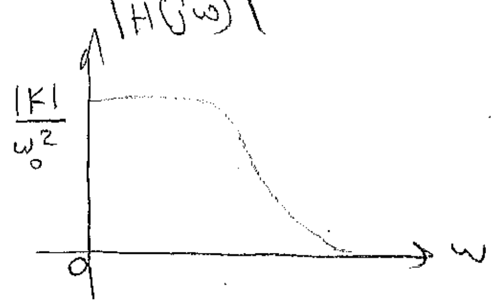
ω_0: resonance frequency

γ: Damping factor



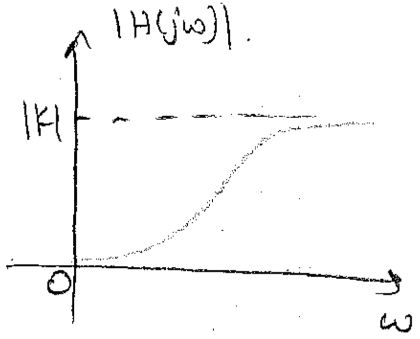
2^{nd} Order Low-Pass!

Today: $H(s) = \dfrac{K}{s^2+2\gamma\omega_0 s+\omega_0^2}$



Later: 2^{nd} Order Highpass

$$H(s) = \frac{Ks^2}{s^2+2\gamma\omega_0 s+\omega_0^2}$$



→ 2^{nd} Order Low-pass

$$H(s)\Big\downarrow_{s=j\omega} = \frac{K}{-\omega^2+\omega_0^2+j2\gamma\omega_0\omega} = \frac{K}{\omega_0^2\left[1-\left(\frac{\omega}{\omega_0}\right)^2+j2\gamma\left(\frac{\omega}{\omega_0}\right)\right]}$$

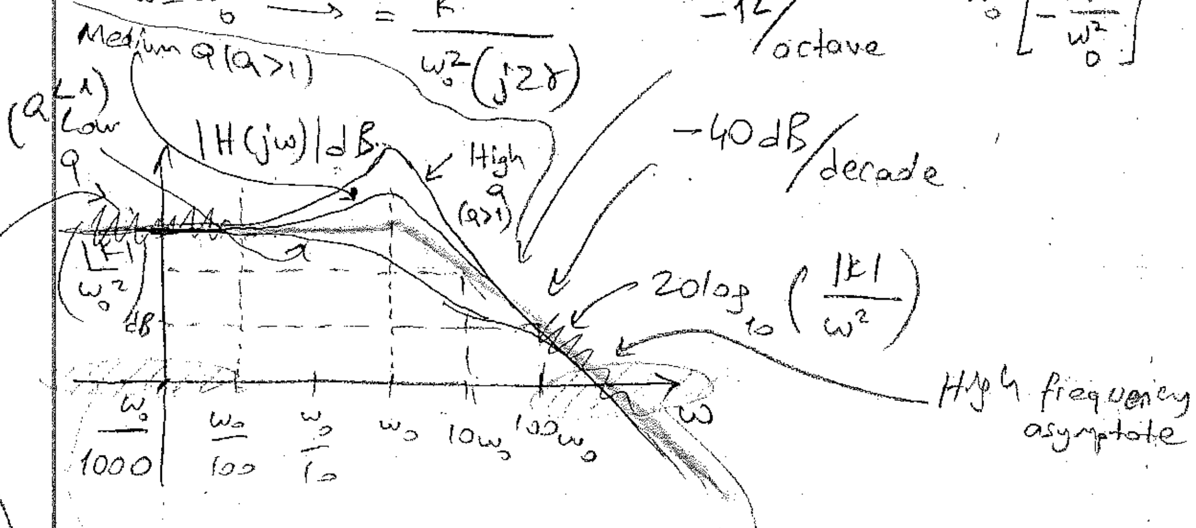
$\omega \ll \omega_0 \quad \left(\frac{\omega}{\omega_0} \ll 1\right)$

$$\approx \frac{K}{\omega_0^2}$$

$\omega \gg \omega_0$

$$\approx \frac{K}{\omega_0^2\left[-\frac{\omega^2}{\omega_0^2}\right]} = \frac{-K}{\omega^2}$$

$\omega = \omega_0 \longrightarrow = \dfrac{K}{\omega_0^2(j2\gamma)}$



Low frequency region

High frequency region

Low frequency asymptote

$|H(j\omega)| = 20\log_{10}\dfrac{|K|}{\omega^2}$

$|H(j2\omega)| = 20\log_{10}\dfrac{|K|}{4\omega_0^2}$

$|H(j\omega)| - 20\log_{10}4$

Let's find the intersection point of Low and High frequency asymptotes:

$$\frac{|K|}{\omega_o^2} = \frac{|K|}{\omega^2} \longrightarrow \omega = \omega_o$$

Intersection point of asymptotes

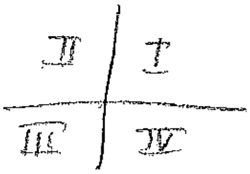
$$\boxed{Q \triangleq \frac{1}{2\gamma}}$$

Quality factor

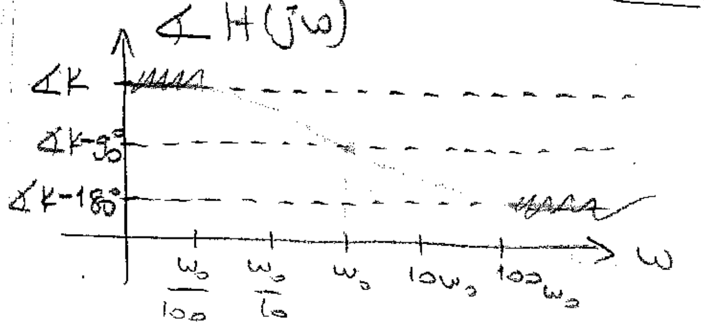
$1+j' \rightarrow \sqrt{2}\,\angle\,\tan^{-1}(1)$

$-(1+j) \leftarrow -1-j \neq \sqrt{2}\,\angle\,\tan^{-1}(-1/-1)$

$-1\cdot\sqrt{2}\,\angle\,\tan^{-1}(\frac{1}{1})$



$\tan^{-1}$ is defined only in 1st and 4th quadrands.



Note: The maxima of $|H(j\omega)|$ of 2nd order low_pass filter is at:

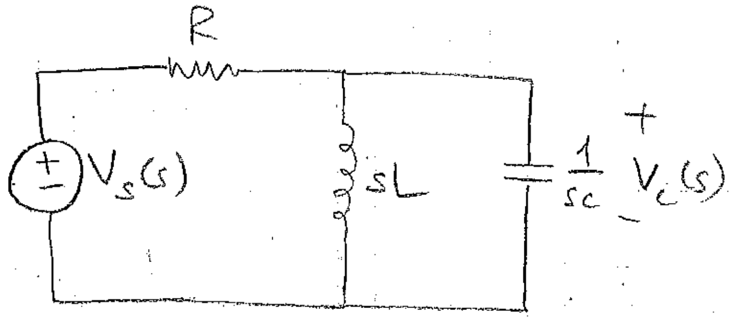
$$\boxed{\omega_m = \omega_o\sqrt{1-\frac{1}{2Q^2}}}$$

Second Order High Pass System (Page : 1.17)

$$H(s) = \frac{Ks^2}{s^2+2\gamma\omega_o s+\omega_o^2} \qquad H(j\omega) = \frac{K(-\omega^2)}{\omega_o^2-\omega^2+j2\gamma\omega_o\omega}$$

$$= -K\frac{(\omega/\omega_o)^2}{1-\left(\frac{\omega}{\omega_o}\right)^2+j\gamma}$$

Ex:



$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{s/RC}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

$$H_2(s) = \frac{V_R(s)}{V_s(s)} \qquad -V_s(s) + V_R(s) + V_c(s) = 0$$

$$H_2(s) = \frac{V_R(s)}{V_s(s)} = 1 - \underbrace{\frac{V_c(s)}{V_s(s)}}_{H(s)}$$

$\rightarrow H_2\left(j\frac{1}{\sqrt{LC}}\right) = 0$; It is a band-stop filter.

$$H(s) = \frac{s/RC}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} = K\frac{s}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$L = \frac{1}{4} H$
$R = 1\ k\Omega$
$C = 1\ \mu F$

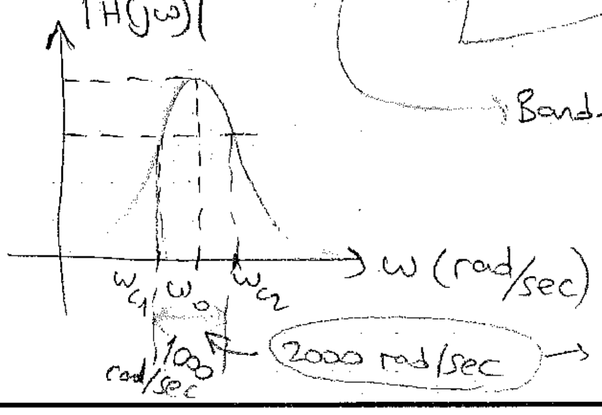
$$= \frac{s1000}{s^2 + \boxed{1000}s + (2000)^2}$$

$\rightarrow \omega_0 = 2000$

$\gamma = \frac{1}{4} \rightarrow \left(Q = \frac{1}{2\gamma} = 2\right)$

$\boxed{2\gamma\omega_0}$

Band-width = 1000 rad/sec



2000 rad/sec $\rightarrow = 2\pi f_0 \rightarrow f_0 = \frac{1000}{\pi}$ Hz

$$\omega_{c_1} = \omega_o \overbrace{\left(-\gamma + \sqrt{1+\gamma^2}\right)}^{\approx 0.10}$$

$$\omega_{c_2} = \omega_o \underbrace{\left(+\gamma + \sqrt{1+\gamma^2}\right)}_{\approx 1.128}$$

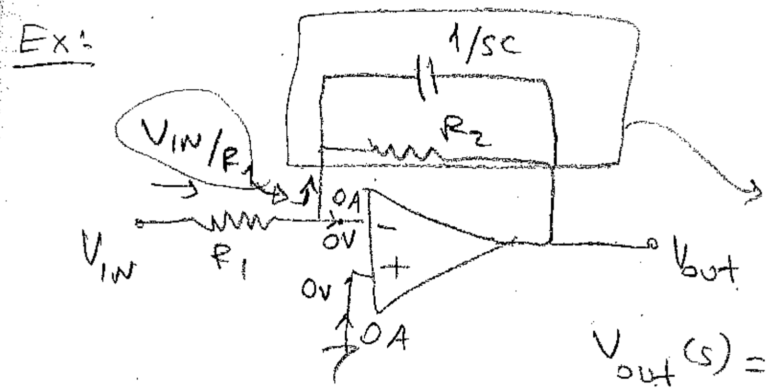
$$\boxed{\omega_0 = \sqrt{\omega_{c_1} \cdot \omega_{c_2}}} \rightarrow \text{Geometric mean}$$

Active Filters:

→ Kirm 12.1 ☆ ☆

Filters constructed with R, L, C components (passive components) and op-amps (active components) are called active filters.

Ex:



Assume ideal op-amp in linear region.

$\rightarrow Z(s) = \frac{1}{sC} // R_2$

$$V_{out}(s) = -Z(s) \cdot \frac{V_{IN}(s)}{R_1}$$

$$= -\frac{1}{R_1} \frac{\frac{1}{sC} \cdot R_2}{\frac{1}{sC} + R_2} V_{IN}(s)$$

$$= \boxed{\frac{-R_2}{R_1}} \frac{1}{1 + sCR_2} V_{IN}(s) = \frac{K}{s+\alpha}$$

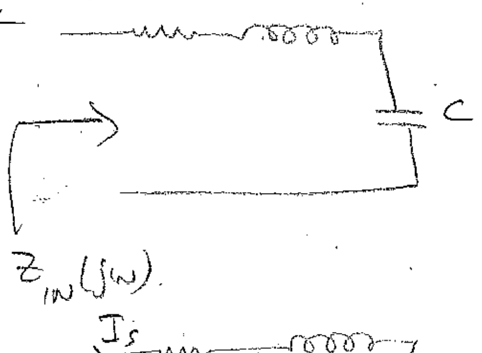
Note: ① $|K| > 1$ by selecting $R_1 \ll R_2$.

② $\angle K = 180°$.

General Definition of Resonance:

The resonance frequency is the frequency for which input impedance $Z(j\omega)$ is purely real.

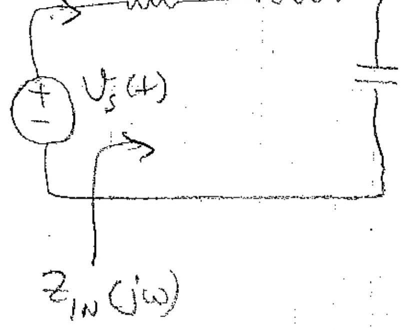
Ex:



$$Z_{IN}(j\omega) = R + j\omega L - \frac{j}{\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\Rightarrow Z_{IN}(j\omega_o) = R \text{ when } \omega_o L - \frac{1}{\omega_o C}$$

$$\rightarrow \omega_o^2 = \frac{1}{LC} \rightarrow \omega_o = \frac{1}{\sqrt{LC}}$$

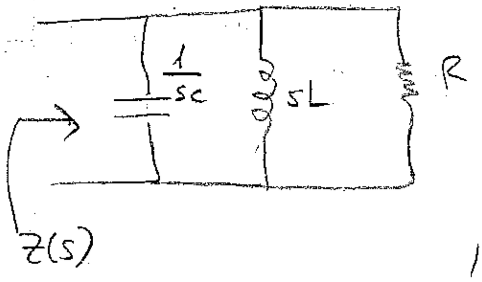


$$I_S^{RMS} = \frac{V_S^{RMS}}{|Z(j\omega)|}$$

$$P_{AVG}^{Resistor} = \left(I_R^{RMS}\right)^2 \cdot R$$

$$= \left(\frac{V_S^{RMS}}{|Z(j\omega)|}\right)^2 \cdot R$$

$P_{AVG}^{R}(\omega)$ is maximum when $\omega = \omega_o$ ← resonance frequency.



$$Z(s) = \frac{1}{\boxed{Y(s)}} \rightarrow \text{admittance}$$

$$Y(s) = \frac{1}{R} + \frac{1}{sL} + sC$$

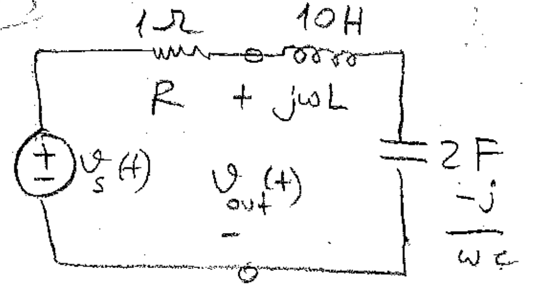
If $Z(s)$ is real $Y(s)$ must be real too.

$$Y(j\omega_0) = \frac{1}{R} + j\underbrace{\left(\omega_0 C - \frac{1}{\omega_0 L}\right)}_{\text{has to be zero at resonance}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Magnitude/Frequency Scaling

Magnitude Scaling



$$H(j\omega) = \frac{V_{out}(j\omega)}{V_s(j\omega)} = \frac{Z_L(j\omega) + Z_C(j\omega)}{R + Z_L(j\omega) + Z_C(j\omega)} = \frac{j\omega L - j/\omega c}{R + j\omega L - j/\omega c}$$

$$H(j\omega) = \frac{k_m}{k_m} H(j\omega) = \frac{j\omega \overset{L'}{\boxed{k_m L}} - j \overset{C'}{\boxed{\frac{1}{\frac{\omega c}{k_m}}}}}{\underset{R'}{\boxed{R k_m}} + j\omega \underset{L'}{\boxed{k_m L}} - j \frac{1}{\omega \underset{C'}{\boxed{\frac{C}{k_m}}}}}$$

Magnitude Scaling:

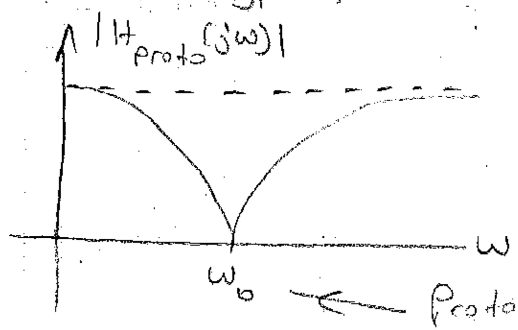
$R \rightarrow k_m R$

$L \rightarrow k_m L$

$C \rightarrow \frac{C}{k_m}$

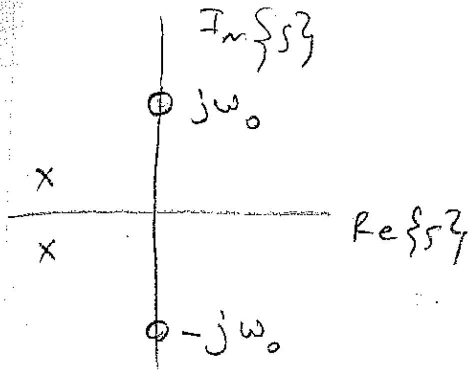
Frequency Scaling

Prototype filter

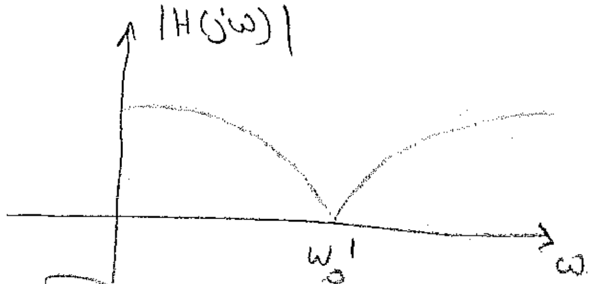


$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$\leftarrow$ Prototype filter has a zero at $s = \mp j\omega_0$



Designed filter has a different band-stop frequency ω_0'.



$$H_{desired}(j\omega) = H_{prototype}\left(j\omega \boxed{\frac{\omega_0}{\omega_0'}}\right) \xrightarrow{} = H_{prototype}\left(j\frac{\omega}{k_f}\right)$$

$$\boxed{\frac{\omega_0}{\omega_0'}} = \frac{1}{k_f}$$

$$\hookrightarrow H_{desired}(j\omega_0') = H_{prototype}\left(j\cancel{\omega_0'} \cdot \frac{\omega_0}{\cancel{\omega_0'}}\right) = 0$$

$$H_{prototype}(j\omega) = \frac{j\omega L - \frac{j}{\omega C}}{R + j\omega L - \frac{j}{\omega C}}$$

$$H_{prototype}\left(\frac{j\omega}{k_f}\right) = \frac{j\omega \boxed{\frac{L}{k_f}}^{L''} - \frac{j}{\boxed{\omega C / k_f}}\; C''}{\underbrace{\boxed{R}}_{R''} + \underbrace{\frac{j\omega L}{\boxed{k_f}}}_{L''} - j\frac{1}{\frac{\omega C}{\boxed{k_f}}}\; C''}$$

Frequency Scaling.

$k_f = \frac{\omega_0'}{\omega_0}$ ← new frequency

$R \rightarrow R$

$L \rightarrow \frac{L}{k_f}$

$C \rightarrow \frac{C}{k_f}$

Bode Plots.

1st Order Bode plots:-

$$H(s) = 12500 \frac{s+10}{(s+50)(s+500)}$$

Step ①: Bring H(s) into standart form.

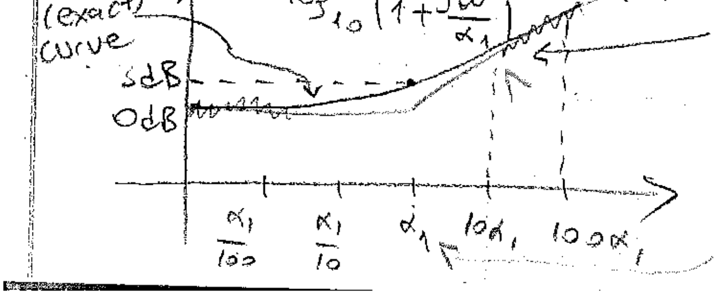
$$H(s) = \frac{K\left(1+\frac{s}{\alpha_1}\right)\left(1+\frac{s}{\alpha_2}\right)\cdots}{\left(1+\frac{s}{\beta_1}\right)\left(1+\frac{s}{\beta_2}\right)\cdots}$$

$$H(s) = 12500 \frac{10\left(1+\frac{s}{10}\right)}{50\left(1+\frac{s}{50}\right) 500\left(1+\frac{s}{500}\right)} = 5 \frac{1+\frac{s}{10}}{\left(1+\frac{s}{50}\right)\left(1+\frac{s}{500}\right)}$$

Step ②: $|H(j\omega)|_{dB} = 20\log_{10} 5 + 20\log_{10}\left|1+\frac{j\omega}{100}\right| - 20\log_{10}\left|1+\frac{j\omega}{50}\right|$

$- 20\log_{10}\left|1+\frac{j\omega}{500}\right|$

let's examine $20\log\left|1+j\frac{\omega}{\alpha_1}\right|$



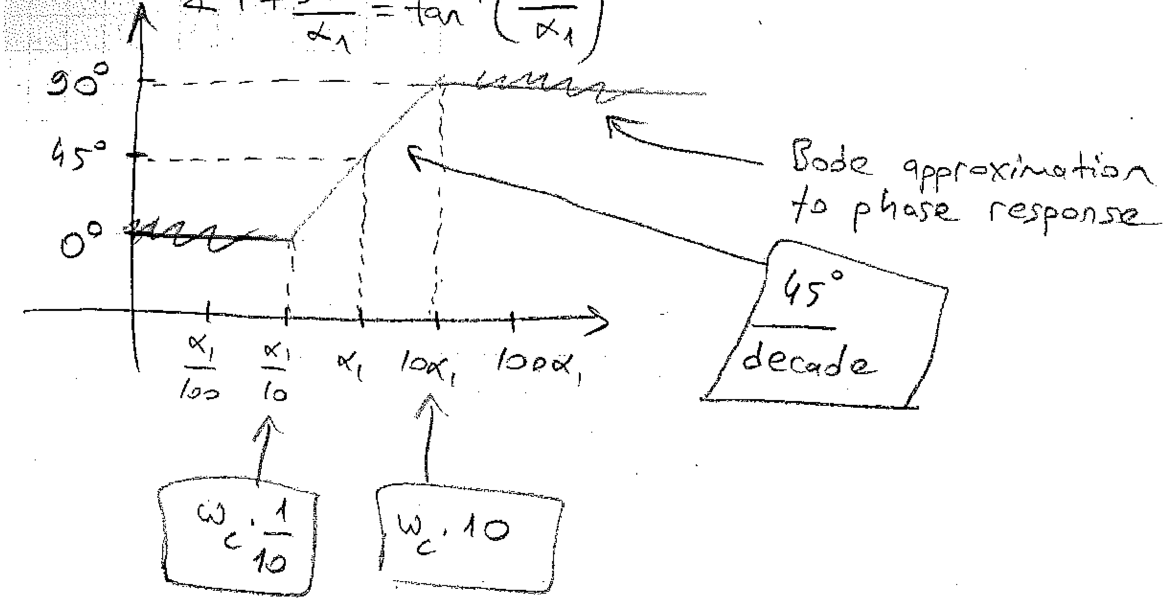
$\frac{20\text{ dB}}{\text{decade}}$

Bode plot/straight line approximation

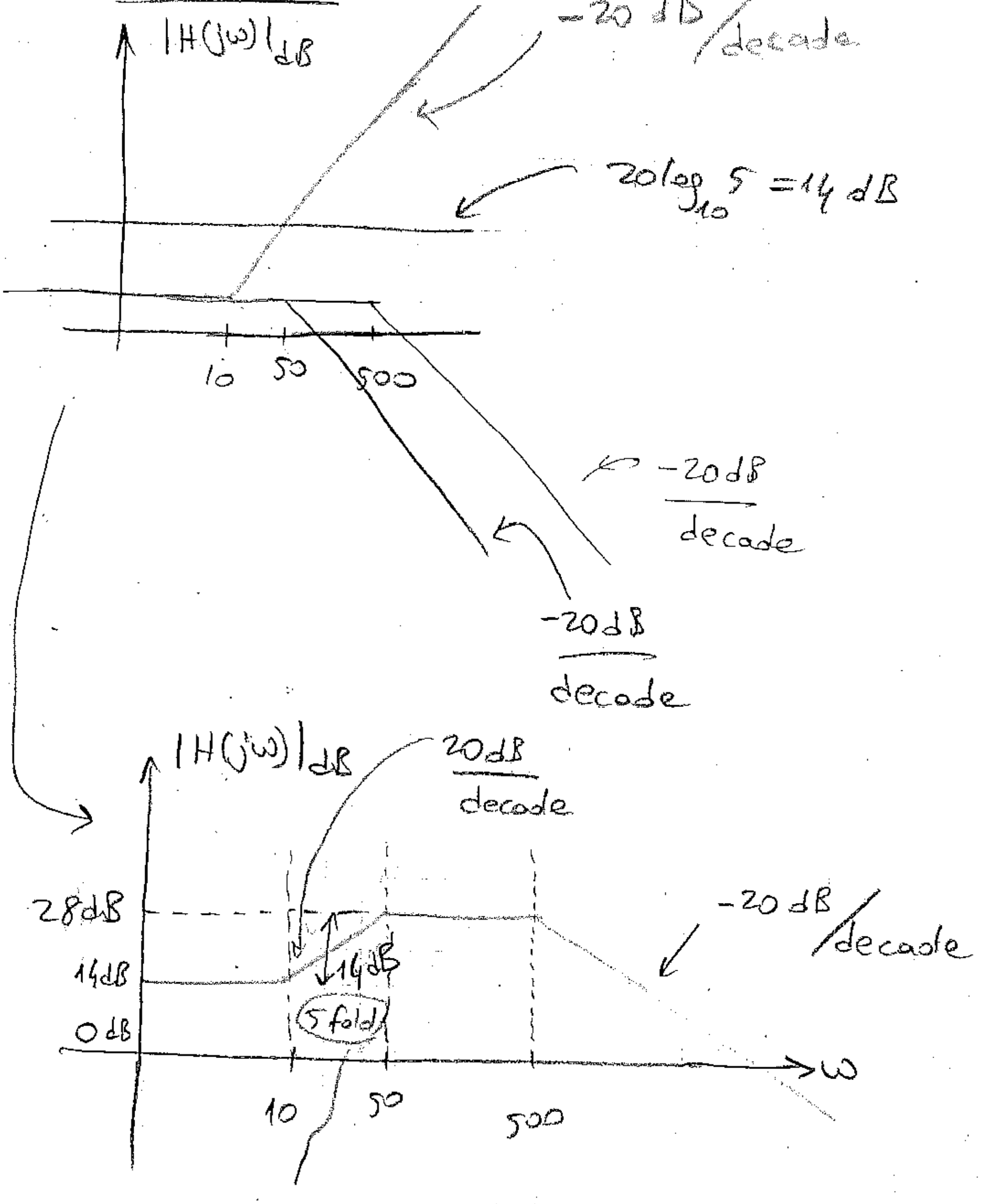
ω_c: $\omega_{cut\text{-}off}$

$\omega_{critical\text{-}frequency}$ $\omega_{corner\text{-}freq}$

$\angle 1 + \frac{j\omega}{\alpha_1} = \tan^{-1}\left(\frac{\omega}{\alpha_1}\right)$

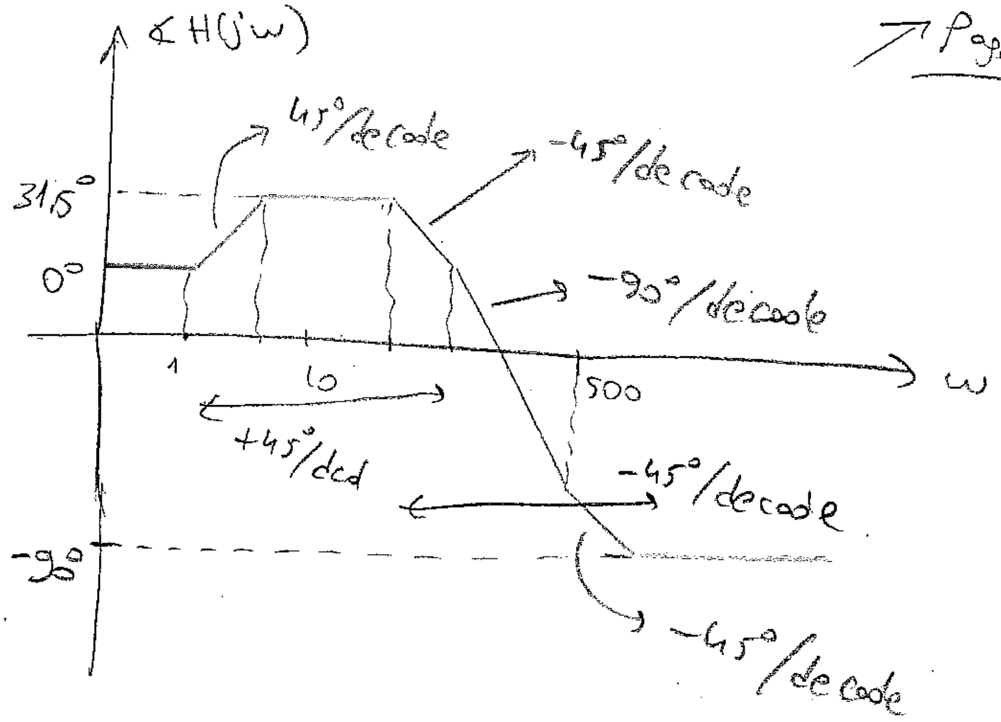


Do the plot



$20\log_{10}(5) = 14$ dB

↗ Page 113


∡H(jω)
45°/decade
-45°/decade
31,5°
0°
-90°/decade
1
10
500
ω
+45°/dcd
-45°/decade
-90°
-45°/decade