CHAPTER 12

Introduction to the Laplace Transform

CHAPTER CONTENTS

✓CHAPTER OBJECTIVES

1. Be able to calculate the Laplace transform of a function using the definition of Laplace transform, the Laplace transform table, and/or a table of operational transforms.
2. Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table.
3. Understand and know how to use the initial value theorem and the final value theorem.

We now introduce a powerful analytical technique that is widely used to study the behavior of linear, lumped-parameter circuits. The method is based on the Laplace transform, which we define mathematically in Section 12.1. Before doing so, we need to explain why another analytical technique is needed. First, we wish to consider the transient behavior of circuits whose describing equations consist of more than a single node-voltage or mesh-current differential equation. In other words, we want to consider multiple-node and multiple-mesh circuits that are described by sets of linear differential equations.

Second, we wish to determine the transient response of circuits whose signal sources vary in ways more complicated than the simple dc level jumps considered in Chapters 7 and 8. Third, we can use the Laplace transform to introduce the concept of the transfer function as a tool for analyzing the steady-state sinusoidal response of a circuit when the frequency of the sinusoidal source is varied. We discuss the transfer function in Chapter 13. Finally, we wish to relate, in a systematic fashion, the time-domain behavior of a circuit to its frequency-domain behavior. Using the Laplace transform will provide a broader understanding of circuit functions.

In this chapter, we introduce the Laplace transform, discuss its pertinent characteristics, and present a systematic method for transforming from the frequency domain to the time domain.

Practical Perspective

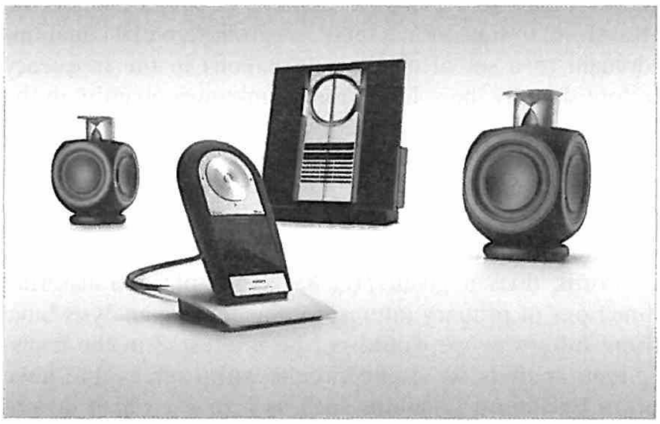
Transient Effects

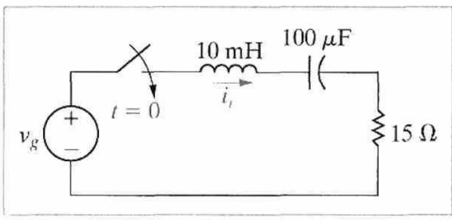
As we learned in Chapter 9, power delivered from electrical wall outlets in the U.S. can be modeled as a sinusoidal voltage or current source, where the frequency of the sinusoid is 60 Hz. The phasor concepts introduced in Chapter 9 allowed us to analyze the steady-state response of a circuit to a sinusoidal source.

It is often important to pay attention to the complete response of a circuit to a sinusoidal source. Remember that the complete response has two parts—the steady-state response that takes the same form as the input to the circuit, and the transient response that decays to zero as time progresses. When the source for a circuit is modeled as a 60 Hz sinusoid, the steady-state response is also a 60 Hz sinusoid whose magnitude and phase angle can be calculated using phasor circuit analysis. The transient response depends on the components that make up the circuit, the values of those components, and the way the components are interconnected. The voltage and current for every component in a circuit is the sum of a transient part and a steady-state part, once the source is switched into the circuit.

While the transient part of the voltage and current eventually decays to zero, initially this transient part, when added to the steady-state part, may exceed the voltage or current rating of the circuit component. This is why it is important to be able to determine the complete response of a circuit. The Laplace transform techniques introduced in this chapter can be used to find the complete response of a circuit to a sinusoidal source.

Consider the *RLC* circuit shown below, comprised of components from Appendix H and powered by a 60 Hz sinusoidal source. As detailed in Appendix H, the 10 mH inductor has a current rating of 40 mA. The amplitude of the sinusoidal source has been chosen so that this rating is met in the steady state (see Problem 12.54). Once we have presented the Laplace transform method, we will be able to determine whether or not this current rating is exceeded when the source is first switched on and both the transient and steady-state components of the inductor current are active.





12.1 Definition of the Laplace Transform

The **Laplace transform** of a function is given by the expression

Laplace transform ▶

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt, \qquad (12.1)$$

where the symbol $\mathscr{L}\{f(t)\}$ is read "the Laplace transform of $f(t)$."

The Laplace transform of $f(t)$ is also denoted $F(s)$; that is,

$$F(s) = \mathscr{L}\{f(t)\}. \qquad (12.2)$$

This notation emphasizes that when the integral in Eq. 12.1 has been evaluated, the resulting expression is a function of s. In our applications, t represents the time domain, and, because the exponent of e in the integral of Eq. 12.1 must be dimensionless, s must have the dimension of reciprocal time, or frequency. The Laplace transform transforms the problem from the time domain to the frequency domain. After obtaining the frequency-domain expression for the unknown, we inverse-transform it back to the time domain.

If the idea behind the Laplace transform seems foreign, consider another familiar mathematical transform. Logarithms are used to change a multiplication or division problem, such as A = BC, into a simpler addition or subtraction problem: log A = log BC = log B + log C. Antilogs are used to carry out the inverse process. The phasor is another transform; as we know from Chapter 9, it converts a sinusoidal signal into a complex number for easier, algebraic computation of circuit values. After determining the phasor value of a signal, we transform it back to its time-domain expression. Both of these examples point out the essential feature of mathematical transforms: They are designed to create a new domain to make the mathematical manipulations easier. After finding the unknown in the new domain, we inverse-transform it back to the original domain. In circuit analysis, we use the Laplace transform to transform a set of integrodifferential equations from the time domain to a set of algebraic equations in the frequency domain. We therefore simplify the solution for an unknown quantity to the manipulation of a set of algebraic equations.

Before we illustrate some of the important properties of the Laplace transform, some general comments are in order. First, note that the integral in Eq. 12.1 is improper because the upper limit is infinite. Thus we are confronted immediately with the question of whether the integral converges. In other words, does a given $f(t)$ have a Laplace transform? Obviously, the functions of primary interest in engineering analysis have Laplace transforms; otherwise we would not be interested in the transform. In linear circuit analysis, we excite circuits with sources that have Laplace transforms. Excitation functions such as t^t or e^{t^2}, which do not have Laplace transforms, are of no interest here.

Second, because the lower limit on the integral is zero, the Laplace transform ignores $f(t)$ for negative values of t. Put another way, $F(s)$ is determined by the behavior of $f(t)$ only for positive values of t. To emphasize that the lower limit is zero, Eq. 12.1 is frequently referred to as the **one-sided**, or **unilateral, Laplace transform**. In the two-sided, or bilateral, Laplace transform, the lower limit is $-\infty$. We do not use the bilateral form here; hence $F(s)$ is understood to be the one-sided transform.

Another point regarding the lower limit concerns the situation when $f(t)$ has a discontinuity at the origin. If $f(t)$ is continuous at the origin—as, for example, in Fig. 12.1(a)—$f(0)$ is not ambiguous. However, if $f(t)$ has a finite discontinuity at the origin—as, for example, in Fig. 12.1(b)—the question arises as to whether the Laplace transform integral should include or exclude the discontinuity. In other words, should we make the lower limit 0^- and include the discontinuity, or should we exclude the discontinuity by making the lower limit 0^+? (We use the notation 0^- and 0^+ to denote values of t just to the left and right of the origin, respectively.) Actually, we may choose either as long as we are consistent. For reasons to be explained later, we choose 0^- as the lower limit.

Because we are using 0^- as the lower limit, we note immediately that the integration from 0^- to 0^+ is zero. The only exception is when the discontinuity at the origin is an impulse function, a situation we discuss in Section 12.3. The important point now is that the two functions shown in Fig. 12.1 have the same unilateral Laplace transform because there is no impulse function at the origin.

The one-sided Laplace transform ignores $f(t)$ for $t < 0^-$. What happens prior to 0^- is accounted for by the initial conditions. Thus we use the Laplace transform to predict the response to a disturbance that occurs after initial conditions have been established.

In the discussion that follows, we divide the Laplace transforms into two types: functional transforms and operational transforms. A **functional transform** is the Laplace transform of a specific function, such as $\sin \omega t$, t, e^{-at}, and so on. An **operational transform** defines a general mathematical property of the Laplace transform, such as finding the transform of the derivative of $f(t)$. Before considering functional and operational transforms, however, we need to introduce the step and impulse functions.

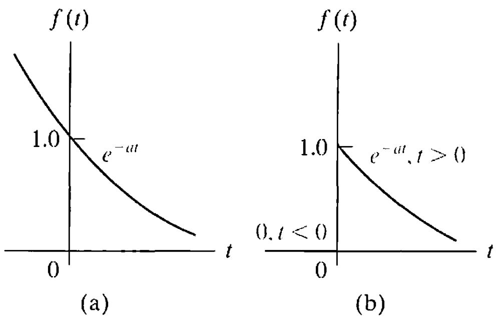


Figure 12.1 ▲ A continuous and discontinuous function at the origin. (a) $f(t)$ is continuous at the origin. (b) $f(t)$ is discontinuous at the origin.

12.2 The Step Function

We may encounter functions that have a discontinuity, or jump, at the origin. For example, we know from earlier discussions of transient behavior that switching operations create abrupt changes in currents and voltages. We accommodate these discontinuities mathematically by introducing the step and impulse functions.

Figure 12.2 illustrates the step function. It is zero for $t < 0$. The symbol for the step function is $Ku(t)$. Thus, the mathematical definition of the **step function** is

$$Ku(t) = 0, \quad t < 0,$$
$$Ku(t) = K, \quad t > 0. \tag{12.3}$$

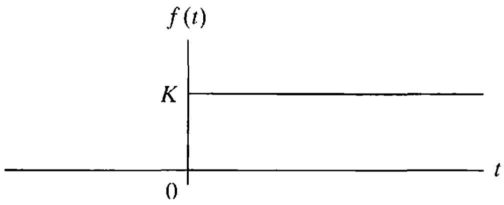


Figure 12.2 ▲ The step function.

If K is 1, the function defined by Eq. 12.3 is the **unit step**.

The step function is not defined at $t = 0$. In situations where we need to define the transition between 0^- and 0^+, we assume that it is linear and that

$$Ku(0) = 0.5K. \tag{12.4}$$

As before, 0^- and 0^+ represent symmetric points arbitrarily close to the left and right of the origin. Figure 12.3 illustrates the linear transition from 0^- to 0^+.

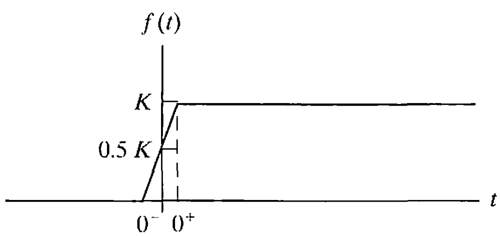


Figure 12.3 ▲ The linear approximation to the step function.

A discontinuity may occur at some time other than $t = 0$; for example, in sequential switching. A step that occurs at $t = a$ is expressed as $Ku(t - a)$. Thus

$$Ku(t - a) = 0, \quad t < a,$$
$$Ku(t - a) = K, \quad t > a. \tag{12.5}$$

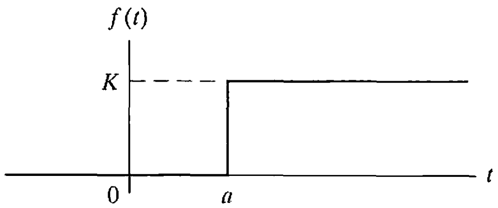


Figure 12.4 ▲ A step function occurring at $t = a$ when $a > 0$.

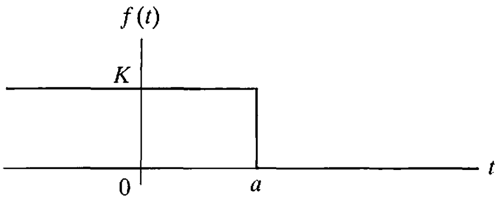


Figure 12.5 ▲ A step function $Ku(a - t)$ for $a > 0$.

If $a > 0$, the step occurs to the right of the origin, and if $a < 0$, the step occurs to the left of the origin. Figure 12.4 illustrates Eq. 12.5. Note that the step function is 0 when the argument $t - a$ is negative, and it is K when the argument is positive.

A step function equal to K for $t < a$ is written as $Ku(a - t)$. Thus

$$\begin{aligned} Ku(a - t) &= K, \quad t < a, \\ Ku(a - t) &= 0, \quad t > a. \end{aligned} \tag{12.6}$$

The discontinuity is to the left of the origin when $a < 0$. Equation 12.6 is shown in Fig. 12.5.

One application of the step function is to use it to write the mathematical expression for a function that is nonzero for a finite duration but is defined for all positive time. One example useful in circuit analysis is a finite-width pulse, which we can create by adding two step functions. The function $K[u(t - 1) - u(t - 3)]$ has the value K for $1 < t < 3$ and the value 0 everywhere else, so it is a finite-width pulse of height K initiated at $t = 1$ and terminated at $t = 3$. In defining this pulse using step functions, it is helpful to think of the step function $u(t - 1)$ as "turning on" the constant value K at $t = 1$, and the step function $-u(t - 3)$ as "turning off" the constant value K at $t = 3$. We use step functions to turn on and turn off linear functions at desired times in Example 12.1.

Example 12.1 Using Step Functions to Represent a Function of Finite Duration

Use step functions to write an expression for the function illustrated in Fig. 12.6.

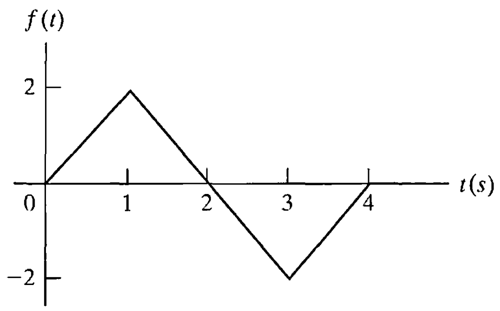


Figure 12.6 ▲ The function for Example 12.1.

Solution

The function shown in Fig. 12.6 is made up of linear segments with break points at 0, 1, 3, and 4 s. To construct this function, we must add and subtract linear functions of the proper slope. We use the step function to initiate and terminate these linear segments at the proper times. In other words, we use the step function to turn on and turn off a straight line with the following equations: $+2t$, on at $t = 0$, off at $t = 1$; $-2t + 4$, on at $t = 1$, off at $t = 3$; and $+2t - 8$, on at $t = 3$, off at $t = 4$. These straight line segments and their equations are shown in Fig. 12.7. The expression for $f(t)$ is

$$\begin{aligned} f(t) = 2t[u(t) - u(t - 1)] + (-2t + 4)[u(t - 1) \\ - u(t - 3)] + (2t - 8)[u(t - 3) - u(t - 4)]. \end{aligned}$$

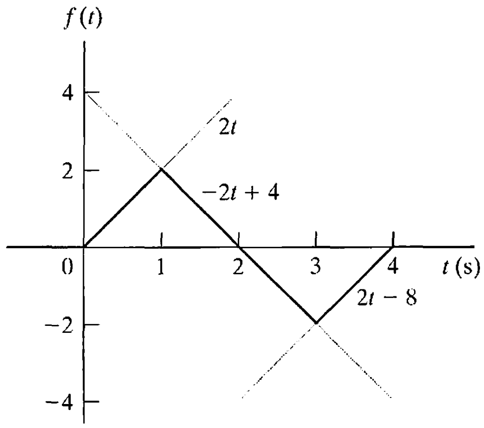


Figure 12.7 ▲ Definition of the three line segments turned on and off with step functions to form the function shown in Fig. 12.6.

NOTE: Assess your understanding of step functions by trying Chapter Problems 12.2 and 12.3.

12.3 The Impulse Function

When we have a finite discontinuity in a function, such as that illustrated in Fig. 12.1(b), the derivative of the function is not defined at the point of the discontinuity. The concept of an impulse function[1] enables us to define the derivative at a discontinuity, and thus to define the Laplace transform of that derivative. An **impulse** is a signal of infinite amplitude and zero duration. Such signals don't exist in nature, but some circuit signals come very close to approximating this definition, so we find a mathematical model of an impulse useful. Impulsive voltages and currents occur in circuit analysis either because of a switching operation or because the circuit is excited by an impulsive source. We will analyze these situations in Chapter 13, but here we focus on defining the impulse function generally.

To define the derivative of a function at a discontinuity, we first assume that the function varies linearly across the discontinuity, as shown in Fig. 12.8, where we observe that as $\epsilon \rightarrow 0$, an abrupt discontinuity occurs at the origin. When we differentiate the function, the derivative between $-\epsilon$ and $+\epsilon$ is constant at a value of $1/2\epsilon$. For $t > \epsilon$, the derivative is $-ae^{-a(t-\epsilon)}$. Figure 12.9 shows these observations graphically. As ϵ approaches zero, the value of $f'(t)$ between $\pm\epsilon$ approaches infinity. At the same time, the duration of this large value is approaching zero. Furthermore, the area under $f'(t)$ between $\pm\epsilon$ remains constant as $\epsilon \rightarrow 0$. In this example, the area is unity. As ϵ approaches zero, we say that the function between $\pm\epsilon$ approaches a **unit impulse function**, denoted $\delta(t)$. Thus the derivative of $f(t)$ at the origin approaches a unit impulse function as ϵ approaches zero, or

$$f'(0) \rightarrow \delta(t) \quad \text{as } \epsilon \rightarrow 0.$$

If the area under the impulse function curve is other than unity, the impulse function is denoted $K\delta(t)$, where K is the area. K is often referred to as the **strength** of the impulse function.

To summarize, an impulse function is created from a variable-parameter function whose parameter approaches zero. The variable-parameter function must exhibit the following three characteristics as the parameter approaches zero:

1. The amplitude approaches infinity.
2. The duration of the function approaches zero.
3. The area under the variable-parameter function is constant as the parameter changes.

Many different variable-parameter functions have the aforementioned characteristics. In Fig. 12.8, we used a linear function $f(t) = 0.5t/\epsilon + 0.5$. Another example of a variable-parameter function is the exponential function:

$$f(t) = \frac{K}{2\epsilon}e^{-|t|/\epsilon}. \tag{12.7}$$

As ϵ approaches zero, the function becomes infinite at the origin and at the same time decays to zero in an infinitesimal length of time. Figure 12.10 illustrates the character of $f(t)$ as $\epsilon \rightarrow 0$. To show that an impulse function

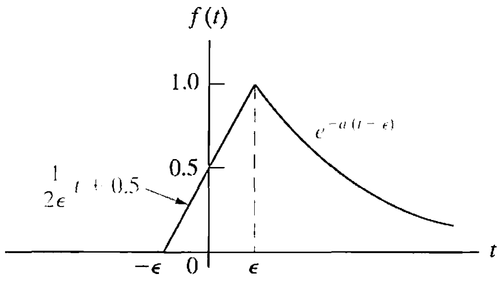


Figure 12.8 ▲ A magnified view of the discontinuity in Fig. 12.1(b), assuming a linear transition between $-\epsilon$ and $+\epsilon$.

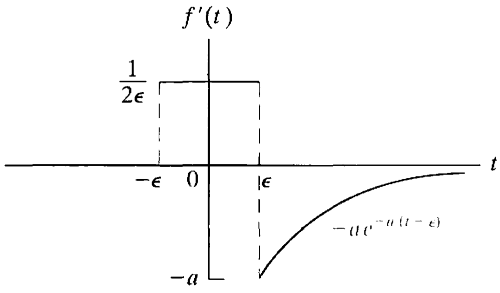


Figure 12.9 ▲ The derivative of the function shown in Fig. 12.8.

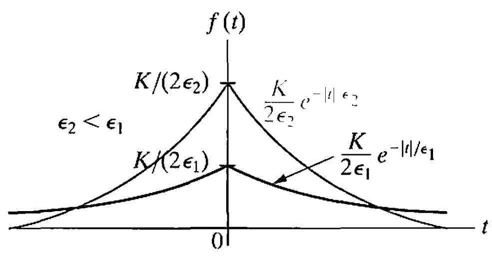


Figure 12.10 ▲ A variable-parameter function used to generate an impulse function.

[1] The impulse function is also known as the Dirac delta function.

is created as $\epsilon \to 0$, we must also show that the area under the function is independent of ϵ. Thus,

$$\begin{aligned}\text{Area} &= \int_{-\infty}^{0} \frac{K}{2\epsilon} e^{t/\epsilon} dt + \int_{0}^{\infty} \frac{K}{2\epsilon} e^{-t/\epsilon} dt \\ &= \frac{K}{2\epsilon} \cdot \frac{e^{t/\epsilon}}{1/\epsilon}\bigg|_{-\infty}^{0} + \frac{K}{2\epsilon} \cdot \frac{e^{-t/\epsilon}}{-1/\epsilon}\bigg|_{0}^{\infty} \\ &= \frac{K}{2} + \frac{K}{2} = K,\end{aligned} \tag{12.8}$$

which tells us that the area under the curve is constant and equal to K units. Therefore, as $\epsilon \to 0$, $f(t) \to K\delta(t)$.

Mathematically, the **impulse function** is defined

$$\int_{-\infty}^{\infty} K\delta(t) dt = K; \tag{12.9}$$

$$\delta(t) = 0, \quad t \neq 0. \tag{12.10}$$

Equation 12.9 states that the area under the impulse function is constant. This area represents the strength of the impulse. Equation 12.10 states that the impulse is zero everywhere except at $t = 0$. An impulse that occurs at $t = a$ is denoted $K\delta(t - a)$.

The graphic symbol for the impulse function is an arrow. The strength of the impulse is given parenthetically next to the head of the arrow. Figure 12.11 shows the impulses $K\delta(t)$ and $K\delta(t - a)$.

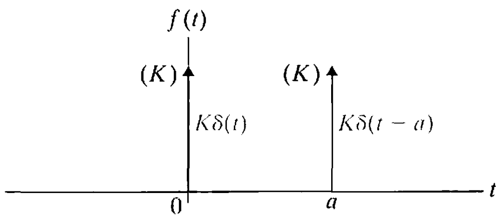


Figure 12.11 ▲ A graphic representation of the impulse $K\delta(t)$ and $K\delta(t - a)$.

An important property of the impulse function is the **sifting property**, which is expressed as

$$\int_{-\infty}^{\infty} f(t)\delta(t - a) dt = f(a), \tag{12.11}$$

where the function $f(t)$ is assumed to be continuous at $t = a$; that is, at the location of the impulse. Equation 12.11 shows that the impulse function sifts out everything except the value of $f(t)$ at $t = a$. The validity of Eq. 12.11 follows from noting that $\delta(t - a)$ is zero everywhere except at $t = a$, and hence the integral can be written

$$I = \int_{-\infty}^{\infty} f(t)\delta(t - a) dt = \int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t - a) dt. \tag{12.12}$$

But because $f(t)$ is continuous at a, it takes on the value $f(a)$ as $t \to a$, so

$$\begin{aligned}I &= \int_{a-\epsilon}^{a+\epsilon} f(a)\delta(t - a) dt = f(a) \int_{a-\epsilon}^{a+\epsilon} \delta(t - a) dt \\ &= f(a).\end{aligned} \tag{12.13}$$

We use the sifting property of the impulse function to find its Laplace transform:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) dt = 1, \tag{12.14}$$

which is an important Laplace transform pair that we make good use of in circuit analysis.

We can also define the derivatives of the impulse function and the Laplace transform of these derivatives. We discuss the first derivative, along with its transform and then state the result for the higher-order derivatives.

The function illustrated in Fig. 12.12(a) generates an impulse function as $\epsilon \to 0$. Figure 12.12(b) shows the derivative of this impulse-generating function, which is defined as the derivative of the impulse $[\delta'(t)]$ as $\epsilon \to 0$. The derivative of the impulse function sometimes is referred to as a moment function, or unit doublet.

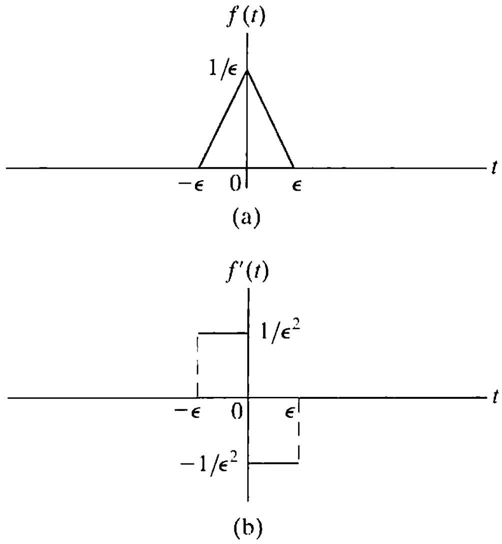


Figure 12.12 ▲ The first derivative of the impulse function. (a) The impulse-generating function used to define the first derivative of the impulse. (b) The first derivative of the impulse-generating function that approaches $\delta'(t)$ as $\epsilon \to 0$.

To find the Laplace transform of $\delta'(t)$, we simply apply the defining integral to the function shown in Fig. 12.12(b) and, after integrating, let $\epsilon \to 0$. Then

$$L\{\delta'(t)\} = \lim_{\epsilon \to 0}\left[\int_{-\epsilon}^{0} \frac{1}{\epsilon^2}e^{-st}\,dt + \int_{0^-}^{\epsilon}\left(-\frac{1}{\epsilon^2}\right)e^{-st}\,dt\right]$$

$$= \lim_{\epsilon \to 0}\frac{e^{s\epsilon} + e^{-s\epsilon} - 2}{s\epsilon^2}$$

$$= \lim_{\epsilon \to 0}\frac{se^{s\epsilon} - se^{-s\epsilon}}{2\epsilon s}$$

$$= \lim_{\epsilon \to 0}\frac{s^2e^{s\epsilon} + s^2e^{-s\epsilon}}{2s}$$

$$= s. \tag{12.15}$$

In deriving Eq. 12.15, we had to use l'Hôpital's rule twice to evaluate the indeterminate form 0/0.

Higher-order derivatives may be generated in a manner similar to that used to generate the first derivative (see Problem 12.6), and the defining integral may then be used to find its Laplace transform. For the nth derivative of the impulse function, we find that its Laplace transform simply is s^n; that is,

$$\mathcal{L}\{\delta^{(n)}(t)\} = s^n. \tag{12.16}$$

Finally, an impulse function can be thought of as a derivative of a step function; that is,

$$\delta(t) = \frac{du(t)}{dt}. \tag{12.17}$$

Figure 12.13 presents the graphic interpretation of Eq. 12.17. The function shown in Fig. 12.13(a) approaches a unit step function as $\epsilon \to 0$. The function shown in Fig. 12.13(b)—the derivative of the function in Fig. 12.13(a)—approaches a unit impulse as $\epsilon \to 0$.

The impulse function is an extremely useful concept in circuit analysis, and we say more about it in the following chapters. We introduced the concept here so that we can include discontinuities at the origin in our definition of the Laplace transform.

NOTE: Assess your understanding of the impulse function by trying Chapter Problems 12.7, 12.9, and 12.10.

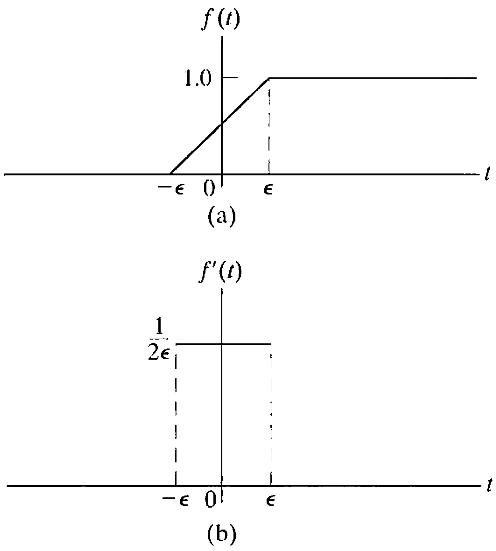


Figure 12.13 ▲ The impulse function as the derivative of the step function: (a) $f(t) \to u(t)$ as $\epsilon \to 0$; and (b) $f'(t) \to \delta(t)$ as $\epsilon \to 0$.

12.4 Functional Transforms

A functional transform is simply the Laplace transform of a specified function of t. Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

We derived one functional transform pair in Section 12.3, where we showed that the Laplace transform of the unit impulse function equals 1; (see Eq. 12.14). A second illustration is the unit step function of Fig. 12.13(a), where

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^-}^{\infty} 1e^{-st}dt \\ &= \frac{e^{-st}}{-s}\bigg|_{0^-}^{\infty} = \frac{1}{s}.\end{aligned} \tag{12.18}$$

Equation 12.18 shows that the Laplace transform of the unit step function is $1/s$.

The Laplace transform of the decaying exponential function shown in Fig. 12.14 is

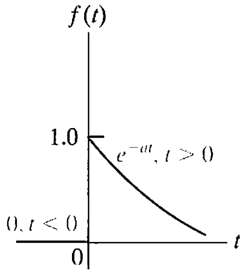


Figure 12.14 ▲ A decaying exponential function.

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(a+s)t}\,dt = \frac{1}{s+a}. \tag{12.19}$$

In deriving Eqs. 12.18 and 12.19, we used the fact that integration across the discontinuity at the origin is zero.

A third illustration of finding a functional transform is the sinusoidal function shown in Fig. 12.15. The expression for $f(t)$ for $t > 0^-$ is $\sin \omega t$; hence the Laplace transform is

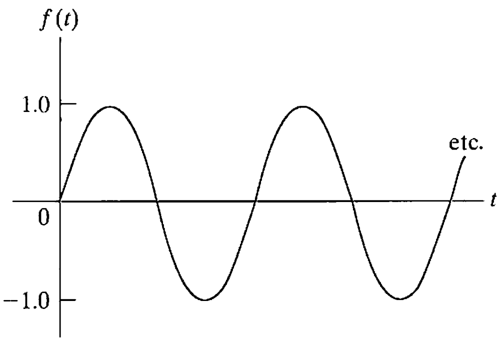


Figure 12.15 ▲ A sinusoidal function for $t > 0$.

$$\begin{aligned}\mathcal{L}\{\sin \omega t\} &= \int_{0^-}^{\infty} (\sin \omega t)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} \frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\,dt \\ &= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) \\ &= \frac{\omega}{s^2+\omega^2}.\end{aligned} \tag{12.20}$$

Table 12.1 gives an abbreviated list of Laplace transform pairs. It includes the functions of most interest in an introductory course on circuit applications.

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)$ $(t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

✓ASSESSMENT PROBLEM

Objective 1—Be able to calculate the Laplace transform of a function using the definition of Laplace transform

12.1 Use the defining integral to
a) find the Laplace transform of $\cosh \beta t$;
b) find the Laplace transform of $\sinh \beta t$.

Answer: (a) $s/(s^2 - \beta^2)$;
(b) $\beta/(s^2 - \beta^2)$.

NOTE: Also try Chapter Problem 12.17.

12.5 Operational Transforms

Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain. The operations of primary interest are (1) multiplication by a constant; (2) addition (subtraction); (3) differentiation; (4) integration; (5) translation in the time domain; (6) translation in the frequency domain; and (7) scale changing.

Multiplication by a Constant

From the defining integral, if

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\mathcal{L}\{Kf(t)\} = KF(s). \qquad (12.21)$$

Thus, multiplication of $f(t)$ by a constant corresponds to multiplying $F(s)$ by the same constant.

Addition (Subtraction)

Addition (subtraction) in the time domain translates into addition (subtraction) in the frequency domain. Thus if

$$\mathscr{L}\{f_1(t)\} = F_1(s),$$

$$\mathscr{L}\{f_2(t)\} = F_2(s),$$

$$\mathscr{L}\{f_3(t)\} = F_3(s),$$

then

$$\mathscr{L}\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s), \tag{12.22}$$

which is derived by simply substituting the algebraic sum of time-domain functions into the defining integral.

Differentiation

Differentiation in the time domain corresponds to multiplying $F(s)$ by s and then subtracting the initial value of $f(t)$—that is, $f(0^-)$—from this product:

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-), \tag{12.23}$$

which is obtained directly from the definition of the Laplace transform, or

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}\,dt. \tag{12.24}$$

We evaluate the integral in Eq. 12.24 by integrating by parts. Letting $u = e^{-st}$ and $dv = [df(t)/dt]\,dt$ yields

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st}f(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-se^{-st}dt). \tag{12.25}$$

Because we are assuming that $f(t)$ is Laplace transformable, the evaluation of $e^{-st}f(t)$ at $t = \infty$ is zero. Therefore the right-hand side of Eq. 12.25 reduces to

$$-f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st}dt = sF(s) - f(0^-).$$

This observation completes the derivation of Eq. 12.23. It is an important result because it states that differentiation in the time domain reduces to an algebraic operation in the s domain.

We determine the Laplace transform of higher-order derivatives by using Eq. 12.23 as the starting point. For example, to find the Laplace transform of the second derivative of $f(t)$, we first let

$$g(t) = \frac{df(t)}{dt}. \tag{12.26}$$

Now we use Eq. 12.23 to write

$$G(s) = sF(s) - f(0^-). \qquad (12.27)$$

But because

$$\frac{dg(t)}{dt} = \frac{d^2f(t)}{dt^2},$$

we write

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = sG(s) - g(0^-). \qquad (12.28)$$

Combining Eqs. 12.26, 12.27, and 12.28 gives

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}. \qquad (12.29)$$

We find the Laplace transform of the nth derivative by successively applying the preceding process, which leads to the general result

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$$

$$- s^{n-3}\frac{d^2f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}. \qquad (12.30)$$

Integration

Integration in the time domain corresponds to dividing by s in the s domain. As before, we establish the relationship by the defining integral:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)\,dx\right]e^{-st}\,dt. \qquad (12.31)$$

We evaluate the integral on the right-hand side of Eq. 12.31 by integrating by parts, first letting

$$u = \int_{0}^{t} f(x)\,dx,$$

$$dv = e^{-st}\,dt.$$

Then

$$du = f(t)\,dt,$$

$$v = -\frac{e^{-st}}{s}.$$

The integration-by-parts formula yields

$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = -\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)\,dx\Bigg|_{0^-}^{\infty} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)\,dt. \quad (12.32)$$

The first term on the right-hand side of Eq. 12.32 is zero at both the upper and lower limits. The evaluation at the lower limit obviously is zero, whereas the evaluation at the upper limit is zero because we are assuming that $f(t)$ has a Laplace transform. The second term on the right-hand side of Eq. 12.32 is $F(s)/s$; therefore

$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s}, \quad (12.33)$$

which reveals that the operation of integration in the time domain is transformed to the algebraic operation of multiplying by $1/s$ in the s domain. Equation 12.33 and Eq. 12.30 form the basis of the earlier statement that the Laplace transform translates a set of integrodifferential equations into a set of algebraic equations.

Translation in the Time Domain

If we start with any function $f(t)u(t)$, we can represent the same function, translated in time by the constant a, as $f(t - a)u(t - a)$.[2] Translation in the time domain corresponds to multiplication by an exponential in the frequency domain. Thus

$$\mathscr{L}\{f(t - a)u(t - a)\} = e^{-as}F(s), \quad a > 0. \quad (12.34)$$

For example, knowing that

$$\mathscr{L}\{tu(t)\} = \frac{1}{s^2},$$

Eq. 12.34 permits writing the Laplace transform of $(t - a)u(t - a)$ directly:

$$\mathscr{L}\{(t - a)u(t - a)\} = \frac{e^{-as}}{s^2}.$$

The proof of Eq. 12.34 follows from the defining integral:

$$\mathscr{L}\{(t - a)u(t - a)\} = \int_{0^-}^{\infty} u(t - a)f(t - a)e^{-st}\,dt$$

$$= \int_{a}^{\infty} f(t - a)e^{-st}\,dt. \quad (12.35)$$

In writing Eq. 12.35, we took advantage of $u(t - a) = 1$ for $t > a$. Now we change the variable of integration. Specifically, we let $x = t - a$. Then

[2] Note that throughout we multiply any arbitrary function $f(t)$ by the unit step function $u(t)$ to ensure that the resulting function is defined for all positive time.

$x = 0$ when $t = a$, $x = \infty$ when $t = \infty$ and $dx = dt$. Thus we write the integral in Eq. 12.35 as

$$\begin{aligned}\mathscr{L}\{f(t-a)u(t-a)\} &= \int_0^\infty f(x)e^{-s(x+a)}\,dx \\ &= e^{-sa}\int_0^\infty f(x)e^{-sx}\,dx \\ &= e^{-as}F(s),\end{aligned}$$

which is what we set out to prove.

Translation in the Frequency Domain

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain:

$$\mathscr{L}\{e^{-at}f(t)\} = F(s+a), \qquad (12.36)$$

which follows from the defining integral. The derivation of Eq. 12.36 is left to Problem 12.13.

We may use the relationship in Eq. 12.36 to derive new transform pairs. Thus, knowing that

$$\mathscr{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2},$$

we use Eq. 12.36 to deduce that

$$\mathscr{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}.$$

Scale Changing

The scale-change property gives the relationship between $f(t)$ and $F(s)$ when the time variable is multiplied by a positive constant:

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0, \qquad (12.37)$$

the derivation of which is left to Problem 12.16. The scale-change property is particularly useful in experimental work, especially where time-scale changes are made to facilitate building a model of a system.

We use Eq. 12.37 to formulate new transform pairs. Thus, knowing that

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1},$$

we deduce from Eq. 12.37 that

$$\mathscr{L}\{\cos\omega t\} = \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} = \frac{s}{s^2+\omega^2}.$$

Table 12.2 gives an abbreviated list of operational transforms.

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

✓ASSESSMENT PROBLEM

Objective 1—Be able to calculate the Laplace transform of a function using the Laplace transform table or a table of operational transforms

12.2 Use the appropriate operational transform from Table 12.2 to find the Laplace transform of each function:

a) t^2e^{-at};

b) $\frac{d}{dt}(e^{-at}\sinh \beta t)$;

c) $t \cos \omega t$.

Answer: (a) $\frac{2}{(s+a)^3}$;

(b) $\frac{\beta s}{(s+a)^2 - \beta^2}$;

(c) $\frac{s^2 - \omega^2}{(s^2+\omega^2)^2}$.

NOTE: Also try Chapter Problems 11.14 and 11.22.

12.6 Applying the Laplace Transform

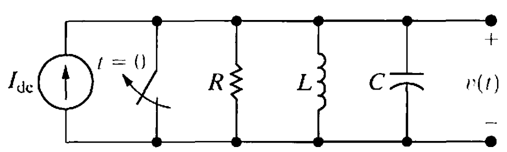


Figure 12.16 ▲ A parallel *RLC* circuit.

We now illustrate how to use the Laplace transform to solve the ordinary integrodifferential equations that describe the behavior of lumped-parameter circuits. Consider the circuit shown in Fig. 12.16. We assume that no initial energy is stored in the circuit at the instant when the switch, which is shorting the dc current source, is opened. The problem is to find the time-domain expression for $v(t)$ when $t \geq 0$.

We begin by writing the integrodifferential equation that $v(t)$ must satisfy. We need only a single node-voltage equation to describe the circuit. Summing the currents away from the top node in the circuit generates the equation:

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t). \tag{12.38}$$

Note that in writing Eq. 12.38, we indicated the opening of the switch in the step jump of the source current from zero to I_{dc}.

After deriving the integrodifferential equations (in this example, just one), we transform the equations to the s domain. We will not go through the steps of the transformation in detail, because in Chapter 13 we will discover how to bypass them and generate the s-domain equations directly. Briefly though, we use three operational transforms and one functional transform on Eq. 12.38 to obtain

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right), \tag{12.39}$$

an algebraic equation in which $V(s)$ is the unknown variable. We are assuming that the circuit parameters R, L, and C, as well as the source current I_{dc}, are known; the initial voltage on the capacitor $v(0^-)$ is zero because the initial energy stored in the circuit is zero. Thus we have reduced the problem to solving an algebraic equation.

Next we solve the algebraic equations (again, just one in this case) for the unknowns. Solving Eq. 12.39 for $V(s)$ gives

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s},$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}. \tag{12.40}$$

To find $v(t)$ we must inverse-transform the expression for $V(s)$. We denote this inverse operation

$$v(t) = \mathscr{L}^{-1}\{V(s)\}. \tag{12.41}$$

The next step in the analysis is to find the inverse transform of the s-domain expression; this is the subject of Section 12.7. In that section we also present a final, critical step: checking the validity of the resulting time-domain expression. The need for such checking is not unique to the Laplace transform; conscientious and prudent engineers always test any derived solution to be sure it makes sense in terms of known system behavior.

Simplifying the notation now is advantageous. We do so by dropping the parenthetical t in time-domain expressions and the parenthetical s in frequency-domain expressions. We use lowercase letters for all time-domain variables, and we represent the corresponding s-domain variables with uppercase letters. Thus

$$\mathscr{L}\{v\} = V \quad \text{or} \quad v = \mathscr{L}^{-1}\{V\},$$

$$\mathscr{L}\{i\} = I \quad \text{or} \quad i = \mathscr{L}^{-1}\{I\},$$

$$\mathscr{L}\{f\} = F \quad \text{or} \quad f = \mathscr{L}^{-1}\{F\},$$

and so on.

NOTE: Assess your understanding of this material by trying Chapter Problem 12.26.

12.7 Inverse Transforms

The expression for $V(s)$ in Eq. 12.40 is a **rational** function of s; that is, one that can be expressed in the form of a ratio of two polynomials in s such that no nonintegral powers of s appear in the polynomials. In fact, for linear, lumped-parameter circuits whose component values are constant, the s-domain expressions for the unknown voltages and currents are always rational functions of s. (You may verify this observation by working Problems 12.28–12.31.) If we can inverse-transform rational functions of s, we can solve for the time-domain expressions for the voltages and currents. The purpose of this section is to present a straight-forward and systematic technique for finding the inverse transform of a rational function.

In general, we need to find the inverse transform of a function that has the form

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}. \quad (12.42)$$

The coefficients a and b are real constants, and the exponents m and n are positive integers. The ratio $N(s)/D(s)$ is called a **proper rational function** if $m > n$, and an **improper rational function** if $m \leq n$. Only a proper rational function can be expanded as a sum of partial fractions. This restriction poses no problem, as we show at the end of this section.

Partial Fraction Expansion: Proper Rational Functions

A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$. Thus $D(s)$ must be in factored form before we can make a partial fraction expansion. For each distinct root of $D(s)$, a single term appears in the sum of partial fractions. For each multiple root of $D(s)$ of multiplicity r, the expansion contains r terms. For example, in the rational function

$$\frac{s + 6}{s(s + 3)(s + 1)^2},$$

the denominator has four roots. Two of these roots are distinct—namely, at $s = 0$ and $s = -3$. A multiple root of multiplicity 2 occurs at $s = -1$. Thus the partial fraction expansion of this function takes the form

$$\frac{s + 6}{s(s + 3)(s + 1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s + 3} + \frac{K_3}{(s + 1)^2} + \frac{K_4}{s + 1}. \quad (12.43)$$

The key to the partial fraction technique for finding inverse transforms lies in recognizing the $f(t)$ corresponding to each term in the sum of partial fractions. From Table 12.1 you should be able to verify that

$$\mathcal{L}^{-1}\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\}$$
$$= (K_1 + K_2e^{-3t} + K_3te^{-t} + K_4e^{-t})u(t). \tag{12.44}$$

All that remains is to establish a technique for determining the coefficients ($K_1, K_2, K_3, \ldots$) generated by making a partial fraction expansion. There are four general forms this problem can take. Specifically, the roots of $D(s)$ are either (1) real and distinct; (2) complex and distinct; (3) real and repeated; or (4) complex and repeated. Before we consider each situation in turn, a few general comments are in order.

We used the identity sign $\equiv$ in Eq. 12.43 to emphasize that expanding a rational function into a sum of partial fractions establishes an identical equation. Thus both sides of the equation must be the same for all values of the variable s. Also, the identity relationship must hold when both sides are subjected to the same mathematical operation. These characteristics are pertinent to determining the coefficients, as we will see.

Be sure to verify that the rational function is proper. This check is important because nothing in the procedure for finding the various Ks will alert you to nonsense results if the rational function is improper. We present a procedure for checking the Ks, but you can avoid wasted effort by forming the habit of asking yourself, "Is $F(s)$ a proper rational function?"

Partial Fraction Expansion: Distinct Real Roots of $D(s)$

We first consider determining the coefficients in a partial fraction expansion when all the roots of $D(s)$ are real and distinct. To find a K associated with a term that arises because of a distinct root of $D(s)$, we multiply both sides of the identity by a factor equal to the denominator beneath the desired K. Then when we evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value. For example,

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}. \tag{12.45}$$

To find the value of K_1, we multiply both sides by s and then evaluate both sides at $s = 0$:

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2s}{s+8}\right|_{s=0} + \left.\frac{K_3s}{s+6}\right|_{s=0},$$

or

$$\frac{96(5)(12)}{8(6)} \equiv K_1 = 120. \tag{12.46}$$

To find the value of K_2, we multiply both sides by $s + 8$ and then evaluate both sides at $s = -8$:

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8}$$
$$\equiv \left.\frac{K_1(s+8)}{s}\right|_{s=-8} + K_2 + \left.\frac{K_3(s+8)}{(s+6)}\right|_{s=-8},$$

or

$$\frac{96(-3)(4)}{(-8)(-2)} = K_2 = -72. \quad (12.47)$$

Then K_3 is

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = K_3 = 48. \quad (12.48)$$

From Eq. 12.45 and the K values obtained,

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}. \quad (12.49)$$

At this point, testing the result to protect against computational errors is a good idea. As we already mentioned, a partial fraction expansion creates an identity; thus both sides of Eq. 12.49 must be the same for all s values. The choice of test values is completely open; hence we choose values that are easy to verify. For example, in Eq. 12.49, testing at either -5 or -12 is attractive because in both cases the left-hand side reduces to zero. Choosing -5 yields

$$\frac{120}{-5} + \frac{48}{1} - \frac{72}{3} = -24 + 48 - 24 = 0,$$

whereas testing -12 gives

$$\frac{120}{-12} + \frac{48}{-6} - \frac{72}{-4} = -10 - 8 + 18 = 0.$$

Now confident that the numerical values of the various Ks are correct, we proceed to find the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t). \quad (12.50)$$

✓ASSESSMENT PROBLEMS

Objective 2—Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table

12.3 Find $f(t)$ if

$$F(s) = \frac{6s^2 + 26s + 26}{(s+1)(s+2)(s+3)}.$$

Answer: $f(t) = (3e^{-t} + 2e^{-2t} + e^{-3t})u(t)$.

12.4 Find $f(t)$ if

$$F(s) = \frac{7s^2 + 63s + 134}{(s+3)(s+4)(s+5)}.$$

Answer: $f(t) = (4e^{-3t} + 6e^{-4t} - 3e^{-5t})u(t)$.

NOTE: Also try Chapter Problems 12.40(a) and (b).

Partial Fraction Expansion: Distinct Complex Roots of $D(s)$

The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that the algebra in the former involves complex numbers. We illustrate by expanding the rational function:

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}. \tag{12.51}$$

We begin by noting that $F(s)$ is a proper rational function. Next we must find the roots of the quadratic term $s^2 + 6s + 25$:

$$s^2 + 6s + 25 = (s + 3 - j4)(s + 3 + j4). \tag{12.52}$$

With the denominator in factored form, we proceed as before:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} \equiv$$

$$\frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}. \tag{12.53}$$

To find K_1, K_2, and K_3, we use the same process as before:

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12, \tag{12.54}$$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^\circ}, \tag{12.55}$$

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^\circ}. \tag{12.56}$$

Then

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4}$$

$$+ \frac{10\angle 53.13^\circ}{s+3+j4}. \tag{12.57}$$

Again, we need to make some observations. First, in physically realizable circuits, complex roots always appear in conjugate pairs. Second, the coefficients associated with these conjugate pairs are themselves conjugates. Note, for example, that K_3 (Eq. 12.56) is the conjugate of K_2

(Eq. 12.55). Thus for complex conjugate roots, you actually need to calculate only half the coefficients.

Before inverse-transforming Eq. 12.57, we check the partial fraction expansion numerically. Testing at -3 is attractive because the left-hand side reduces to zero at this value:

$$F(s) = \frac{-12}{3} + \frac{10\,\angle{-53.13^\circ}}{-j4} + \frac{10\,\angle{53.13^\circ}}{j4}$$

$$= -4 + 2.5\,\angle{36.87^\circ} + 2.5\,\angle{-36.87^\circ}$$

$$= -4 + 2.0 + j1.5 + 2.0 - j1.5 = 0.$$

We now proceed to inverse-transform Eq. 12.57:

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t}$$

$$+ 10e^{j53.13^\circ}e^{-(3+j4)t})u(t). \qquad (12.58)$$

In general, having the function in the time domain contain imaginary components is undesirable. Fortunately, because the terms involving imaginary components always come in conjugate pairs, we can eliminate the imaginary components simply by adding the pairs:

$$10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t}$$

$$= 10e^{-3t}\left(e^{j(4t-53.13^\circ)} + e^{-j(4t-53.13^\circ)}\right)$$

$$= 20e^{-3t}\cos(4t - 53.13^\circ), \qquad (12.59)$$

which enables us to simplify Eq. 12.58:

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}$$

$$= [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t). \qquad (12.60)$$

Because distinct complex roots appear frequently in lumped-parameter linear circuit analysis, we need to summarize these results with a new transform pair. Whenever $D(s)$ contains distinct complex roots—that is, factors of the form $(s + \alpha - j\beta)(s + \alpha + j\beta)$—a pair of terms of the form

$$\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta} \qquad (12.61)$$

appears in the partial fraction expansion, where the partial fraction coefficient is, in general, a complex number. In polar form,

$$K = |K|e^{j\theta} = |K|\angle{\theta^\circ}, \qquad (12.62)$$

where $|K|$ denotes the magnitude of the complex coefficient. Then

$$K^* = |K|e^{-j\theta} = |K|\underline{/-\theta^\circ}. \quad (12.63)$$

The complex conjugate pair in Eq. 12.61 always inverse-transforms as

$$\mathcal{L}^{-1}\left\{\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right\}$$
$$= 2|K|e^{-\alpha t}\cos(\beta t + \theta). \quad (12.64)$$

In applying Eq. 12.64 it is important to note that K is defined as the coefficient associated with the denominator term $s + \alpha - j\beta$, and K^* is defined as the coefficient associated with the denominator $s + \alpha + j\beta$.

✓ASSESSMENT PROBLEM

Objective 2—Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table

12.5 Find $f(t)$ if

$$F(s) = \frac{10(s^2 + 119)}{(s+5)(s^2 + 10s + 169)}.$$

Answer: $f(t) = (10e^{-5t} - 8.33e^{-5t}\sin 12t)u(t)$.

NOTE: Also try Chapter Problems 12.40(c) and (d).

Partial Fraction Expansion: Repeated Real Roots of $D(s)$

To find the coefficients associated with the terms generated by a multiple root of multiplicity r, we multiply both sides of the identity by the multiple root raised to its rth power. We find the K appearing over the factor raised to the rth power by evaluating both sides of the identity at the multiple root. To find the remaining $(r - 1)$ coefficients, we differentiate both sides of the identity $(r - 1)$ times. At the end of each differentiation, we evaluate both sides of the identity at the multiple root. The right-hand side is always the desired K, and the left-hand side is always its numerical value. For example,

$$\frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}. \quad (12.65)$$

We find K_1 as previously described; that is,

$$K_1 = \left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = \frac{100(25)}{125} = 20. \quad (12.66)$$

To find K_2, we multiply both sides by $(s + 5)^3$ and then evaluate both sides at -5:

$$\left.\frac{100(s+25)}{s}\right|_{s=-5} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + K_3(s+5)|_{s=-5}$$
$$+ K_4(s+5)^2\Big|_{s=-5}, \quad (12.67)$$

$$\frac{100(20)}{(-5)} = K_1 \times 0 + K_2 + K_3 \times 0 + K_4 \times 0$$
$$= K_2 = -400. \quad (12.68)$$

To find K_3 we first must multiply both sides of Eq. 12.65 by $(s + 5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s = -5$:

$$\frac{d}{ds}\left[\frac{100(s + 25)}{s}\right]_{s=-5} = \frac{d}{ds}\left[\frac{K_1(s + 5)^3}{s}\right]_{s=-5}$$

$$+ \frac{d}{ds}[K_2]_{s=-5}$$

$$+ \frac{d}{ds}[K_3(s + 5)]_{s=-5}$$

$$+ \frac{d}{ds}[K_4(s + 5)^2]_{s=-5}, \quad (12.69)$$

$$100\left[\frac{s - (s + 25)}{s^2}\right]_{s=-5} = K_3 = -100. \quad (12.70)$$

To find K_4 we first multiply both sides of Eq. 12.65 by $(s + 5)^3$. Next we differentiate both sides twice with respect to s and then evaluate both sides at $s = -5$. After simplifying the first derivative, the second derivative becomes

$$100\frac{d}{ds}\left[-\frac{25}{s^2}\right]_{s=-5} = K_1\frac{d}{ds}\left[\frac{(s + 5)^2(2s - 5)}{s^2}\right]_{s=-5}$$

$$+ 0 + \frac{d}{ds}[K_3]_{s=-5} + \frac{d}{ds}[2K_4(s + 5)]_{s=-5},$$

or

$$-40 = 2K_4. \quad (12.71)$$

Solving Eq. 12.71 for K_4 gives

$$K_4 = -20. \quad (12.72)$$

Then

$$\frac{100(s + 25)}{s(s + 5)^3} = \frac{20}{s} - \frac{400}{(s + 5)^3} - \frac{100}{(s + 5)^2} - \frac{20}{s + 5}. \quad (12.73)$$

At this point we can check our expansion by testing both sides of Eq. 12.73 at $s = -25$. Noting both sides of Eq. 12.73 equal zero when $s = -25$ gives us confidence in the correctness of the partial fraction expansion. The inverse transform of Eq. 12.73 yields

$$\mathcal{L}^{-1}\left\{\frac{100(s + 25)}{s(s + 5)^3}\right\}$$

$$= [20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t}]u(t). \quad (12.74)$$

✓ASSESSMENT PROBLEM

Objective 2—Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table

12.6 Find $f(t)$ if

$$F(s) = \frac{(4s^2 + 7s + 1)}{s(s + 1)^2}.$$

Answer: $f(t) = (1 + 2te^{-t} + 3e^{-t})u(t)$.

NOTE: Also try Chapter Problems 12.41(a), (b), and (d).

Partial Fraction Expansion: Repeated Complex Roots of $D(s)$

We handle repeated complex roots in the same way that we did repeated real roots; the only difference is that the algebra involves complex numbers. Recall that complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugates, so that only half the Ks need to be evaluated. For example,

$$F(s) = \frac{768}{(s^2 + 6s + 25)^2}. \tag{12.75}$$

After factoring the denominator polynomial, we write

$$F(s) = \frac{768}{(s + 3 - j4)^2(s + 3 + j4)^2}$$

$$= \frac{K_1}{(s + 3 - j4)^2} + \frac{K_2}{s + 3 - j4}$$

$$+ \frac{K_1^*}{(s + 3 + j4)^2} + \frac{K_2^*}{s + 3 + j4}. \tag{12.76}$$

Now we need to evaluate only K_1 and K_2, because K_1^* and K_2^* are conjugate values. The value of K_1 is

$$K_1 = \left.\frac{768}{(s + 3 + j4)^2}\right|_{s=-3+j4}$$

$$= \frac{768}{(j8)^2} = -12. \tag{12.77}$$

The value of K_2 is

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s + 3 + j4)^2}\right]_{s=-3+j4}$$

$$= -\left.\frac{2(768)}{(s + 3 + j4)^3}\right|_{s=-3+j4}$$

$$= -\frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\,\underline{/-90^\circ}. \tag{12.78}$$

From Eqs. 12.77 and 12.78,

$$K_1^* = -12, \tag{12.79}$$

$$K_2^* = j3 = 3\,\underline{/90^\circ}. \tag{12.80}$$

We now group the partial fraction expansion by conjugate terms to obtain

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2}\right]$$

$$+ \left(\frac{3\,\underline{/-90^\circ}}{s+3-j4} + \frac{3\,\underline{/90^\circ}}{s+3+j4}\right). \tag{12.81}$$

We now write the inverse transform of $F(s)$:

$$f(t) = [-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90^\circ)]u(t). \tag{12.82}$$

Note that if $F(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form

$$\frac{K}{(s+a)^r}.$$

The inverse transform of this term is

$$\mathscr{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t). \tag{12.83}$$

If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}.$$

The inverse transform of this pair is

$$\mathscr{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\}$$

$$= \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t + \theta)\right]u(t). \tag{12.84}$$

Equations 12.83 and 12.84 are the key to being able to inverse-transform any partial fraction expansion by inspection. One further note regarding these two equations: In most circuit analysis problems, r is seldom greater than 2. Therefore, the inverse transform of a rational function can be handled with four transform pairs. Table 12.3 lists these pairs.

TABLE 12.3 Four Useful Transform Pairs

Pair Number	Nature of Roots	$F(s)$	$f(t)$
1	Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\|K\|e^{-\alpha t}\cos(\beta t+\theta)u(t)$
4	Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\|K\|e^{-\alpha t}\cos(\beta t+\theta)u(t)$

Note: In pairs 1 and 2, K is a real quantity, whereas in pairs 3 and 4, K is the complex quantity $|K|\angle\theta$.

✓ASSESSMENT PROBLEM

Objective 2—Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table

12.7 Find $f(t)$ if

$$F(s) = \frac{40}{(s^2 + 4s + 5)^2}.$$

Answer: $f(t) = (-20te^{-2t}\cos t + 20e^{-2t}\sin t)u(t)$.

NOTE: Also try Chapter Problem 12.41(e).

Partial Fraction Expansion: Improper Rational Functions

We conclude the discussion of partial fraction expansions by returning to an observation made at the beginning of this section, namely, that improper rational functions pose no serious problem in finding inverse transforms. An improper rational function can always be expanded into a polynomial plus a proper rational function. The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions. The proper rational function is inverse-transformed by the techniques outlined in this section. To illustrate the procedure, we use the function

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}. \qquad (12.85)$$

Dividing the denominator into the numerator until the remainder is a proper rational function gives

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}, \qquad (12.86)$$

where the term $(30s + 100)/(s^2 + 9s + 20)$ is the remainder.

Next we expand the proper rational function into a sum of partial fractions:

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s + 4)(s + 5)} = \frac{-20}{s + 4} + \frac{50}{s + 5}. \qquad (12.87)$$

Substituting Eq. 12.87 into Eq. 12.86 yields

$$F(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}. \tag{12.88}$$

Now we can inverse-transform Eq. 12.88 by inspection. Hence

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t). \tag{12.89}$$

✓ASSESSMENT PROBLEMS

Objective 2—Be able to calculate the inverse Laplace transform using partial fraction expansion and the Laplace transform table

12.8 Find $f(t)$ if

$$F(s) = \frac{(5s^2 + 29s + 32)}{(s+2)(s+4)}.$$

Answer: $f(t) = 5\delta(t) - (3e^{-2t} - 2e^{-4t})u(t)$.

NOTE: Also try Chapter Problem 12.42(c).

12.9 Find $f(t)$ if

$$F(s) = \frac{(2s^3 + 8s^2 + 2s - 4)}{(s^2 + 5s + 4)}.$$

Answer: $f(t) = 2\frac{d\delta(t)}{dt} - 2\delta(t) + 4e^{-4t}u(t)$.

12.8 Poles and Zeros of $F(s)$

The rational function of Eq. 12.42 also may be expressed as the ratio of two factored polynomials. In other words, we may write $F(s)$ as

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)}, \tag{12.90}$$

where K is the constant a_n/b_m. For example, we may also write the function

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

as

$$F(s) = \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)} = \frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}. \tag{12.91}$$

The roots of the denominator polynomial, that is, $-p_1, -p_2, -p_3, \ldots, -p_m$, are called the **poles of $F(s)$**; they are the values of s at which $F(s)$ becomes infinitely large. In the function described by Eq. 12.91, the poles of $F(s)$ are -1, -2, -3, and -4.

The roots of the numerator polynomial, that is, $-z_1, -z_2, -z_3, \ldots, -z_n$, are called the **zeros of $F(s)$**; they are the values of s at which $F(s)$ becomes zero. In the function described by Eq. 12.91, the zeros of $F(s)$ are -5 and -10.

In what follows, you may find that being able to visualize the poles and zeros of $F(s)$ as points on a complex s plane is helpful. A complex plane is needed because the roots of the polynomials may be complex. In the complex s plane, we use the horizontal axis to plot the real values of s and the vertical axis to plot the imaginary values of s.

As an example of plotting the poles and zeros of $F(s)$, consider the function

$$F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}. \quad (12.92)$$

The poles of $F(s)$ are at $0, -10, -6+j8$, and $-6-j8$. The zeros are at -5, $-3+j4$, and $-3-j4$. Figure 12.17 shows the poles and zeros plotted on the s plane, where X's represent poles and O's represent zeros.

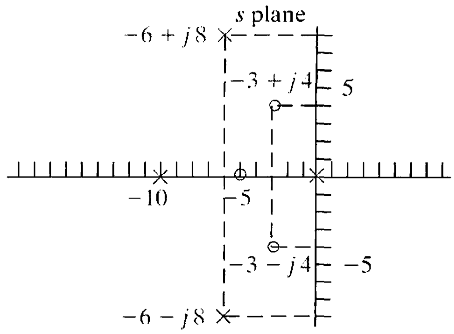


Figure 12.17 ▲ Plotting poles and zeros on the s plane.

Note that the poles and zeros for Eq. 12.90 are located in the finite s plane. $F(s)$ can also have either an rth-order pole or an rth-order zero at infinity. For example, the function described by Eq. 12.91 has a second-order zero at infinity, because for large values of s the function reduces to $4/s^2$, and $F(s) = 0$ when $s = \infty$. In this text, we are interested in the poles and zeros located in the finite s plane. Therefore, when we refer to the poles and zeros of a rational function of s, we are referring to the finite poles and zeros.

12.9 Initial- and Final-Value Theorems

The initial- and final-value theorems are useful because they enable us to determine from $F(s)$ the behavior of $f(t)$ at 0 and ∞. Hence we can check the initial and final values of $f(t)$ to see if they conform with known circuit behavior, before actually finding the inverse transform of $F(s)$.

The initial-value theorem states that

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s), \quad (12.93)$$

◀ **Initial value theorem**

and the final-value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \quad (12.94)$$

◀ **Final value theorem**

The initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions. In Eq. 12.94, we must add the restriction that the theorem is valid only if the poles of $F(s)$, except for a first-order pole at the origin, lie in the left half of the s plane.

To prove Eq. 12.93, we start with the operational transform of the first derivative:

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt. \quad (12.95)$$

Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty} [sF(s) - f(0^-)] = \lim_{s\to\infty} \int_{0}^{\infty} \frac{df}{dt} e^{-st}\, dt. \quad (12.96)$$

Observe that the right-hand side of Eq. 12.96 may be written as

$$\lim_{s\to\infty}\left(\int_{0^-}^{0^+}\frac{df}{dt}e^{0}dt + \int_{0^+}^{\infty}\frac{df}{dt}e^{-st}\,dt\right).$$

As $s \to \infty$, $(df/dt)e^{-st} \to 0$; hence the second integral vanishes in the limit. The first integral reduces to $f(0^+) - f(0^-)$, which is independent of s. Thus the right-hand side of Eq. 12.96 becomes

$$\lim_{s\to\infty}\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt = f(0^+) - f(0^-). \tag{12.97}$$

Because $f(0^-)$ is independent of s, the left-hand side of Eq. 12.96 may be written

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}[sF(s)] - f(0^-). \tag{12.98}$$

From Eqs. 12.97 and 12.98,

$$\lim_{s\to\infty} sF(s) = f(0^+) = \lim_{t\to 0^+} f(t),$$

which completes the proof of the initial-value theorem.

The proof of the final-value theorem also starts with Eq. 12.95. Here we take the limit as $s \to 0$:

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right). \tag{12.99}$$

The integration is with respect to t and the limit operation is with respect to s, so the right-hand side of Eq. 12.99 reduces to

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right) = \int_{0^-}^{\infty}\frac{df}{dt}dt. \tag{12.100}$$

Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

$$\int_{0^-}^{\infty}\frac{df}{dt}dt = \lim_{t\to\infty}\int_{0^-}^{t}\frac{df}{dy}dy, \tag{12.101}$$

where we use y as the symbol of integration to avoid confusion with the upper limit on the integral. Carrying out the integration process yields

$$\lim_{t\to\infty}[f(t) - f(0^-)] = \lim_{t\to\infty}[f(t)] - f(0^-). \tag{12.102}$$

Substituting Eq. 12.102 into Eq. 12.99 gives

$$\lim_{s\to 0}[sF(s)] - f(0^-) = \lim_{t\to\infty}[f(t)] - f(0^-). \tag{12.103}$$

Because $f(0^-)$ cancels, Eq. 12.103 reduces to the final-value theorem, namely,

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t).$$

The final-value theorem is useful only if $f(\infty)$ exists. This condition is true only if all the poles of $F(s)$, except for a simple pole at the origin, lie in the left half of the s plane.

The Application of Initial- and Final-Value Theorems

To illustrate the application of the initial- and final-value theorems, we apply them to a function we used to illustrate partial fraction expansions. Consider the transform pair given by Eq. 12.60. The initial-value theorem gives

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{100s^2[1 + (3/s)]}{s^3[1 + (6/s)][1 + (6/s) + (25/s^2)]} = 0,$$

$$\lim_{t\to 0^+} f(t) = [-12 + 20\cos(-53.13°)](1) = -12 + 12 = 0.$$

The final-value theorem gives

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{100s(s + 3)}{(s + 6)(s^2 + 6s + 25)} = 0,$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13°)]u(t) = 0.$$

In applying the theorems to Eq. 12.60, we already had the time-domain expression and were merely testing our understanding. But the real value of the initial- and final-value theorems lies in being able to test the s-domain expressions before working out the inverse transform. For example, consider the expression for $V(s)$ given by Eq. 12.40. Although we cannot calculate $v(t)$ until the circuit parameters are specified, we can check to see if $V(s)$ predicts the correct values of $v(0^+)$ and $v(\infty)$. We know from the statement of the problem that generated $V(s)$ that $v(0^+)$ is zero. We also know that $v(\infty)$ must be zero because the ideal inductor is a perfect short circuit across the dc current source. Finally, we know that the poles of $V(s)$ must lie in the left half of the s plane because R, L, and C are positive constants. Hence the poles of $sV(s)$ also lie in the left half of the s plane.

Applying the initial-value theorem yields

$$\lim_{s\to\infty} sV(s) = \lim_{s\to\infty} \frac{s(I_{dc}/C)}{s^2[1 + 1/(RCs) + 1/(LCs^2)]} = 0.$$

Applying the final-value theorem gives

$$\lim_{s\to 0} sV(s) = \lim_{s\to 0} \frac{s(I_{dc}/C)}{s^2 + (s/RC) + (1/LC)} = 0.$$

The derived expression for $V(s)$ correctly predicts the initial and final values of $v(t)$.

✓ASSESSMENT PROBLEM

Objective 3—Understand and know how to use the initial value theorem and the final value theorem

12.10 Use the initial- and final-value theorems to find the initial and final values of $f(t)$ in Assessment Problems 12.4, 12.6, and 12.7.

Answer: 7, 0; 4, 1; and 0, 0.

NOTE: Also try Chapter Problem 12.50.

Practical Perspective

Transient Effects

The circuit introduced in the Practical Perspective at the beginning of the chapter is repeated in Fig. 12.18 with the switch closed and the chosen sinusoidal source.

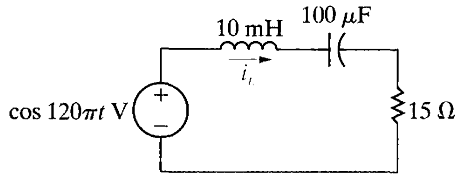


Figure 12.18 ▲ A series *RLC* circuit with a 60 Hz sinusoidal source.

We use the Laplace methods to determine the complete response of the inductor current, $i_L(t)$. To begin, use KVL to sum the voltages drops around the circuit, in the clockwise direction:

$$15i_L(t) + 0.01\frac{di_L(t)}{dt} + \frac{1}{100 \times 10^{-6}}\int_0^t i_L(x)dx = \cos 120\pi t \tag{12.104}$$

Now we take the Laplace transform of Eq. 12.104, using Tables 12.1 and 12.2:

$$15I_L(s) + 0.01sI_L(s) + 10^4\frac{I_L(s)}{s} = \frac{s}{s^2 + (120\pi)^2} \tag{12.105}$$

Next, rearrange the terms in Eq. 12.105 to get an expression for $I_L(s)$:

$$I_L(s) = \frac{100s^2}{\left[s^2 + 1500s + 10^6\right]\left[s^2 + (120\pi^2)\right]} \tag{12.106}$$

Note that the expression for $I_L(s)$ has two complex conjugate pairs of poles, so the partial fraction expansion of $I_L(s)$ will have four terms:

$$I_L(s) = \frac{K_1}{(s + 750 - j661.44)} + \frac{K_1^*}{(s + 750 + j661.44)} + \frac{K_2}{(s - j120\pi)} + \frac{K_2^*}{(s + j120\pi)} \tag{12.107}$$

Determine the values of K_1 and K_2:

$$K_1 = \left.\frac{100s^2}{\left[s + 750s + j661.44\right]\left[s^2 + (120\pi)^2\right]}\right|_{s=-750+j661.44} = 0.07357\angle -97.89^\circ$$

$$K_2 = \left.\frac{100s^2}{\left[s^2 + 1500s + 10^6\right]\left[s + j120\pi\right]}\right|_{s=j120\pi} = 0.018345\angle 56.61^\circ \tag{12.108}$$

Finally, we can use Table 12.3 to calculate the inverse Laplace transform of Eq. 12.107 to give $i_L(t)$:

$$i_L(t) = 147.14e^{-750t}\cos(661.44t - 97.89^\circ) + 36.69\cos(120\pi t + 56.61^\circ)\,\text{mA} \tag{12.109}$$

The first term of Eq. 12.109 is the transient response, which will decay to essentially zero in about 7 ms. The second term of Eq. 12.109 is the steady-state response, which has the same frequency as the 60 Hz sinusoidal source and will persist so long as this source is connected in the circuit. Note that the amplitude of the steady-state response is 36.69 mA, which is less than the 40 mA current rating of the inductor. But the transient response has an initial amplitude of 147.14 mA, far greater than the 40 mA current rating. Calculate the value of the inductor current at $t = 0$:

$$i_L(0) = 147.14(1)\cos(-97.89°) + 36.69\cos(56.61°) = -6.21\mu\text{A}$$

Clearly, the transient part of the response does not cause the inductor current to exceed its rating initially. But we need a plot of the complete response to determine whether or not the current rating is ever exceeded, as shown in Fig. 12.19. The plot suggests we check the value of the inductor current at 1 ms:

$$i_L(0.001) = 147.14e^{-0.75}\cos(-59.82°) + 36.69\cos(78.21°) = 42.6\text{ mA}$$

Thus, the current rating is exceeded in the inductor, at least momentarily. If we determine that we never want to exceed the current rating, we should reduce the magnitude of the sinusoidal source. This example illustrates the importance of considering the complete response of a circuit to a sinusoidal input, even if we are satisfied with the steady-state response.

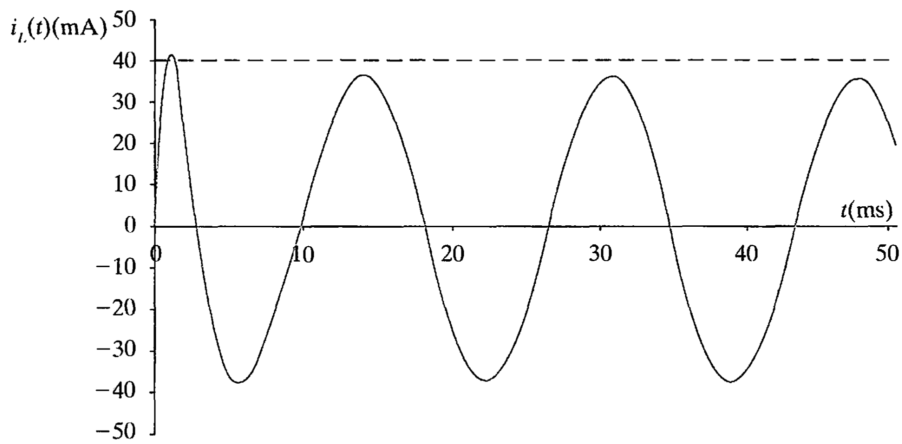


Figure 12.19 ▲ Plot of the inductor current for the circuit in Fig. 12.18.

NOTE: Access your understanding of the Practical Perspective by trying Chapter Problems 12.55 and 12.56.

Summary

- The **Laplace transform** is a tool for converting time-domain equations into frequency-domain equations, according to the following general definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s),$$

  where $f(t)$ is the time-domain expression, and $F(s)$ is the frequency-domain expression. (See page 430.)
- The **step function** $Ku(t)$ describes a function that experiences a discontinuity from one constant level to another at some point in time. K is the magnitude of the jump; if $K = 1$, $Ku(t)$ is the **unit step function**. (See page 431.)
- The **impulse function** $K\delta(t)$ is defined

$$\int_{-\infty}^{\infty} K\delta(t)dt = K,$$

$$\delta(t) = 0, \quad t \neq 0.$$

  K is the strength of the impulse; if $K = 1$, $K\delta(t)$ is the **unit impulse function**. (See page 433.)
- A **functional transform** is the Laplace transform of a specific function. Important functional transform pairs are summarized in Table 12.1. (See page 436.)
- **Operational transforms** define the general mathematical properties of the Laplace transform. Important operational transform pairs are summarized in Table 12.2. (See page 437.)
- In linear lumped-parameter circuits, $F(s)$ is a rational function of s. (See page 444.)
- If $F(s)$ is a proper rational function, the inverse transform is found by a partial fraction expansion. (See page 444.)
- If $F(s)$ is an improper rational function, it can be inverse-transformed by first expanding it into a sum of a polynomial and a proper rational function. (See page 453.)

- $F(s)$ can be expressed as the ratio of two factored polynomials. The roots of the denominator are called **poles** and are plotted as Xs on the complex s plane. The roots of the numerator are called **zeros** and are plotted as 0s on the complex s plane. (See page 454.)
- The initial-value theorem states that

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s).$$

The theorem assumes that $f(t)$ contains no impulse functions. (See page 455.)

- The final-value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s).$$

The theorem is valid only if the poles of $F(s)$, except for a first-order pole at the origin, lie in the left half of the s plane. (See page 455.)

- The initial- and final-value theorems allow us to predict the initial and final values of $f(t)$ from an s-domain expression. (See page 457.)

Problems

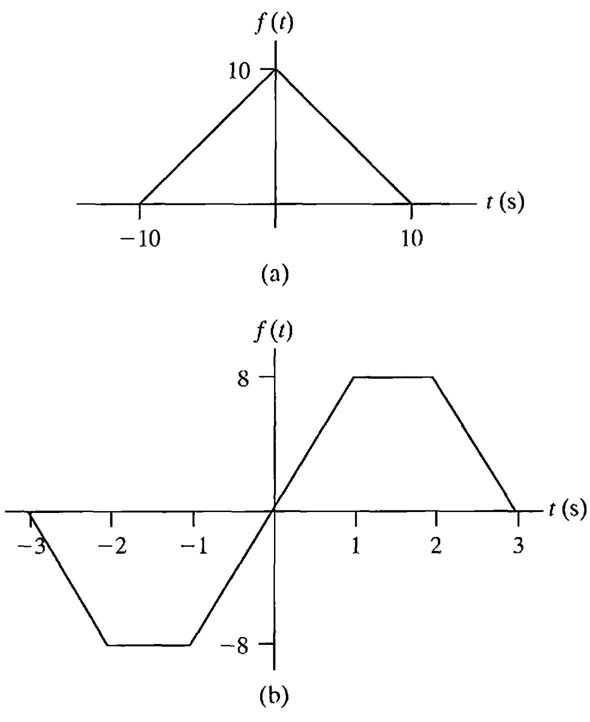
Section 12.2

12.1 Make a sketch of $f(t)$ for $-10\text{ s} \le t \le 30\text{ s}$ when $f(t)$ is given by the following expression:

$$\begin{aligned} f(t) = {} & (10t + 100)u(t + 10) - (10t + 50)u(t + 5) \\ & + (50 - 10t)u(t - 5) \\ & - (150 - 10t)u(t - 15) + (10t - 250)u(t - 25) \\ & - (10t - 300)u(t - 30) \end{aligned}$$

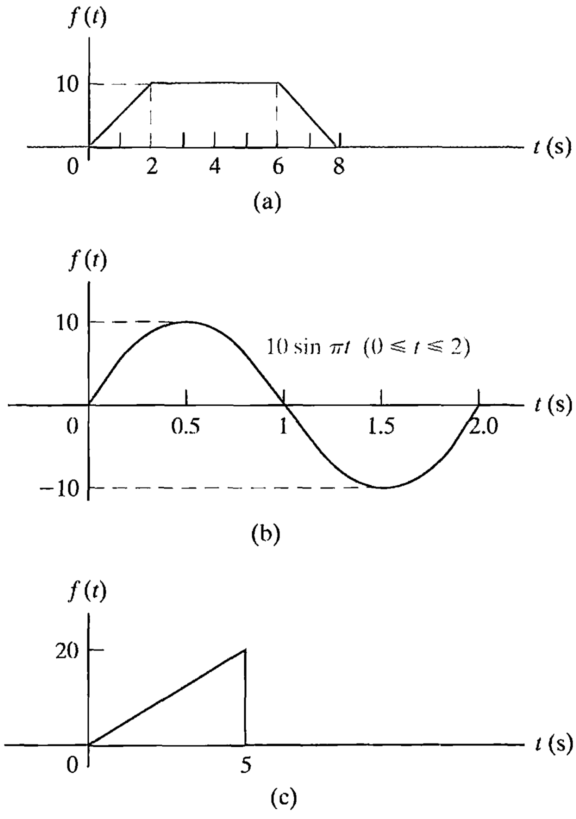
12.2 Use step functions to write the expression for each of the functions shown in Fig. P12.2.

Figure P12.2



12.3 Use step functions to write the expression for each function shown in Fig. P12.3.

Figure P12.3



12.4 Step functions can be used to define a *window* function. Thus $u(t - 1) - u(t - 4)$ defines a window 1 unit high and 3 units wide located on the time axis between 1 and 4.

A function $f(t)$ is defined as follows:

$$
\begin{aligned}
f(t) &= 0, & t &\leq 0 \\
&= -20t, & 0 &\leq t \leq 1\text{ s} \\
&= -20, & 1\text{ s} &\leq t \leq 2\text{ s} \\
&= 20\cos\frac{\pi}{2}t, & 2\text{ s} &\leq t \leq 4\text{ s}; \\
&= 100 - 20t & 4\text{ s} &\leq t \leq 5\text{ s} \\
&= 0, & 5\text{ s} &\leq t < \infty.
\end{aligned}
$$

a) Sketch $f(t)$ over the interval $-1\text{ s} \leq t \leq 6\text{ s}$.
b) Use the concept of the window function to write an expression for $f(t)$.

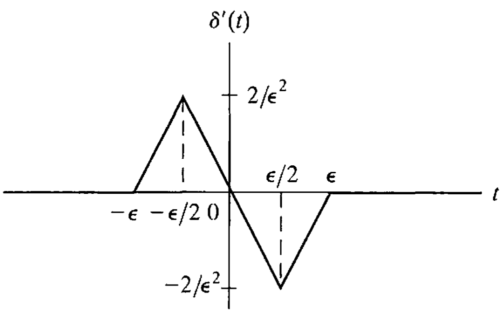
Section 12.3

12.5 Explain why the following function generates an impulse function as $\epsilon \rightarrow 0$:

$$f(t) = \frac{\epsilon/\pi}{\epsilon^2 + t^2}, \quad -\infty \leq t \leq \infty.$$

12.6 The triangular pulses shown in Fig. P12.6 are equivalent to the rectangular pulses in Fig. 12.12(b), because they both enclose the same area ($1/\epsilon$) and they both approach infinity proportional to $1/\epsilon^2$ as $\epsilon \rightarrow 0$. Use this triangular-pulse representation for $\delta'(t)$ to find the Laplace transform of $\delta''(t)$.

Figure P12.6



12.7 a) Find the area under the function shown in Fig. 12.12(a).
b) What is the duration of the function when $\epsilon = 0$?
c) What is the magnitude of $f(0)$ when $\epsilon = 0$?

12.8 In Section 12.3, we used the sifting property of the impulse function to show that $\mathscr{L}\{\delta(t)\} = 1$. Show that we can obtain the same result by finding the Laplace transform of the rectangular pulse that exists between $\pm\epsilon$ in Fig. 12.9 and then finding the limit of this transform as $\epsilon \rightarrow 0$.

12.9 Evaluate the following integrals:

a) $I = \int_{-1}^{3} (t^3 + 2)[\delta(t) + 8\delta(t - 1)]\,dt.$

b) $I = \int_{-2}^{2} t^2[\delta(t) + \delta(t + 1.5) + \delta(t - 3)]\,dt.$

12.10 Find $f(t)$ if

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{jt\omega}\,d\omega,$$

and

$$F(\omega) = \frac{4 + j\omega}{9 + j\omega}\pi\delta(\omega).$$

12.11 Show that

$$\mathscr{L}\{\delta^{(n)}(t)\} = s^n.$$

12.12 a) Show that

$$\int_{-\infty}^{\infty} f(t)\delta'(t - a)\,dt = -f'(a).$$

(*Hint:* Integrate by parts.)

b) Use the formula in (a) to show that

$$\mathscr{L}\{\delta'(t)\} = s.$$

Sections 12.4–12.5

12.13 Show that

$$\mathscr{L}\{e^{-at}f(t)\} = F(s + a).$$

12.14 a) Find $\mathscr{L}\left\{\frac{d}{dt}\sin\omega t\right\}$.

b) Find $\mathscr{L}\left\{\frac{d}{dt}\cos\omega t\right\}$.

c) Find $\mathscr{L}\left\{\frac{d^3}{dt^3}t^2u(t)\right\}$.

d) Check the results of parts (a), (b), and (c) by first differentiating and then transforming.

12.15 a) Find the Laplace transform of

$$\int_{0^-}^{t} x\,dx$$

by first integrating and then transforming.

b) Check the result obtained in (a) by using the operational transform given by Eq. 12.33.

12.16 Show that

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

12.17 Find the Laplace transform of each of the following functions:

a) $f(t) = te^{-at}$;

b) $f(t) = \sin \omega t$;

c) $f(t) = \sin(\omega t + \theta)$;

d) $f(t) = t$;

e) $f(t) = \cosh(t + \theta)$.

(*Hint*: See Assessment Problem 12.1.)

12.18 Find the Laplace transform (when $\epsilon \to 0$) of the derivative of the exponential function illustrated in Fig. 12.8, using each of the following two methods:

a) First differentiate the function and then find the transform of the resulting function.

b) Use the operational transform given by Eq. 12.23.

12.19 Find the Laplace transform of each of the following functions:

a) $f(t) = 40e^{-8(t-3)}u(t-3)$.

b) $f(t) = (5t - 10)[u(t-2) - u(t-4)] + (30 - 5t)[u(t-4) - u(t-8)] + (5t - 50)[u(t-8) - u(t-10)]$.

12.20 a) Find the Laplace transform of te^{-at}.

b) Use the operational transform given by Eq. 12.23 to find the Laplace transform of $\frac{d}{dt}(te^{-at})$.

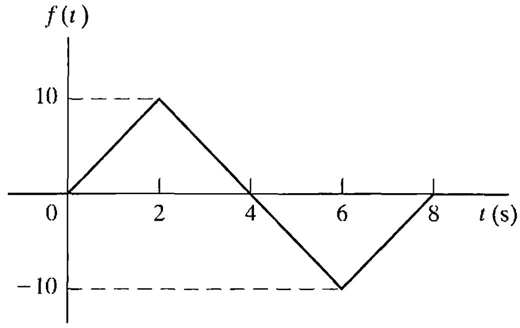
c) Check your result in part (b) by first differentiating and then transforming the resulting expression.

12.21 a) Find the Laplace transform of the function illustrated in Fig. P12.21.

b) Find the Laplace transform of the first derivative of the function illustrated in Fig. P12.21.

c) Find the Laplace transform of the second derivative of the function illustrated in Fig. P12.21.

Figure P12.21



12.22 a) Find $\mathcal{L}\left\{\int_{0-}^{t} e^{-ax}\,dx\right\}$.

b) Check the results of (a) by first integrating and then transforming.

12.23 a) Given that $F(s) = \mathcal{L}\{f(t)\}$, show that

$$-\frac{dF(s)}{ds} = \mathcal{L}\{tf(t)\}.$$

b) Show that

$$(-1)^n \frac{d^nF(s)}{ds^n} = \mathcal{L}\{t^n f(t)\}.$$

c) Use the result of (b) to find $\mathcal{L}\{t^5\}$, $\mathcal{L}\{t \sin \beta t\}$, and $\mathcal{L}\{te^{-t}\cosh t\}$.

12.24 a) Show that if $F(s) = \mathcal{L}\{f(t)\}$, and $\{f(t)/t\}$ is Laplace-transformable, then

$$\int_s^{\infty} F(u)du = \mathcal{L}\left\{\frac{f(t)}{t}\right\}.$$

(*Hint:* Use the defining integral to write

$$\int_s^{\infty} F(u)du = \int_s^{\infty}\left(\int_{0^-}^{\infty} f(t)e^{-ut}dt\right)du$$

and then reverse the order of integration.)

b) Start with the result obtained in Problem 12.23(c) for $\mathcal{L}\{t \sin \beta t\}$ and use the operational transform given in (a) of this problem to find $\mathcal{L}\{\sin \beta t\}$.

12.25 Find the Laplace transform for (a) and (b).

a) $f(t) = \frac{d}{dt}(e^{-at} \sin \omega t)$.

b) $f(t) = \int_{0^-}^{t} e^{-ax} \cos \omega x \, dx$.

c) Verify the results obtained in (a) and (b) by first carrying out the indicated mathematical operation and then finding the Laplace transform.

Section 12.6

12.26 In the circuit shown in Fig. 12.16, the dc current source is replaced with a sinusoidal source that delivers a current of $1.2 \cos t$ A. The circuit components are $R = 1\ \Omega$, $C = 625$ mF, and $L = 1.6$ H. Find the numerical expression for $V(s)$.

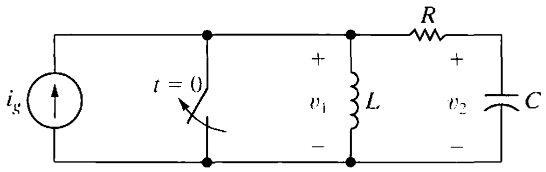
12.27 There is no energy stored in the circuit shown in Fig. P12.27 at the time the switch is opened.

a) Derive the integrodifferential equations that govern the behavior of the node voltages v_1 and v_2.

b) Show that

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}.$$

Figure P12.27



12.28 The switch in the circuit in Fig. P12.28 has been open for a long time. At $t = 0$, the switch closes.

a) Derive the integrodifferential equation that governs the behavior of the voltage v_o for $t \geq 0$.

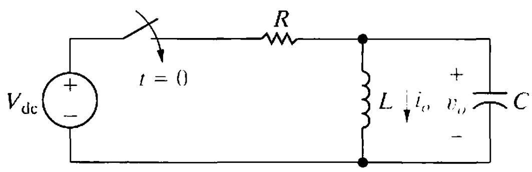
b) Show that

$$V_o(s) = \frac{V_{dc}/RC}{s^2 + (1/RC)s + (1/LC)}.$$

c) Show that

$$I_o(s) = \frac{V_{dc}/RLC}{s[s^2 + (1/RC)s + (1/LC)]}.$$

Figure P12.28



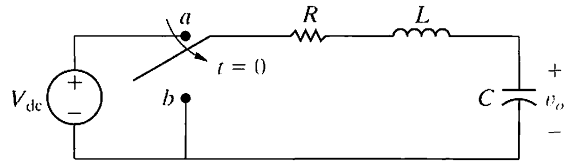
12.29 The switch in the circuit in Fig. P12.29 has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.

a) Derive the integrodifferential equation that governs the behavior of the voltage v_o for $t \geq 0^+$.

b) Show that

$$V_o(s) = \frac{V_{dc}[s + (R/L)]}{[s^2 + (R/L)s + (1/LC)]}.$$

Figure P12.29



12.30 There is no energy stored in the circuit shown in Fig. P12.30 at the time the switch is opened.

a) Derive the integrodifferential equation that governs the behavior of the voltage v_o.

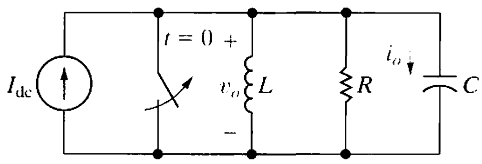
b) Show that

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

c) Show that

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}.$$

Figure P12.30



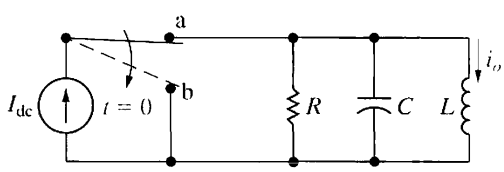
12.31 The switch in the circuit in Fig. P12.31 has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.

a) Derive the integrodifferential equation that governs the behavior of the current i_o for $t \geq 0^+$.

b) Show that

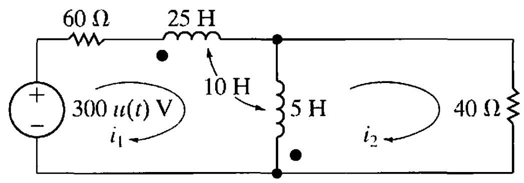
$$I_o(s) = \frac{I_{dc}[s + (1/RC)]}{[s^2 + (1/RC)s + (1/LC)]}.$$

Figure P12.31



12.32 PSPICE MULTISIM
a) Write the two simultaneous differential equations that describe the circuit shown in Fig. P12.32 in terms of the mesh currents i_1 and i_2.
b) Laplace-transform the equations derived in (a). Assume that the initial energy stored in the circuit is zero.
c) Solve the equations in (b) for $I_1(s)$ and $I_2(s)$.

Figure P12.32



Section 12.7

12.33 Find $v(t)$ in Problem 12.26.

12.34 PSPICE MULTISIM
The circuit parameters in the circuit in Fig. P12.27 are $R = 2500\ \Omega$; $L = 500$ mH; and $C = 0.5\ \mu$F. If $i_g(t) = 15$ mA, find $v_2(t)$.

12.35 PSPICE MULTISIM
The circuit parameters in the circuit in Fig. P12.28 are $R = 5\ \text{k}\Omega$; $L = 200$ mH; and $C = 100$ nF. If V_{dc} is 35 V, find
a) $v_o(t)$ for $t \geq 0$
b) $i_o(t)$ for $t \geq 0$

12.36 The circuit parameters in the circuit in Fig. P12.29 are $R = 250\ \Omega$, $L = 50$ mH, and $C = 5\ \mu$F. If $V_{dc} = 48$ V, find $v_o(t)$ for $t \geq 0$.

12.37 PSPICE MULTISIM
The circuit parameters in the circuit seen in Fig. P12.30 have the following values: $R = 1\ \text{k}\Omega$, $L = 12.5$ H, $C = 2\ \mu$F, and $I_{dc} = 30$ mA.
a) Find $v_o(t)$ for $t \geq 0$.
b) Find $i_o(t)$ for $t \geq 0$.
c) Does your solution for $i_o(t)$ make sense when $t = 0$? Explain.

12.38 PSPICE MULTISIM
The circuit parameters in the circuit in Fig. P12.31 are $R = 500\ \Omega$, $L = 250$ mH, and $C = 250$ nF. If $I_{dc} = 5$ mA, find $i_o(t)$ for $t \geq 0$.

12.39 Use the results from Problem 12.32 and the circuit shown in Fig P12.32 to
a) Find $i_1(t)$ and $i_2(t)$.
b) Find $i_1(\infty)$ and $i_2(\infty)$.
c) Do the solutions for i_1 and i_2 make sense? Explain.

12.40 Find $f(t)$ for each of the following functions:

a) $$F(s) = \frac{8s^2 + 37s + 32}{(s + 1)(s + 2)(s + 4)}.$$

b) $$F(s) = \frac{13s^3 + 134s^2 + 392s + 288}{s(s + 2)(s^2 + 10s + 24)}.$$

c) $$F(s) = \frac{20s^2 + 16s + 12}{(s + 1)(s^2 + 2s + 5)}.$$

d) $$F(s) = \frac{250(s + 7)(s + 14)}{s(s^2 + 14s + 50)}.$$

12.41 Find $f(t)$ for each of the following functions.

a) $$F(s) = \frac{100}{s^2(s + 5)}.$$

b) $$F(s) = \frac{50(s + 5)}{s(s + 1)^2}.$$

c) $$F(s) = \frac{100(s + 3)}{s^2(s^2 + 6s + 10)}.$$

d) $$F(s) = \frac{5(s + 2)^2}{s(s + 1)^3}.$$

e) $$F(s) = \frac{400}{s(s^2 + 4s + 5)^2}.$$

12.42 Find $f(t)$ for each of the following functions.

a) $$F(s) = \frac{5s^2 + 38s + 80}{s^2 + 6s + 8}.$$

b) $$F(s) = \frac{10s^2 + 512s + 7186}{s^2 + 48s + 625}.$$

c) $$F(s) = \frac{s^3 + 5s^2 - 50s - 100}{s^2 + 15s + 50}.$$

12.43 Find $f(t)$ for each of the following functions.

a) $$F(s) = \frac{100(s + 1)}{s^2(s^2 + 2s + 5)}.$$

b) $$F(s) = \frac{500}{s(s + 5)^3}.$$

c) $F(s) = \dfrac{40(s + 2)}{s(s + 1)^3}$

d) $F(s) = \dfrac{(s + 5)^2}{s(s + 1)^4}$

12.44 Derive the transform pair given by Eq. 12.64.

12.45 a) Derive the transform pair given by Eq. 12.83.
b) Derive the transform pair given by Eq. 12.84.

Sections 12.8–12.9

12.46 a) Use the initial-value theorem to find the initial value of v in Problem 12.26.

b) Can the final-value theorem be used to find the steady-state value of v? Why?

12.47 Use the initial- and final-value theorems to check the initial and final values of the current and voltage in Problem 12.28.

12.48 Use the initial- and final-value theorems to check the initial and final values of the current and voltage in Problem 12.30.

12.49 Use the initial- and final-value theorems to check the initial and final values of the current in Problem 12.31.

12.50 Apply the initial- and final-value theorems to each transform pair in Problem 12.40.

12.51 Apply the initial- and final-value theorems to each transform pair in Problem 12.41.

12.52 Apply the initial- and final-value theorems to each transform pair in Problem 12.42.

12.53 Apply the initial- and final-value theorems to each transform pair in Problem 12.43.

Sections 12.1–12.9

12.54 a) Use phasor circuit analysis techniques from Chapter 9 to determine the steady-state expression for the inductor current in Fig. 12.18.

b) How does your result in part (a) compare to the complete response as given in Eq. 12.109?

12.55 Find the maximum magnitude of the sinusoidal source in Fig. 12.18 such that the complete response of the inductor current does not exceed the 40 mA current rating at $t = 1$ ms.

12.56 Suppose the input to the circuit in Fig 12.18 is a damped ramp of the form Kte^{-100t} V. Find the largest value of K such that the inductor current does not exceed the 40 mA current rating.