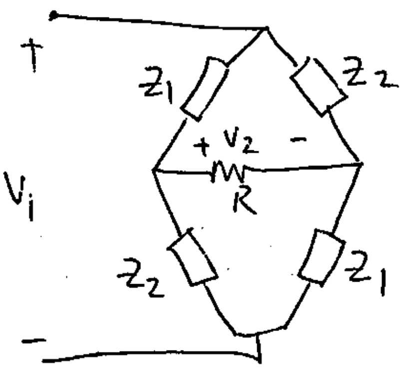
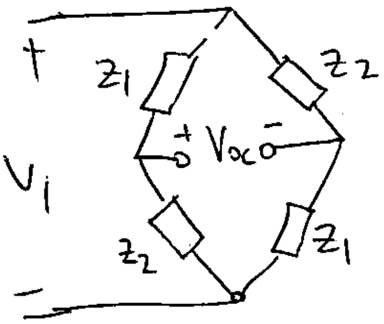
2^{nd} Order All-Pass Circuit:



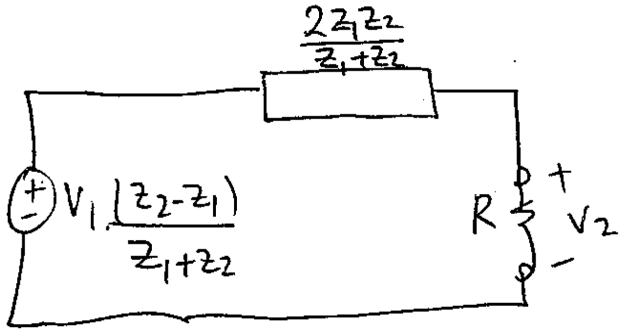
i) Z_{Th} seen by R $\Rightarrow (Z_1 \| Z_2)2$.

ii)



$$V_{oc} = V_1\left(\frac{Z_2 - Z_1}{Z_1 + Z_2}\right)$$

Then.



$$\frac{V_2(s)}{V_1(s)} = ?$$

$$\frac{V_2(s)}{V_1(s)} = \frac{(Z_2^{(s)} - Z_1(s))}{Z_1^{(s)} + Z_2(s)} \cdot \frac{R}{R + \frac{2Z_1Z_2}{Z_1+Z_2}} = (Z_2 - Z_1) \cdot \frac{R}{R(Z_1+Z_2) + 2Z_1Z_2}$$

$$= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + 2\frac{Z_1Z_2}{R}}$$

$\boxed{\text{Let } Z_1Z_2 = R^2}$ $\xrightarrow{\text{then}}$

$$= \frac{(Z_2 - Z_1)}{(Z_1 + Z_2) + 2R}$$

$$= \frac{Z_2(Z_2 - Z_1)}{Z_2\{(Z_1 + Z_2) + 2R\}}$$

$$\boxed{A(s) = \frac{Z_2 - R}{Z_2 + R}} = \frac{(Z_2 - R)(Z_2 + R)}{(Z_2 + R)^2} = \frac{Z_2^2 - R^2}{Z_2^2 + 2RZ_2 + R^2}$$

then

$$H(j\omega) = \frac{Z_2(j\omega) - R}{Z_2(j\omega) + R},$$

then ① $Z_2(j\omega)$: purely real $\forall \omega \xrightarrow[Z_2 = R_2]{} H(j\omega) = \dfrac{R_2 - R}{R_2 + R}.$

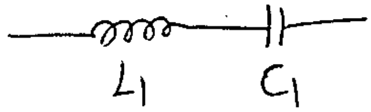
$\rightarrow |H(j\omega)|$ is not function of $\omega \rightarrow$ but system does not contain any dynamic elements. Therefore it is just a voltage divider, not a filter.

② $Z_2(j\omega)$: purely imaginary $\forall \omega \xrightarrow[Z_2 = jX_2]{} H(j\omega) = \dfrac{jX_2 - R}{jX_2 + R}$

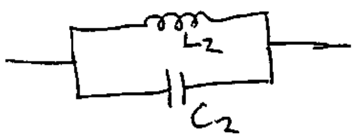
$\rightarrow |H(j\omega)| = 1 \rightarrow$ we have dynamic system with an all-pass structure.

~~Then~~ Example: How to select Z_1 and Z_2 such that $Z_1 Z_2 = R^2$ is satisfied?

Z_1 :



Z_2 :



$$\left\{ \begin{array}{l} Z_1(s) = sL_1 + 1/sC_1 \\ Z_2(s) = \left(\dfrac{1}{sL_2} + sC_2\right)^{-1} \end{array} \right.$$

$$Z_1 \cdot Z_2 = R^2 \rightarrow Z_1 = \frac{R^2}{Z_2} \rightarrow sL_1 + \frac{1}{sC_1} = R^2\left(\frac{1}{sL_2} + sC_2\right)$$

$\downarrow$

Let

$$\boxed{\begin{array}{l} L_1 = R^2 C_2 \\ C_1 = L_2 / R^2 \end{array}}$$

then $H(s) = \dfrac{Z_2 - R}{Z_2 + R} = \dfrac{1 - RY_2}{1 + RY_2}$ $\left(Z_2^{-1} = Y_2 = \dfrac{1}{sL_2} + sC_2 \right)$

$$= \frac{1 - \dfrac{R}{sL_2} - sRC_2}{1 + \dfrac{R}{sL_2} + sRC_2}$$

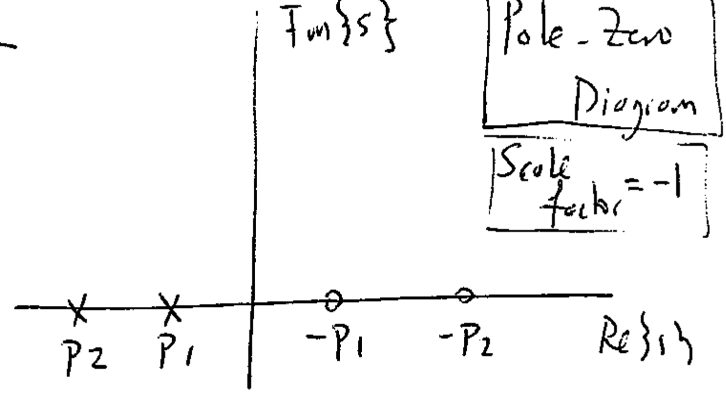
$$= - \frac{s^2 - \dfrac{1}{RC_2}s + \dfrac{1}{L_2C_2}}{s^2 + \underbrace{\dfrac{1}{RC_2}}_{2\alpha w_0}s + \underbrace{\dfrac{1}{L_2C_2}}_{w_0^2}}$$

$$H(s) = - \frac{s^2 - 2\alpha w_0 s + w_0^2}{s^2 + 2\alpha w_0 s + w_0^2}$$

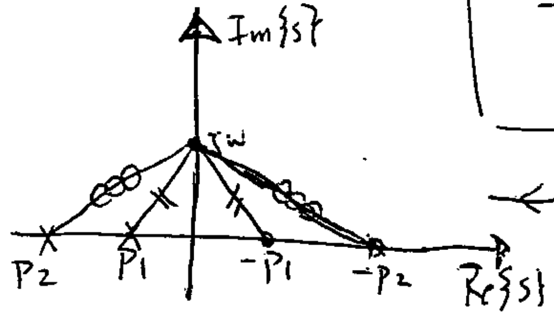
<u>Note</u>: If s_1, s_2 are roots of $s^2 + 2\alpha w_0 s + w_0^2 = 0$.
then: $-s_1, -s_2$ are the roots of $s^2 - 2\alpha w_0 s + w_0^2 = 0$.
(Why?: (Hint: $\mp 2\alpha w_0$ = sum of roots ; w_0^2: product of roots))

then.

$$H(s) = - \frac{(s - P_1)(s - P_2)}{(s + P_1)(s + P_2)}$$

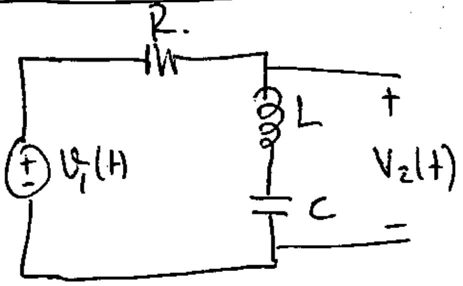


Note:



← this picture shows that system is all-pass!

Series RLC Band-Stop Filter

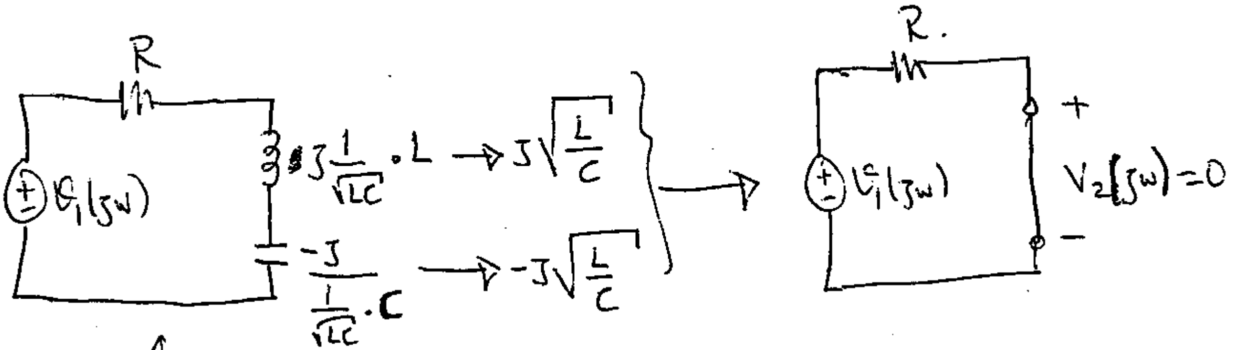


$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{sL + 1/sC}{sL + 1/sC + R}$$

$$= \frac{s^2 + 1/LC}{s^2 + \frac{R}{L}s + 1/LC}.$$

$$H(j\omega) = \frac{\left(1/LC - \omega^2\right)}{\left(1/LC - \omega^2\right) + j\omega\frac{R}{L}} \longrightarrow \text{Note: } |H(j\omega)| = 0 \text{ for } \left\{\omega = \pm\sqrt{1/LC}\right\}$$

Let's redraw the circuit at $\omega = 1/\sqrt{LC}$

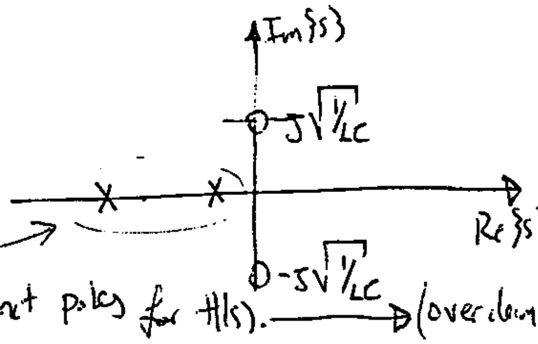


$\omega = \frac{1}{\sqrt{LC}}$

Note that $v_1(j\omega)$ source sees purely resistive circuit at $\omega = \frac{1}{\sqrt{LC}}$

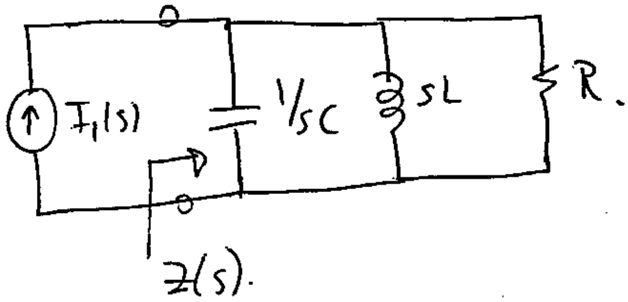
This phenomena is called <u>resonance</u>. (more on this later).

Pole-Zero Diagram



Assuming two distinct poles for $H(s)$. $\longrightarrow$ (overdamped system).

Parallel RLC Resonance Circuit:



$$Z(s) = \frac{1}{1/R + 1/sL + sC} = \frac{s/C}{s^2 + \frac{1}{RC}s + 1/LC}$$

Definition: The frequency for which $Z(j\omega)$ is purely real is called the resonance frequency.

$$Z(j\omega) = \frac{j\omega/C}{-\omega^2 + 1/LC + j\omega\, 1/RC} = \frac{\omega/C}{j(\omega^2 - 1/LC) + \omega\, 1/RC}$$

is zero when $\omega = 1/\sqrt{LC}$ Resonance freq.

(We have studied this system under the title of 2nd Order Band-Pass circuits

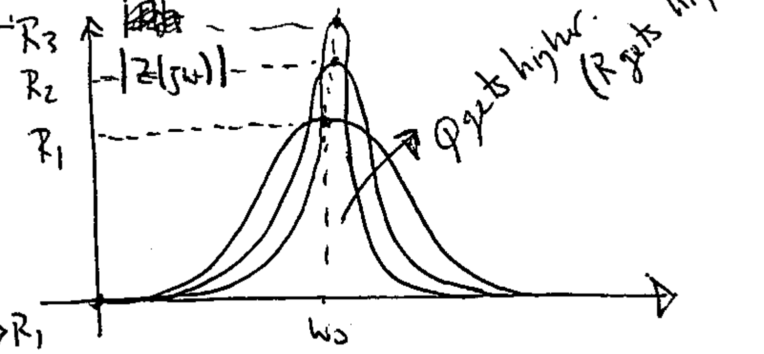
$$\underbrace{s^2 + \frac{1}{RC}}_{2\gamma\omega_0} s + \underbrace{1/LC}_{\omega_0^2} \longrightarrow$$

$$BW = \underbrace{\omega_2 - \omega_1}_{-3\text{dB points}} = 2\gamma\omega_0 = 1/RC$$

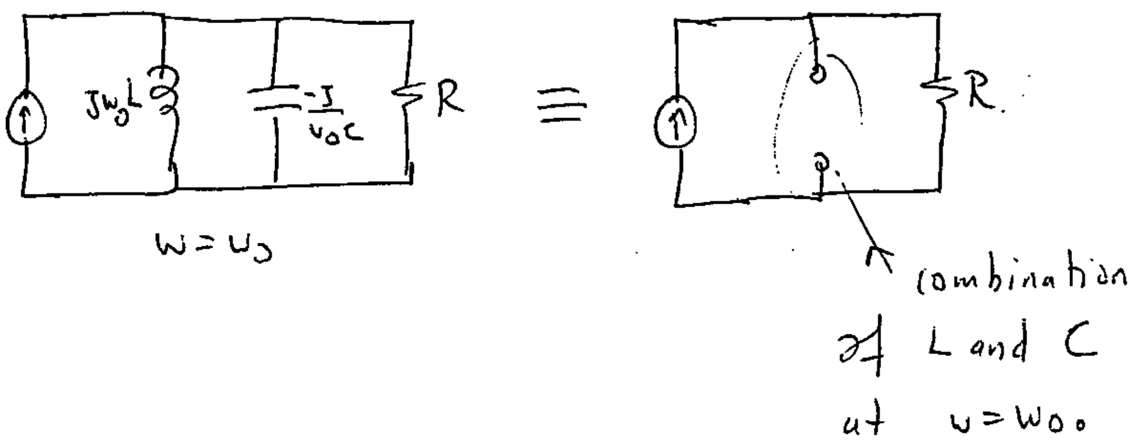
$$Q = \frac{\omega_0}{BW} = \frac{1}{2\gamma} = R\sqrt{\frac{C}{L}}$$

$$\omega_0 = \sqrt{1/LC}\,, \quad 2\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Picture shows the impedance seen by the source for 3 parallel RLC circuits with $R_3 > R_2 > R_1$ and the same L and C's.



The maximum impedance is seen at resonance freq. (for parallel RLC) and it is equal to R. Note that



Another Interpretation for Q: ← Quality factor.

For parallel RLC circuit, the average energy stored in capacitor is $E_C = \frac{1}{2} C V_{eff}^2$; similarly the average magnetic energy stored in inductor is $E_L = \frac{1}{2} L I_{eff}^2 \Rightarrow$

Let's calculate the total energy stored in L and C.

$$E_T = E_C + E_L = \frac{1}{2} C V_{eff}^2 + \frac{1}{2} L \left(\frac{v_{eff}}{wL}\right)^2$$

$$= \frac{1}{2}\left(C + \frac{1}{w^2 L}\right) V_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \frac{1}{w^2 \underbrace{LC}_{w_0^2}}\right) v_{eff}^2$$

$$= \frac{1}{2} C \left(1 + \frac{w_0^2}{w^2}\right) v_{eff}^2$$

$$= \frac{1}{2} C \frac{w_0}{w}\left(\frac{w}{w_0} + \frac{w_0}{w}\right) v_{eff}^2$$

We will now show that at $w = w_0$

4A

$$Q = 2\pi \cdot \frac{E_T}{T P_R} \quad \left(\substack{Q: \\ \text{unitless}}\right) \text{ where } \quad T = \frac{2\pi}{w_0}$$

where P_R is the average ~~power~~ power consumed by R and $T.P_R$ is the energy dissipated over R in a period.

Now

$$2\pi \frac{E_T}{T.P_R} = w \frac{E_T}{P_R} = w \frac{\frac{1}{2} C \frac{w_0}{w}\left(\frac{w}{w_0} + \frac{w_0}{w}\right) \cancel{V_{eff}^2}}{\frac{\cancel{V_{eff}^2}}{R}}$$

$$= \frac{RCw_0}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

$1/\sqrt{LC}$

$$= \frac{R\sqrt{\frac{C}{L}}}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

$$= \frac{Q}{2}\left(\frac{w}{w_0} + \frac{w_0}{w}\right)$$

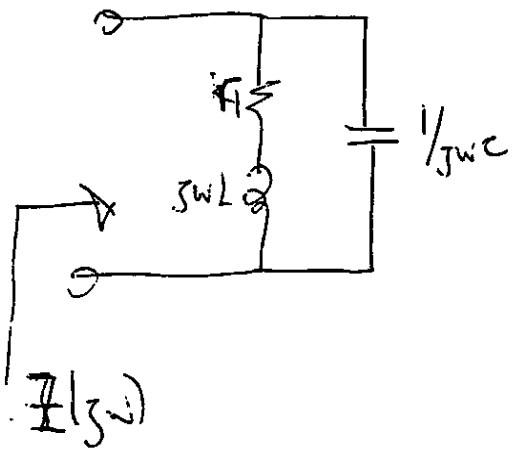
Result we're trying to show!

for $w = w_0 \longrightarrow 2\pi \frac{E_T}{T.P_R} = Q$, $\quad T = \frac{2\pi}{w_0}$.

Note: Resonant freq. is the freq. where maximum of $|Z(jw)|$ for parallel RLC circuit; BUT this is not true in general. If w_m is the freq. of maxima of $|Z(jw)|$ $w_m \neq w_0$ in general.

Finite Q capacitors and Inductors:

In practice, it is not possible to have inductors which does not have any internal resistance. (Remember inductor ≡ coil) Then any resonance circuit, that is containing both capacitors and inductors are effected by the internal resistance value. Since resonance circuits are operated around the resonant freq, our analysis is focused on the behaviour around resonant freq.



$$Z(jw) = \frac{1}{\frac{1}{r_1 + jwL} + jwC}$$

$$Z(s) = \frac{1}{\frac{1}{r_1 + sL} + sC}$$

$$Z(s) = \frac{s/C + r_1/LC}{s^2 + s\underbrace{(R_1/L)}_{2\gamma w_a} + \underbrace{(1/LC)}_{w_a^2}}$$

$$s^2 + s\frac{R_1}{L} + \frac{1}{LC} = s^2 + (2\gamma w_a)s + w_a^2$$

$BW = \frac{w_a}{Q}$

Q of the system. $(Q = 1/2\gamma)$

w_a is not the resonance freq. or it is not the frequency for which $Z(jw)$ has a maxima.

(5A)

$$Z(s)\Big|_{s=jw} = \frac{jw/C + r_1/LC}{\underbrace{(1/LC}_{w_a^2} - w^2) + jw \underbrace{(R_1/L)}_{w_a/Q}} = \frac{\frac{1}{C}\left(jw + \overbrace{r_1/L}^{w_a/Q}\right)}{(w_a^2 - w^2) + jw\, w_a/Q}$$

$$= \frac{\frac{1}{C}\left(jw + w_a/Q\right)}{(w_a^2 - w^2) + jw\, w_a/Q}$$

$\boxed{\text{Resonant Freq. Calcolation:}}$

for $Z(jw)$ to be purely real, angle of numerator and denominator of $Z(jw)$ should be same $\rightarrow$ so

$$\frac{\cancel{w}}{w_a/Q} = \frac{\cancel{w}\, w_a/Q}{w_a^2 - w^2} \rightarrow \left(\frac{w_a}{Q}\right)^2 = w_a^2 - w^2$$

$$\downarrow$$

$$w^2 = w_a^2\left(1 - 1/Q^2\right)$$

$$\boxed{w = w_a\sqrt{1 - 1/Q^2}}$$

then $\boxed{w_o = w_a\sqrt{1 - 1/Q^2}}$ is the resonant freq.

$$Z(jw_o) = \frac{1}{C}\,\frac{\cancel{(jw_o + w_a/Q)}}{w_a/Q\,\cancel{\left(jw_o + \frac{w_a^2 - w^2}{w_a/Q}\right)}} = \frac{Q \cdot 1}{C w_a} = \frac{Q \cdot w_a^2 L \cancel{C}}{\cancel{C} w_a} = Q \cdot \frac{w_a\, r_1}{r_1/L}\, Q = Q^2 \cdot r_1$$

Impedane at resonance

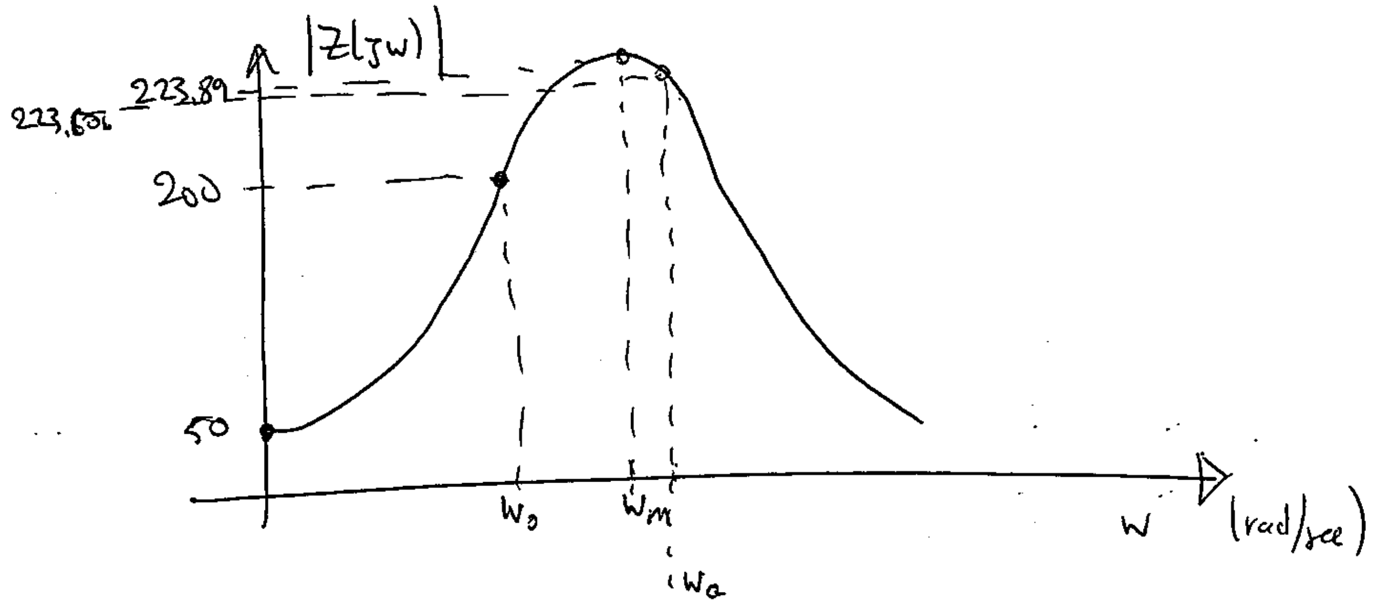
$\sqrt{\frac{1}{LC}}$

Ex: a) $r_1 = 50\,\Omega$, $L = 10\,mH$, $C = 1\,\mu F$

then
$$\begin{cases} \omega_a = 1/\sqrt{LC} = 10^4 = 10\,k \text{ rad/sec.} \\ \varphi = \dfrac{\omega_a}{r_1/L} = 2 \quad (\text{unitless}). \\ \omega_0 = \omega_a\sqrt{1 - 1/\varphi^2} = 10^4\sqrt{1 - 1/4} = 8.66\,k \text{ rad/sec} \end{cases}$$

$Z(j0) = 50\,\Omega$, $Z(j\omega_0) = \varphi^2 \cdot r_1 = 200\,\Omega$; $Z(j\omega_a) = 223.606\angle{-26^\circ}$

$Z(j\omega_m) = 223.89\angle{-23}$



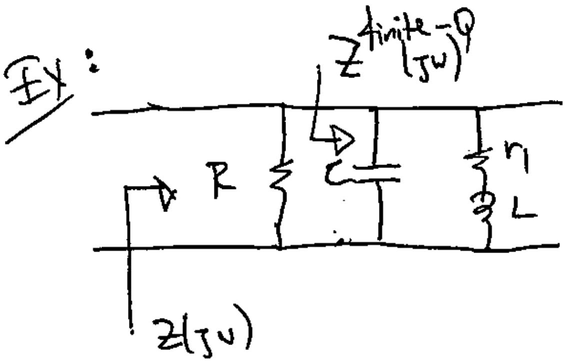
$\omega_a = \sqrt{1/LC}$

ω_m: $\rightarrow$ should be found by taking derivative of $|Z(j\omega)|$ wrt. ω.

$\omega_0 = \omega_a\sqrt{1 - 1/\varphi^2}$ $\leftarrow$ resonant freq. of finite-φ circuit.

We observe that even for $Q = 2$; ω_a is sufficiently close to ω_m. As $Q \to \infty$, ω_a and ω_m approaches ω_0. But note that even $Q = 2$ is close enough for many purposes in this example.

Ex:

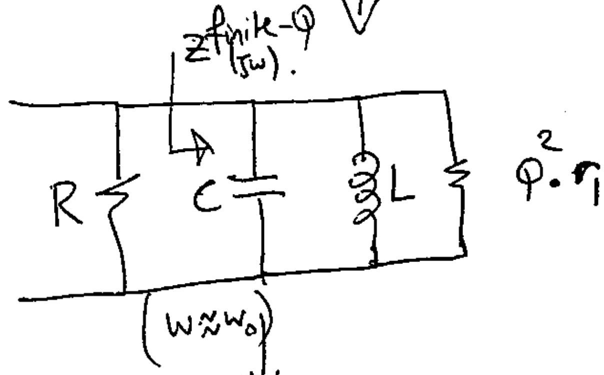


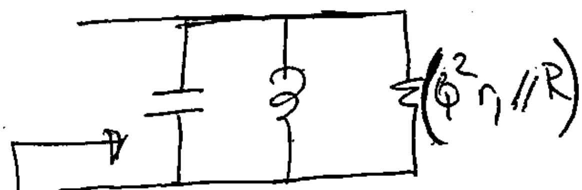
Find $Z(j\omega)$, and assume that $\omega \approx$ resonat freq of this system.

ω: frequency band of interest

high Q filter

if $\omega \approx \omega_0 \approx \omega_a \approx \omega_m$





Parallel
Standard resonance circuit