Steady-State Response of RC Circuit to Periodic Square Wave Input

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Abstract

The steady-state response of RC circuit to the square wave input is examined. The solution is given by three different approaches. The goal is to emphasize the generality, practicality and therefore the importance of the Fourier series based approach. The notes have been prepared to be distributed as an handout in EE 202 (Circuit Theory II) course.

1 RC Circuit : A Leaky Integrator

The first order parallel RC circuit can be considered as a integrator with a leakage term. The leakage is due to the resistance which absorbs some part of the incoming current, as shown in Figure 1. For the given circuit, the capacitor current is $i_s(t) - v_c(t)/R$; and the capacitor voltage is:

$$v_c(t) = V_0 + \frac{1}{C}\int_{0^-}^{t} \left(i_s(t') - \frac{v_c(t')}{R} \right) dt' \tag{1}$$

where $v_c(0^-) = V_0$. The change in capacitor voltage in between t_1 and t_2 can be expressed as follows:

$$v_c(t_2) - v_c(t_1) = \frac{1}{C}\int_{t_1}^{t_2} \left(i_s(t') - \frac{v_c(t')}{R} \right) dt'$$

If t_2 is set as $t_1 + \epsilon$ ($\epsilon > 0$), then the direction of change at t_1 (whether the capacitor voltage is increasing or decreasing) can be written as follows:

$$v_c(t_1 + \epsilon) - v_c(t_1) = \frac{1}{C}\int_{t_1}^{t_1+\epsilon} \left(i_s(t') - \frac{v_c(t')}{R} \right) dt' \approx \frac{1}{C}\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right)\epsilon \tag{2}$$

The part of the equation above having approximately equal sign is due to Newton's calculus, which says that under fairly general conditions (Riemann integrable functions) the integral can be approximated with a rectangle of height $\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right)$ and width ϵ. (Even though these results are not critical for the presented RC circuit study, make sure that you understand the nature of these approximations (infinitesimal calculus) and internalize these ideas as much as possible.)

It is clear from the last relation that $v_c(t)$ increases at $t = t_1$, that is $v_c(t_1 + \epsilon) > v_c(t_1)$; if $\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right) > 0$ or equivalently $i_s(t_1) > \frac{v_c(t_1)}{R}$. There are no surprises in this argument, since the current source produces $i_s(t_1)$ amperes at $t = t_1$ and the amount of current reaching the capacitor $\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right)$. If this value is greater than zero, there is some charge deposited on the plates of the capacitor due to the applied current. If this current is being applied for ϵ seconds, then the total charge deposited is $\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right)\epsilon$. Since $Q = CV$ (the defining equation for the capacitor), we can get the voltage increase due to the application of the current for ϵ seconds as $\frac{1}{C}\left(i_s(t_1) - \frac{v_c(t_1)}{R} \right)\epsilon$. This is exactly equal to the right hand side of (2). We know that such infinitesimal arguments can be formally proved and turn into exact arguments as $\epsilon \to 0$.

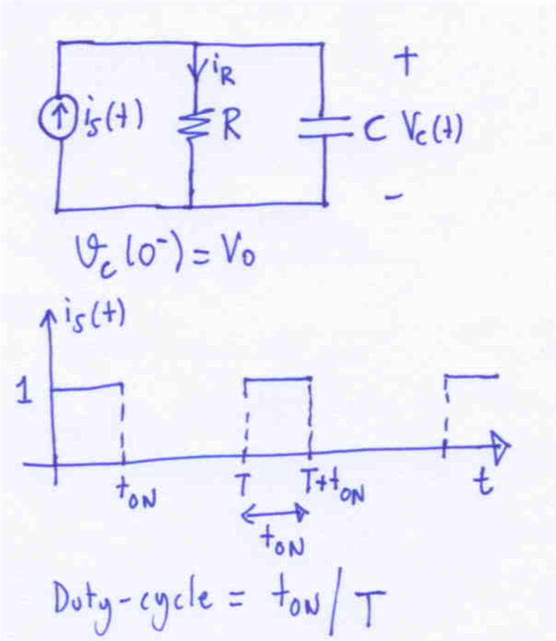

Figure 1: RC circuit with square wave input

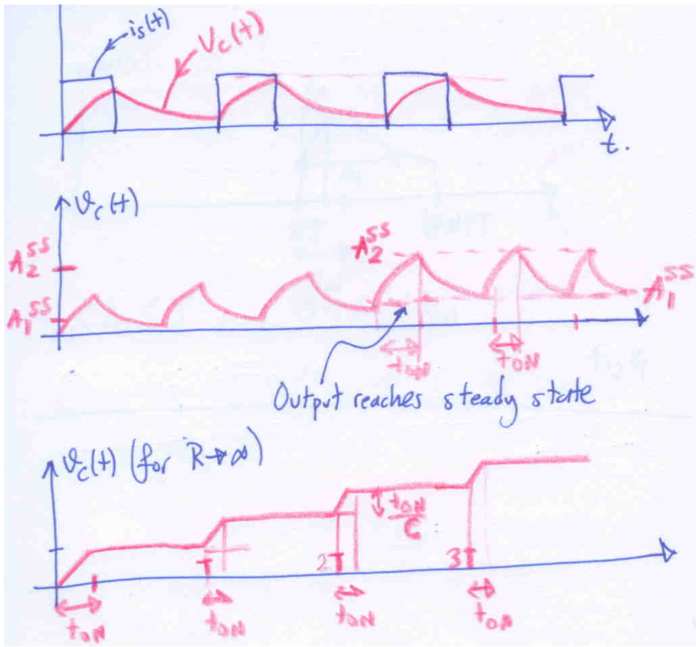

Figure 2: Zero-state response

In these notes, we examine the response due to the square wave of of a given duty cycle. The DC current source in Figure 1 can be considered to be periodically turned ON and OFF. As illustrated in the same figure, the time that the source is ON is much shorter than the time that is OFF. The ratio of ON time and the period (ON time plus OFF time) is called the duty cycle. During the application of 1 Ampere input, the capacitor is being charged; that is its voltages increases (assuming that it is initially less than 1 V) and during the OFF cycle, the voltage of the capacitor decays towards zero. You can consider the described system as a system with a periodic push with large gaps between the pushes. Every push inserts some energy into the system, when the push is over the energy in the system decays towards zero.

If the resistor in the circuit is disconnected, $R \to \infty$; then the system becomes lossless and every push results in a potential increment of $\frac{1}{C}t_{\mathrm{ON}}$ volts for the capacitor. The system for $R \to \infty$ is a perfect integrator. The response is unbounded (see Figure 2).

When the resistance is placed back into the system; the integrator becomes leaky, that is some of the part of the energy leaks out of the system (dissipated as heat over the resistor).

The described electrical system can be considered to be analogous to the leaky bucket system. Assume that there is a leaky bucket in our possession and we are turning on the faucet to fill the bucket; since the bucket is leaky, some of the stored water leaks out. When the bucket is close to full, the leakage is faster; since the weight of the water in the bucket exerts a bigger pressure to the hole. An illustration of the analogy is given in Figure 3.

The laws of nature, or we should say the mathematics of associated with the nature, treat the first order RC circuit and the leaky bucket system identically. If there is no hole in the bucket and we turn on and off the faucet periodically; the amount of water in the bucket increases by time. Assume that there is Δ kg's of water put into the bucket at every faucet opening. Then after N openings, the total amount of mass in the bucket is $N\Delta$ and the tip of the water level is $h = \frac{N\Delta}{\rho \mathrm{Area}}$. Here the base area of the bucket is denoted by Area (assume bucket is in the form of a cylinder, i.e. not the one in Figure 3) and the density of the water is taken as $\rho = 1\,\mathrm{gr/cm}^3$. The center of mass of the water is at the level of $\frac{h}{2}$. The total stored energy in the bucket, through mgh relation, is $N\Delta g \frac{N\Delta}{2\rho\mathrm{Area}} = \frac{1}{2} g \mathrm{Area} \rho (\frac{N\Delta}{\rho\mathrm{Area}})^2 = \frac{1}{2}Kh^2$. This relation is almost like $\frac{1}{2}CV_c^2$ where the the bucket capacitance(!) K is Area $\times\, g \times \rho$ and the height of water level h is the bucket voltage(!). We stop at this point, since we do not want to confuse EE 202 students with the intricate details of the bucket analysis, even though this analogy is quite fun.

Going back to the RC circuit, by taking the derivative of both sides of (1), we can convert the

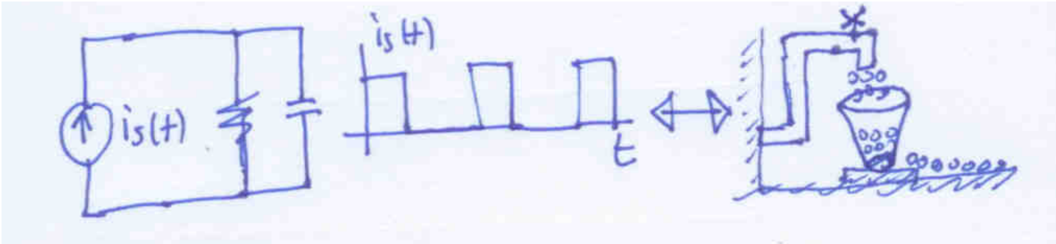

Figure 3: RC Circuit and the leaky bucket

integral equation into the differential equation given below:

$$C\frac{d}{dt}v_c(t) = i_s(t) - \frac{v_c(t)}{R}$$

Now, we can easily write the step-response of the circuit as follows:

$$v_c^{\text{step}}(t) = R(1 - e^{-t/RC})u(t)$$

Next, we use the LTI property of the circuit to write the zero-state response due to pulse of t_{ON} seconds ($\text{pulse}(t) = u(t) - u(t - t_{\text{ON}})$) as follows:

$$v_c^{\text{pulse}}(t) = v_c^{\text{step}}(t) - v_c^{\text{step}}(t - t_{\text{ON}})$$

Finally, the response due to the square wave input can be written as

$$v_c(t) = \sum_{k=0}^{\infty} v_c^{\text{pulse}}(t - kT). \tag{3}$$

This concludes the zero-state of analysis of RC circuit with the square wave input. In the following sections, we approach the steady-state part of the response from three different viewpoints. The first approach tries to evaluate the steady-state response by examining the asymptotical behaviour of (3). The second approaches uses the periodicity of the solution to get the steady state part. The third approach is very general. It is the solution by the Fourier series.

2 Steady-State Response to Square Wave Input

The square wave input injects energy into the system during the ON cycle. Some part of the energy in the circuit is dissipated over the resistor during the OFF cycle. As time progresses, the cycle of energy increase and decrease over the capacitor balances each other, that is the capacitor voltage reaches a maximum value which is show as A_2^{ss} in Figure 2 (the level reached at the end of ON-cycle); and dissipates its energy to A_1^{ss} which is the level at the end of OFF-cycle. This operation repeats itself over the next cycles. In this section, we examine the described steady-state response of this system swinging between A_1^{ss} and A_2^{ss} levels as shown in Figure 2.

2.1 Steady-State Response by Asymptotical Behaviour of the Zero-state Response

We have already given the zero-state response to the square wave input in (3). The steady-state solution is the zero-state solution as $t \to \infty$. We can write the equation (3) more explicitly as follows:

$$v_c^{zs}(t) = R\sum_{k=0}^{\infty}\Big((1 - e^{-\frac{t-kT}{RC}})u(t-kT) - (1 - e^{-\frac{t-t_{\text{ON}}-kT}{RC}})u(t - t_{\text{ON}} - kT)\Big)$$

We examine the response after the application N'th pulse as shown in Figure 4. The response after N'th pulse has two segments. The segment during which the input is ON and OFF. These segments are examined in two different cases.

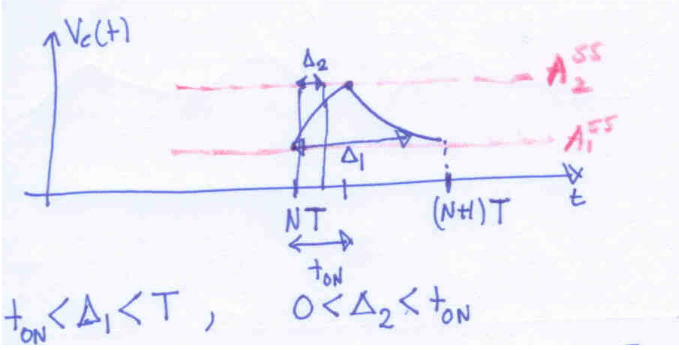

Figure 4: Definitions for Δ_1 and Δ_2

Case 1: OFF-Cycle ($t_{\mathbf{ON}} < \Delta_1 < T$)

In this case we examine the response during the OFF-cycle of the input. We assume that the N'th pulse starts at $t = NT$ and the pulse energies the system for the next t_{ON} seconds and then the OFF-cycle starts. We examine the OFF-cycle in this case.

The OFF-cycle of the N'th pulse is between $NT + t_{\mathrm{ON}}$ and $(N+1)T$ seconds. For $NT + t_{\mathrm{ON}} < t < (N+1)T$, the response can be written as follows:

$$\begin{aligned}
v_c^{zs}(NT+\Delta_1) &= R\sum_{k=0}^{N}\left((1-e^{-\frac{NT+\Delta_1-kT}{RC}}) - (1-e^{-\frac{NT+\Delta_1-t_{\mathrm{ON}}-kT}{RC}})\right) \\
&= Re^{-\frac{\Delta_1}{RC}}\sum_{k=0}^{N}\left(e^{-\frac{(N-k)T-t_{\mathrm{ON}}}{RC}} - e^{\frac{-(N-k)T}{RC}}\right) \\
&= Re^{-\frac{\Delta_1}{RC}}\sum_{k=0}^{N}e^{\frac{-(N-k)T}{RC}}\left(e^{\frac{t_{\mathrm{ON}}}{RC}}-1\right) \\
&= Re^{-\frac{\Delta_1}{RC}}\left(e^{\frac{t_{\mathrm{ON}}}{RC}}-1\right)\sum_{k=0}^{N}e^{\frac{-(N-k)T}{RC}} \\
&= Re^{-\frac{\Delta_1}{RC}}\left(e^{\frac{t_{\mathrm{ON}}}{RC}}-1\right)\sum_{k=0}^{N}e^{\frac{-kT}{RC}} \\
&= Re^{-\frac{\Delta_1}{RC}}\left(e^{\frac{t_{\mathrm{ON}}}{RC}}-1\right)\frac{1-e^{\frac{-(N+1)T}{RC}}}{1-e^{\frac{-T}{RC}}}
\end{aligned}$$

For the steady-state solution, we need $N \to \infty$. For $N \to \infty$, the value of the steady-state solution after Δ_1 seconds from the start of the ON-cycle is:

$$v_c^{ss}(\Delta_1) = Re^{-\frac{\Delta_1}{RC}}\left(e^{\frac{t_{\mathrm{ON}}}{RC}}-1\right)\frac{1}{1-e^{\frac{-T}{RC}}}, \qquad \text{where } t_{\mathrm{ON}} < \Delta_1 < T$$

For simplicity, we examine the 50% duty-cycle square wave and set $t_{\mathrm{ON}} = T/2$. The steady state solution for this choice is

$$v_c^{ss}(\Delta_1) = Re^{-\frac{\Delta_1}{RC}}\frac{e^{\frac{T}{2RC}}-1}{1-e^{\frac{-T}{RC}}}, \qquad \text{where } t_{\mathrm{ON}} < \Delta_1 < T$$

From the last equation and the value of A_1^{ss} can be evaluated. For the 50% duty cycle input, $v_c^{ss}(T^-)$

can be evaluated as follows:

$$
\begin{aligned}
A_1^{ss} = v_c^{ss}(T^-) &= R\frac{e^{-\frac{T}{2RC}} - e^{\frac{-T}{RC}}}{1 - e^{\frac{-T}{RC}}} \\
&= Re^{-\frac{T}{2RC}}\frac{1 - e^{\frac{-T}{2RC}}}{1 - e^{\frac{-T}{RC}}} \\
&= R\frac{e^{-\frac{T}{2RC}}}{1 + e^{\frac{-T}{2RC}}} \qquad (4)
\end{aligned}
$$

The value of A_2^{ss} for the 50% duty cycle input can be evaluated from $v_c^{ss}((\frac{T}{2})^+)$:

$$
\begin{aligned}
A_2^{ss} = v_c^{ss}((\frac{T}{2})^+) &= R\frac{1 - e^{\frac{-T}{2RC}}}{1 - e^{\frac{-T}{RC}}} \\
&= R\frac{1}{1 + e^{\frac{-T}{2RC}}} \qquad (5)
\end{aligned}
$$

Case 2: ON-Cycle ($0 < \Delta_2 < T_{\mathbf{ON}}$)

We examine the response during the ON cycle. We introduce Δ_2 parameter ($0 < \Delta_2 < T_{\text{ON}}$) as shown in Figure 4.

$$
\begin{aligned}
v_c^{zs}(NT + \Delta_2) &= R\sum_{k=0}^{N}\left((1 - e^{-\frac{NT+\Delta_2-kT}{RC}})\right) - R\sum_{k=0}^{N-1}\left((1 - e^{-\frac{NT+\Delta_2-t_{\text{ON}}-kT}{RC}})\right) \qquad (6) \\
&= R\left((1 - e^{-\frac{\Delta_2}{RC}})\right) + R\sum_{k=0}^{N-1}\left((1 - e^{-\frac{NT+\Delta_2-kT}{RC}}) - (1 - e^{-\frac{NT+\Delta_2-t_{\text{ON}}-kT}{RC}})\right) \\
&= R\left((1 - e^{-\frac{\Delta_2}{RC}})\right) + Re^{-\frac{\Delta_2}{RC}}\left(e^{\frac{t_{\text{ON}}}{RC}} - 1\right)\sum_{k=1}^{N} e^{\frac{-kT}{RC}} \\
&= R\left((1 - e^{-\frac{\Delta_2}{RC}})\right) + Re^{-\frac{\Delta_2}{RC}}\left(e^{\frac{t_{\text{ON}}}{RC}} - 1\right)\frac{1 - e^{\frac{-NT}{RC}}}{1 - e^{\frac{-T}{RC}}}e^{\frac{-T}{RC}}
\end{aligned}
$$

To reach the steady-state solution, we let $N \to \infty$ and treat $v_c^{ss}(\Delta_2)$ as the steady-state response after Δ_2 seconds from the start of an input pulse. Here $0 < \Delta_2 < T_{\text{ON}}$, therefore the response is limited to the ON cycle only.

$$
v_c^{ss}(\Delta_2) = R\left(1 + \frac{e^{-\frac{\Delta_2}{RC}}(e^{\frac{t_{\text{ON}}-T}{RC}} - 1)}{1 - e^{\frac{-T}{RC}}}\right), \qquad \text{where } 0 < \Delta_2 < t_{\text{ON}} \qquad (7)
$$

For the 50% duty cycle input, $v_c^{ss}(\Delta_2)$ simplifies as follows:

$$
v_c^{ss}(\Delta_2) = R\left(1 + \frac{e^{-\frac{\Delta_2}{RC}}(e^{\frac{-T}{2RC}} - 1)}{1 - e^{\frac{-T}{RC}}}\right), \qquad \text{where } 0 < \Delta_2 < t_{\text{ON}} = \frac{T}{2} \qquad (8)
$$

The values of A_1^{ss} and A_2^{ss} can be derived by evaluating $v_c(0+)$ and $v_c((T/2)^-)$ using (8). These values are identical to the earlier findings given in (4) and (5) as expected.

2.2 Steady-State Response Through Imposed Periodicity

The ON-cycle of the square wave injects energy into the system. At the end of the ON-cycle, the response reaches the peak value of A_2^{ss} then the capacitor discharges to A_1^{ss} level. From that point on, a new ON-cycle starts and charges the capacitor to the same A_2^{ss} level which is the maximum level reached for the steady-state solution. This is the steady-state operation of the system.

Assume that $v_c(NT) = A_1^{ss}$ and the capacitor voltage increases during the next $t_{\text{ON}} = \frac{T}{2}$ seconds due to applied current, as shown in Figure 4. The capacitor voltage during the ON cycle can be easily written as follows:

$$v_c(t) = R - (R - A_1^{ss})e^{-(t-NT)/RC}, \qquad NT < t < NT + T/2$$

Then A_2^{ss} value (the value reached at the end of ON-cycle) can be written as follows:

$$v_c(NT + T/2) = A_2^{ss} = R - (R - A_1^{ss})e^{-T/2RC} \tag{9}$$

During the OFF-cycle, that is for $NT + T/2 < t < (N+1)T$, the capacitor discharges from A_2^{ss} level according to the relation shown below:

$$v_c(t) = A_2^{ss} e^{-\frac{t-(NT+T/2)}{RC}} \qquad NT + T/2 < t < (N+1)T$$

At the end of OFF-cycle, the capacitor voltage reaches A_1^{ss} level for the steady-state solution.

$$v_c((N+1)T) = A_1^{ss} = A_2^{ss} e^{-T/2RC} \tag{10}$$

Substituting A_2^{ss} from (9) into (10), we get the following equation for A_1^{ss}:

$$A_1^{ss} = (R - (R - A_1^{ss})e^{-T/2RC})e^{-T/2RC} \tag{11}$$

and we can solve for A_1^{ss} from this equation. The result is identical to our earlier findings given (4). Next, we can find A_2^{ss} from (10) which is also identical to (5).

2.3 Steady-state Response Calculation Through the Fourier Series

Fourier series is used to express (almost) any periodic signal in terms of sines and cosines. We have not studied this important result in this course, but we utilize the expansion to produce the steady-state solution to the problem.

The square wave with 50% duty cycle can be expanded in Fourier series as follows:

$$i_s(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1,\, k:\text{odd}}^{\infty} \frac{1}{k} \sin(\omega_0 k t) \tag{12}$$

Here $\omega_0 = \frac{2\pi}{T}$. Figures 5 to 8 show the Fourier series expansion of the square wave input given in (12). It can be noted that the Fourier series converges to the square wave for the values of t for which the function is continuous. The convergence of Fourier series requires a careful analysis, but for the sake of circuit analysis; it is clear that we can approximate the input with the series without much worry.

The frequency response of RC circuit in question is lowpass and the associated transfer function can be written as follows:

$$H(s) = \frac{1/C}{s + 1/RC} \tag{13}$$

Then the steady-state response at the output can be immediately written as follows:

$$v_c^{ss}(t) = \frac{1}{2}H(0) + \frac{2}{\pi} \sum_{k=1, k:\text{odd}}^{\infty} \frac{|H(j\omega_0 k)|}{k} \sin(\omega_0 k t + \angle H(j\omega_0 k)) \tag{14}$$

This result follows from the fundamental result shown in Figure 9. (The result shown in Figure 9 is also valid for the sine functions, since ϕ in the Figure can be taken as $-90°$.)

It is easy to verify the following calculation:

$$\begin{aligned} |H(j\omega)| &= \frac{1/C}{\sqrt{w^2 + (1/RC)^2}} \\ \angle H(j\omega) &= -\tan^{-1}(\omega RC) \end{aligned}$$

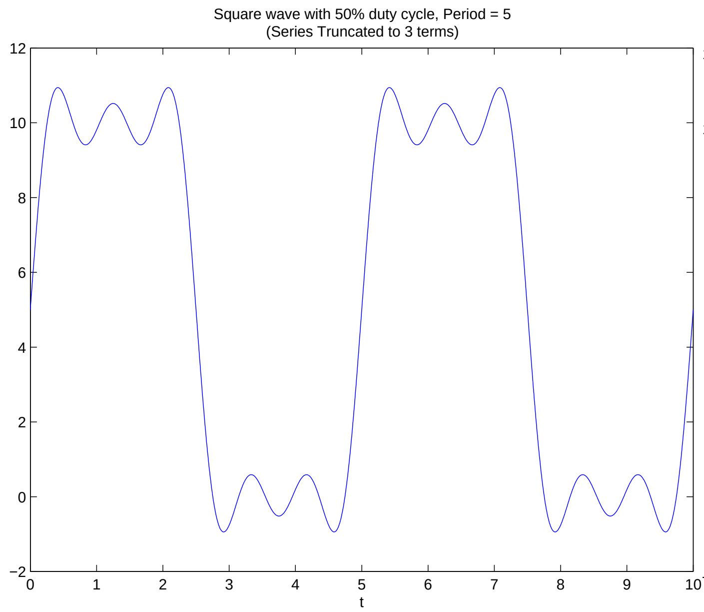

Figure 5: The first 3 terms of the Fourier Series

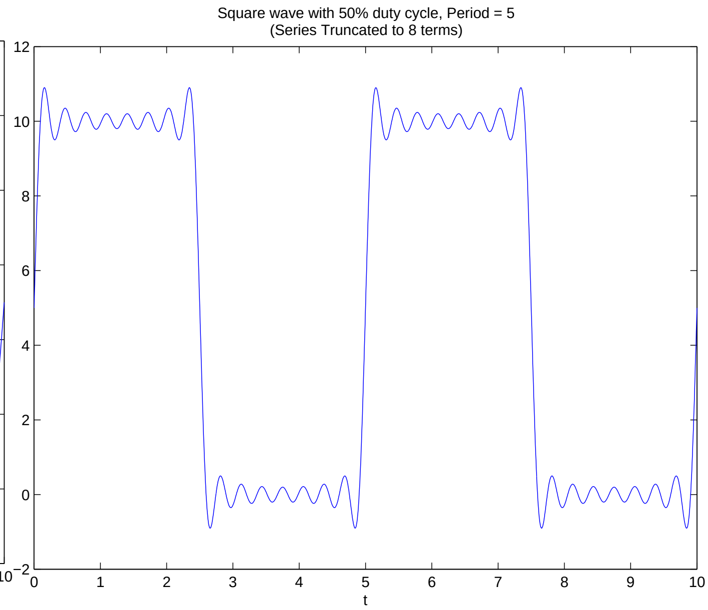

Figure 6: The first 8 terms of the Fourier Series

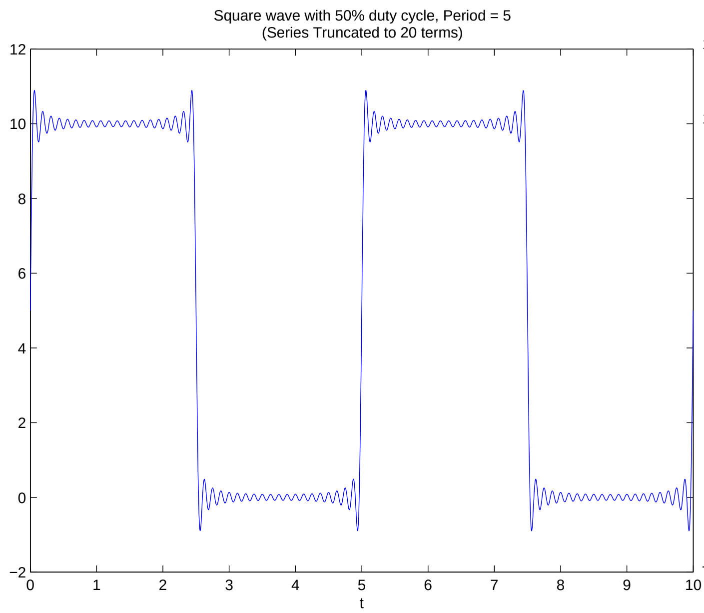

Figure 7: The first 20 terms of the Fourier Series

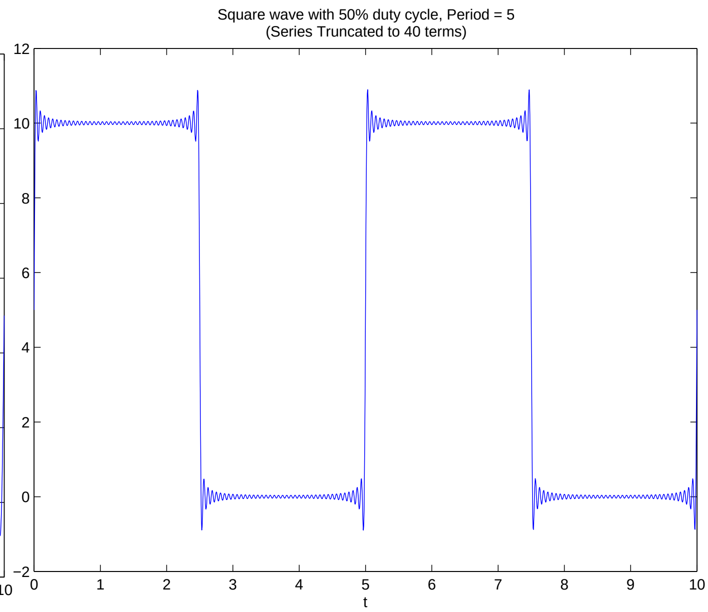

Figure 8: The first 40 terms of the Fourier Series

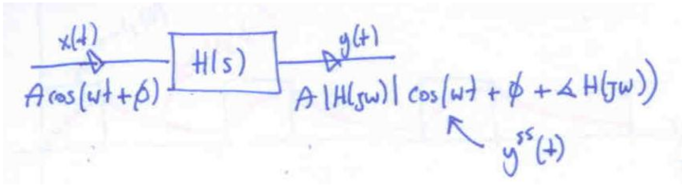

Figure 9: The fundamental result of LTI systems

We can now write, very easily, the steady-state solution in the series form as follows:

$$v_c^{ss}(t) = \frac{1}{2}R + \frac{2}{\pi}\sum_{k=1,k:\text{odd}}^{\infty} \frac{1/C}{k\sqrt{(w_0 k)^2 + (1/RC)^2}} \sin(\omega_0 k t - \tan^{-1}(\omega_0 k RC)) \tag{15}$$

Figures 10 to 17 shows the response calculated through the Fourier Series method. In these figures, the input is also shown and A_1^{ss} and A_2^{ss} levels (calculated using the other methods) are also indicated. The samples of the complete solution is also shown by cross signs. In all figures $R = 1$ and C is varied to change the time-constant of the system.

Figure 10 shows a system with a time-constant of 8 seconds, which is quite large, considering the fact that the ON-cycle is 5 seconds in all figures. Therefore the capacitor requires many ON-cycles to reach the A_2^{ss} level. The following figures show the circuit with gradually smaller time constants. Therefore the steady-state level (or the maximum voltage for the capacitor) is reached earlier in comparison to Figure 10.

Figure 17 shows a system with the time-constant of 1/8 seconds. Evident from the associated figure, the system is capable of following the input closely, that is the deviation of the output and the input curves is very little. This is due to almost immediate charging of capacitor during the ON cycle and again almost immediate discharging of the capacitor during the OFF cycle.

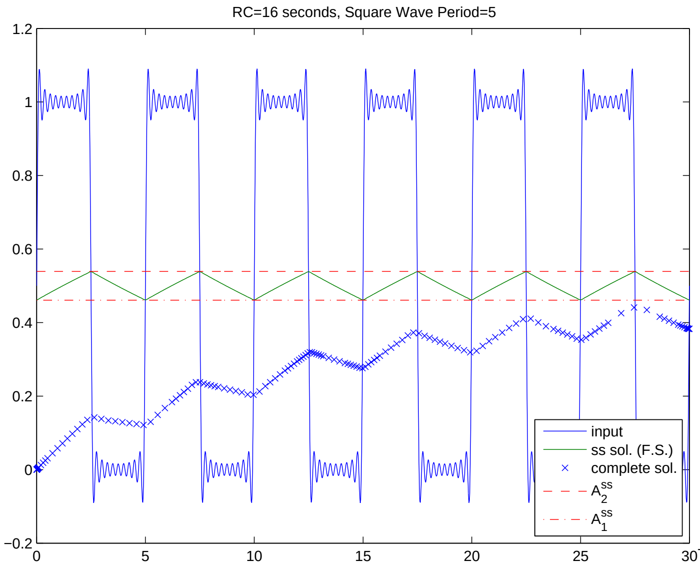

Figure 10: Time-constant: 16 seconds.

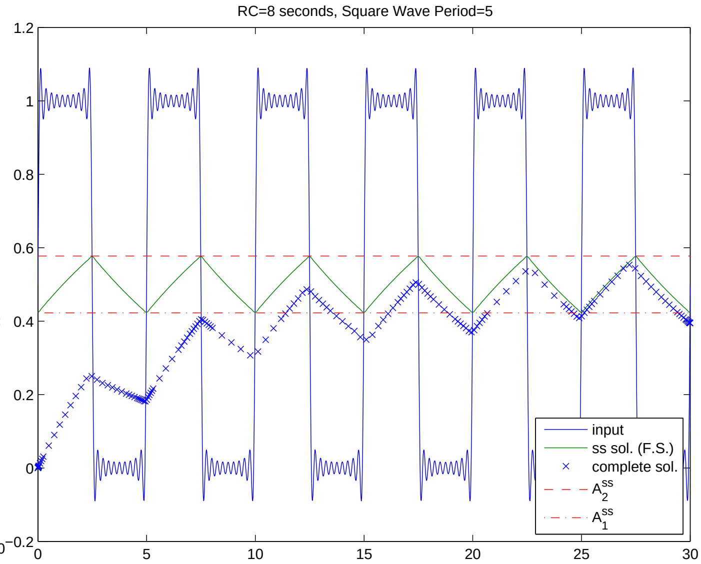

Figure 11: Time-constant: 8 seconds.

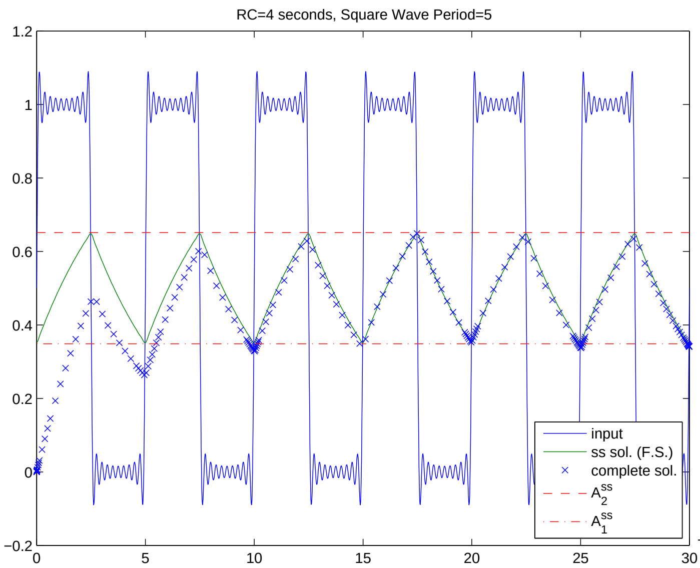

Figure 12: Time-constant: 4 seconds.

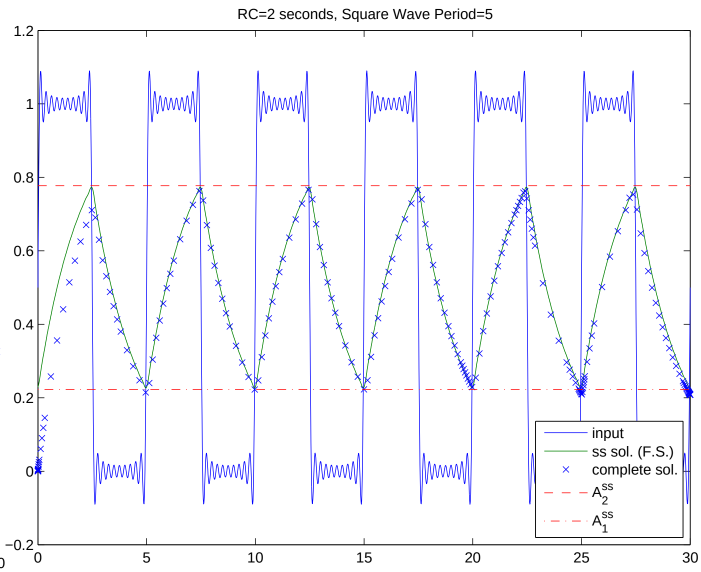

Figure 13: Time-constant: 2 seconds.

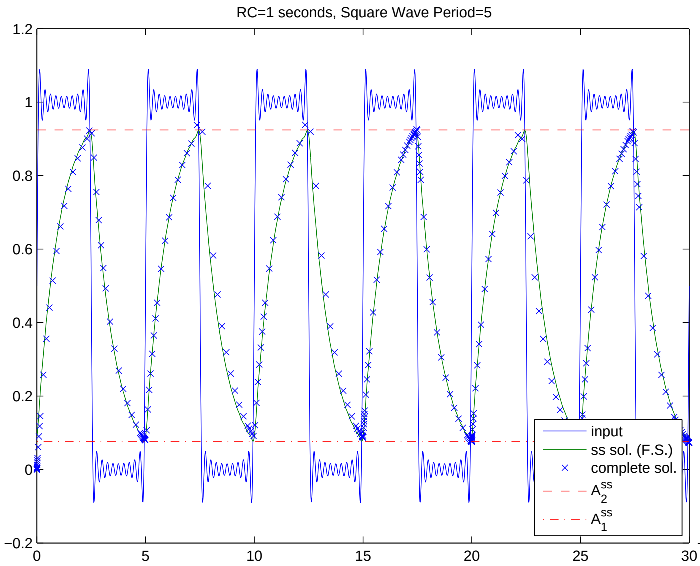

Figure 14: Time-constant: 1 second.

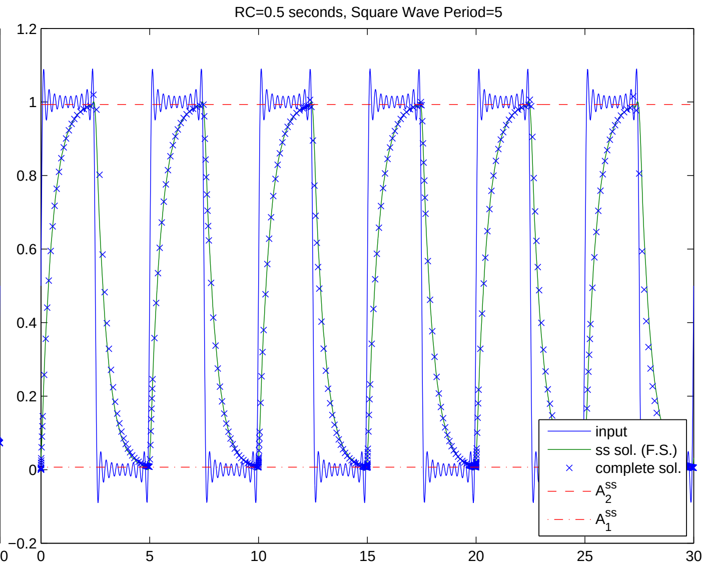

Figure 15: Time-constant: 1/2 seconds.

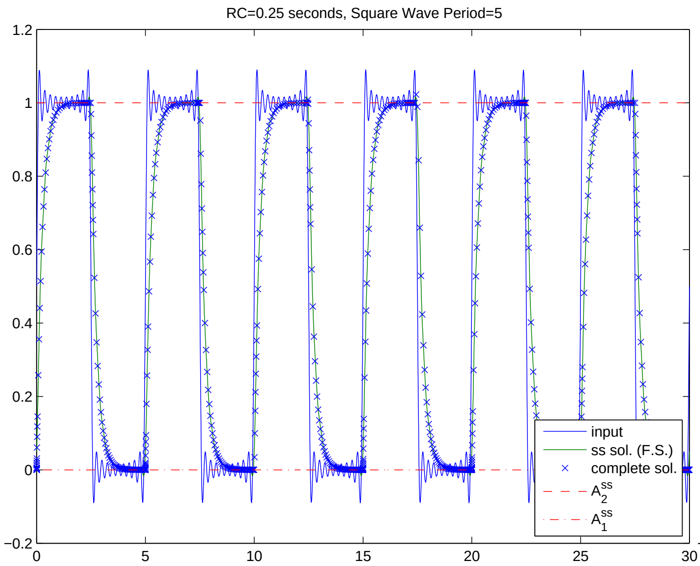

Figure 16: Time-constant: 1/4 seconds.

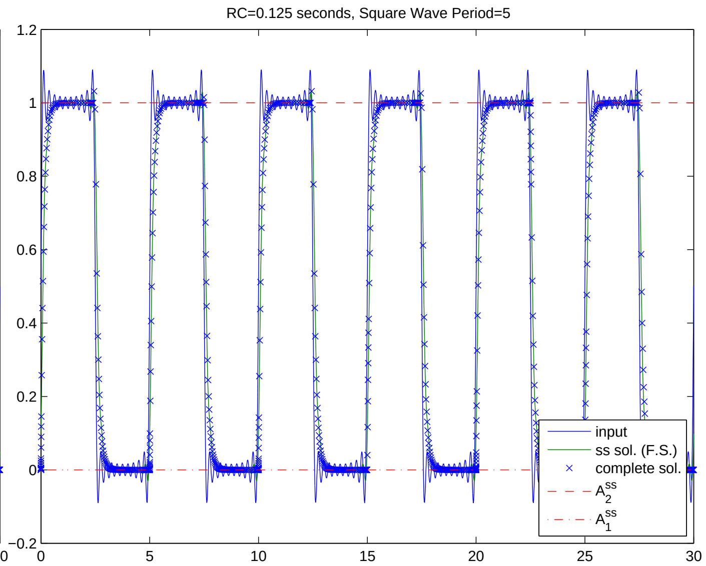

Figure 17: Time-constant: 1/8 seconds.

Figures 18 to 21 show the frequency response of the system and the magnitude spectrum of the input. In these figures, the magnitude spectrum of the input is the same (the spectrum of 50% duty-cycle square wave) but the system changes, since the capacitance in the circuit is varied.

Figure 18 shows the system with the time-constant of 16 seconds. It can be noted that the passband of this system includes only the DC component of the square wave. The other components are significantly attenuated. For example the term of fundamental frequency is attenuated by 25 dB. The higher order harmonics are attenuated even more. Figure 10 shows that during the ON cycle, the increase has almost a constant slope. This is due to integration of the DC term of the input. The other terms does not effect the output as much as the DC term. Again from the same figure, we can also note that A_1^{ss} and A_2^{ss} values are also quite close to each other. Therefore the AC terms at the output does not cause a large swing over DC value.

Figure 19 shows a system with larger passband. The time-domain picture given in Figure 13 shows that the DC term and the term of fundamental frequency is dominant at the output, that is the time-domain picture is almost like a sinusoid with a DC bias. The frequency domain picture given in Figure 19 shows that the fundamental frequency is attenuated by roughly 6 dB. Since the fundamental tone has the power of 12 dB, the term appears at the output with the reduced power of 6 dB. The DC term appears at the output with the power of 10 dB. This simple and rough calculation shows that the DC term and the fundamental frequency has a power difference of 4 dB. In other words, if the DC term at the output takes the value of A, the term of fundamental frequency has an amplitude $\frac{A}{10^{4/20}} = 0.63A$. From Figure 13, it can be observed that the DC level is 0.5. Then $A = 0.5$ and the fundamental tone has an amplitude around $0.6 \times 0.5 = 0.3$ which is a result matching the curve in Figure 13. (The curve in Figure 13 swings between (0.5+0.3) and (0.5-0.3) volts.)

Figures 20 and 21 show the response of the filters having wider passbands. As can be observed from these figures, many components of the Fourier series representing the input pass through the system with a little attenuation in comparison to the earlier systems. This is especially evident in Figure 21. Figure 21 shows that the first 4 terms of the series reside in the passband and the 5th component (the first out-of-band component) is attenuated by 3.5 dB which is an insignificant attenuation in comparison to the earlier systems. From the frequency domain picture, we can expect that the response at the output should be quite similar to the input. Figures 16 and 17 (especially Figure 17) show that the output response is quite fast and follows the input closely. This is also expected when one considers the time-constants of the systems.

3 Conclusions

In these notes we have examined the steady-state solution when square wave input is applied to the first order RC filter. These notes have been prepared to illustrate the concepts of

- steady-state solution
- AC steady-state solution
- relation between AC steady-state solution and transfer function
- importance of Fourier series for the calculation of steady-state solution for arbitrary periodic input

We would like to note that when the fundamental result of LTI systems shown in Figure 9 and the Fourier series expansion are combined, the steady-state solution of an *arbitrary LTI system* to an *arbitrary periodic input* can be easily found.

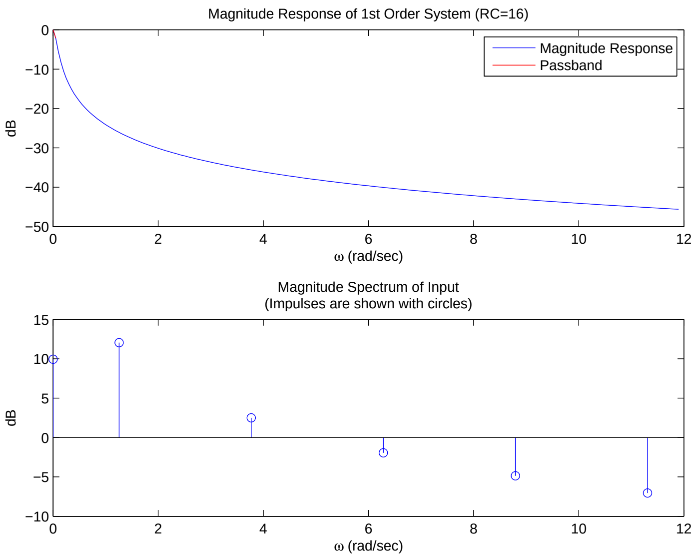

Figure 18: Time-constant: 16 sec. ($\omega_c = \frac{1}{16}$)

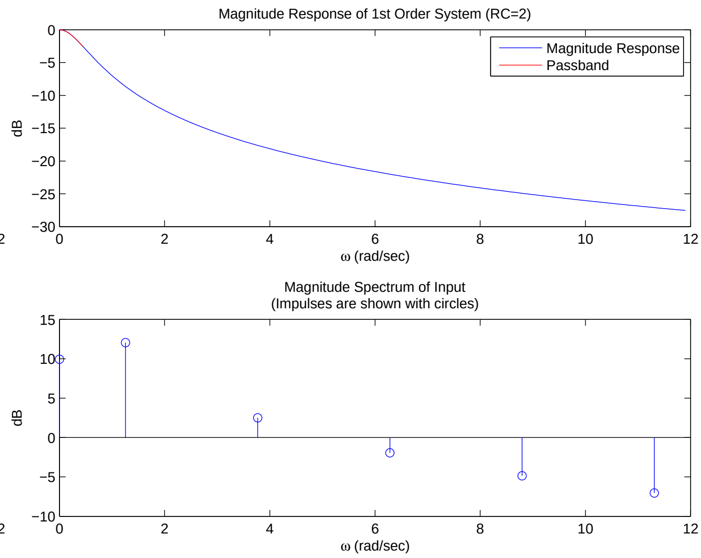

Figure 19: Time-constant: 2 sec. ($\omega_c = \frac{1}{2}$)

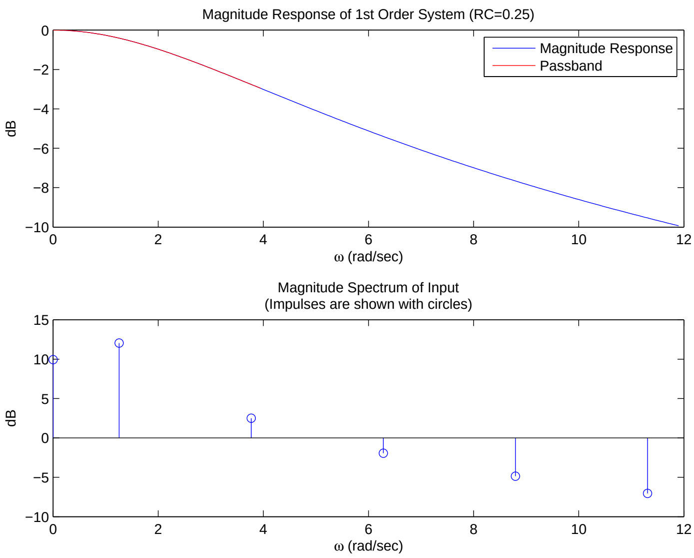

Figure 20: Time-constant: 1/4 sec. ($\omega_c = 4$)

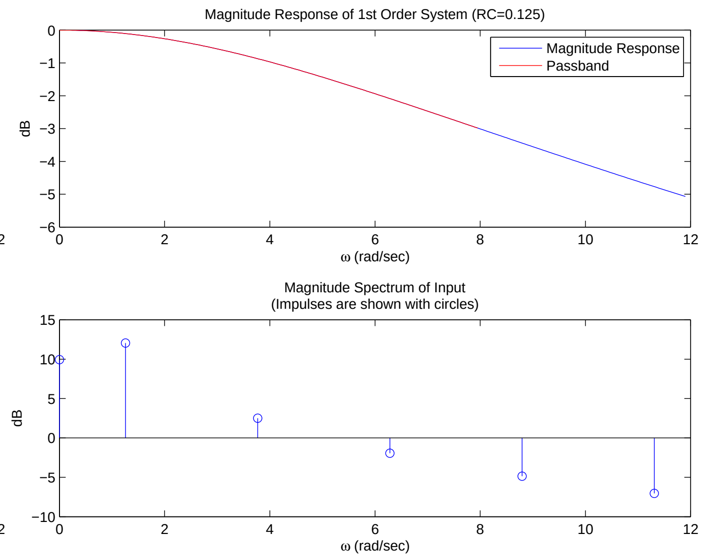

Figure 21: Time-constant: 1/8 sec. ($\omega_c = 8$)

A Appendix

A.1 Matlab Code Generating Figures 10-17

```
%%% SAVE THIS FILE AS sqwaveresp.m

T=5;              %period of square wave
maxt=30;          %maximum solution time
numterms=5;       %number of terms in the Fourier Series
A=1;              %Peak value of square wave
R=1;              %R in ohms
C=16;             %C in Farads

%do not change below unless you know what you are doing
t=linspace(0,maxt,2048);
omega0=2*pi/T;
x=A/2*ones(size(t));
sqw=A/2*ones(size(t));
y=x;
for k=1:2:(2*numterms-1),
    x=x+2*A/pi/k*sin(omega0*k*t);
    H=R/(1+j*omega0*k*C*R);
    y=y+abs(H)*2*A/pi/k*sin(omega0*k*t+angle(H));
    sqw= sqw + 2*A/pi/k*sin(omega0*k*t);
end;
figure(1),plot(t,sqw),
title(['Square wave with 50% duty cycle, Period = ' num2str(T) ...
      char(10) '(Series Truncated to ' num2str(numterms) ' terms)']);
xlabel('t');
figure(2),
plot(t,[x; y]); hold on;

A1ss=A*exp(-T/2/R/C)/(1+exp(-T/2/R/C)),
A2ss=A*1/(1+exp(-T/2/R/C)),

[tout,ycomp] = ode45(@(t,ycomp) rccircuit(t,ycomp,R,C,A,T),[0 max(t)],0);
plot(tout,ycomp,'x');
plot(tout,A2ss*ones(size(tout)),'--r');
plot(tout,A1ss*ones(size(tout)),'-.r');
hold off;
title(['RC=' num2str(R*C) ' seconds, Square Wave Period=' num2str(T) ]);

legend('input','ss sol. (F.S.)', ...
       'complete sol.','A_2^{ss}','A_1^{ss}','location','southeast');
```

```
function dx=rccircuit(t,x,R,C,A,T)
% Save this file as rccircuit.m
%Used in sqwaveresp.m ---
%

in = A*(rem(t,T)<=(T/2)); % input:50% duty cycle square wave
%dx = 1/R/C*(in-x);
dx = 1/C*(in - x/R);
```

A.2 Matlab Code Generating Figures 18-21

```matlab
%Save this file as freqresp_sqwave.m

T=5;              %period of square wave
maxt=30;          %maximum solution time
numterms=5;       %number of terms in the Fourier Series
A=1;              %Peak value of square wave
R=1;              %R in ohms
C=1/8;            %C in Farads

%do not change below
t=linspace(0,maxt,2048);
omega=0:0.05:2*pi*(2*numterms-1+0.5)/T;

omega0=2*pi/T;
x=A/2*ones(size(t));
sqw=A/2*ones(size(t));
y=x;

Hf = R./(1+j*omega*C*R);
inp=1/2;omega0vec=0;
for k=1:2:(2*numterms-1),
    omega0vec=[omega0vec omega0*k];
    inp=[inp 2/k/pi];
end;
inp=2*pi*inp;

figure(1),
subplot(211)
plot(omega,20*log10(abs(Hf)),'-'); hold on
ind=omega<(1/R/C);
plot(omega(ind),20*log10(abs(Hf(ind))),'-r');
title(['Magnitude Response of 1st Order System (RC=' num2str(R*C) ')']);
ylabel('dB'); xlabel('\omega (rad/sec)');
legend('Magnitude Response','Passband');
hold off

subplot(212)
stem(omega0vec, 20*log10(abs(inp))),
title(['Magnitude Spectrum of Input' char(10) '(Impulses are shown with circles)']);
ylabel('dB'); xlabel('\omega (rad/sec)');
```