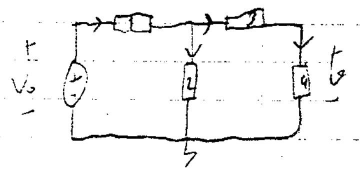
Ömer Özdemir, Spring 2010, Sec. Çağatay Candan.

— EE 262 —

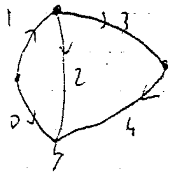
18/02/2010
Thursday

N^{th} ORDER DYNAMIC CIRCUITS

Review Graph Theoretical Node Analysis



$$\underline{\underline{A}} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

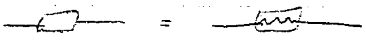


$$\underline{\underline{A}}\,\underline{J} = \underline{0}$$

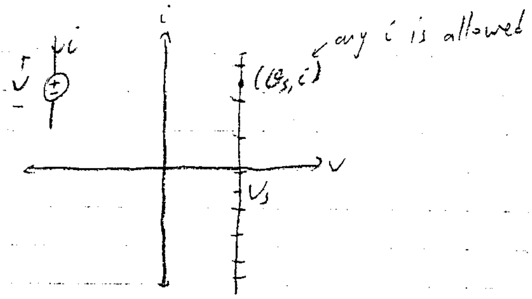
$\uparrow$ Branch current vector $\begin{bmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}$

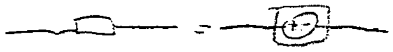
$J_k = ?$

$J_k = \dfrac{V_k}{R_k}$ for Resistors $\qquad J_k = I_s$ for current sources

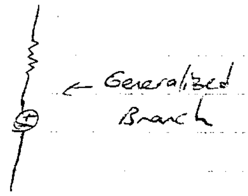


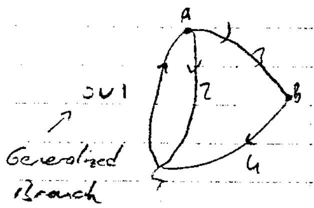
$J_k = ?$ for voltage sources





Generalized Branch





$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

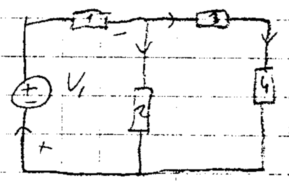
$$v = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} -e_A \\ e_B \\ e_A - e_B \\ e_B \end{bmatrix}$$

$\underline{\underline{A}}\,\underline{J} = \underline{0}$ KCL

$\underline{J} = \underline{\underline{G}}\,\underline{v}$ Terminal eq

$\underline{v} = \underline{\underline{A}}^T \underline{e}$

$\uparrow$ Branch voltages $\qquad \uparrow$ Node Voltages

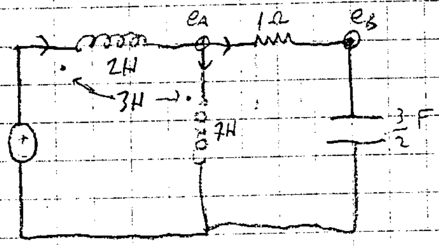


$\Rightarrow$ Insert $v = A^T e$ in other equations.

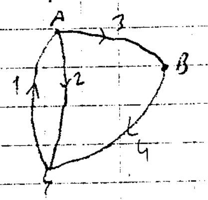
$$J = G v \Rightarrow J = G A^T e$$

$$\Rightarrow A J = A G A^T e = 0$$

Ex



Graph theoretical node analysis



$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$J_3 = \frac{V_3}{1\Omega} = V_3$$

$$J_4 = \frac{3}{2} \frac{d V_4(t)}{dt}$$

$$\begin{bmatrix} v_{2H}(t) \\ v_{7H}(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_{2H}(t) \\ \frac{d}{dt} i_{7H}(t) \end{bmatrix} \Rightarrow \begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} D i_{2H}(t) \\ D i_{7H}(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} D i_{2H}(t) \\ D i_{7H}(t) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} \rightarrow \text{apply } D^{-1} \text{ to both sides.}$$

$\hookrightarrow \Delta = 2 \times 7 - 3 \times 3 = 5$

$$D^{-1}(D i_{2H}(t)) = \int_0^t \frac{d}{dt} i_{2H}(\tau) d\tau = i_{2H}(t) - i_{2H}(0)$$

$$\Rightarrow \begin{bmatrix} \overset{J_1}{\boxed{i_{2H}(t)}} - i_{2H}(0) \\ \underset{J_2}{\boxed{i_{7H}(t)}} - i_{7H}(0) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} \int_0^t V_{2H}(\tau) d\tau \\ \int_0^t V_{7H}(\tau) d\tau \end{bmatrix}$$

$$\Rightarrow J_1 = i_{2H}(0) + \frac{7}{5} D^{-1} \overset{V_1 + V_s}{\boxed{v_{2H}(t)}} - \frac{3}{5} D^{-1} \overset{V_2}{\boxed{v_{7H}(t)}}$$

$$J_2 = i_{7H}(0) - \frac{3}{5} D^{-1} v_{2H}(t) + \frac{2}{5} D^{-1} v_{7H}(t)$$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} D^{-1} & -\frac{3}{5} D^{-1} & 0 & 0 \\ -\frac{3}{5} D^{-1} & \frac{2}{5} D^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2} D \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} i_{2H}(0) \\ i_{7H}(0) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{5} D^{-1} \\ -\frac{3}{5} D^{-1} \\ 0 \\ 0 \end{bmatrix} V_s(t)$$

$$J = G v + i(0) + v_s$$

$$A J = 0 \qquad v = A^T e$$

$$A G v + A i(0) + A v_s = 0$$

$$A G A^T e = -A (i(0) + v_s)$$

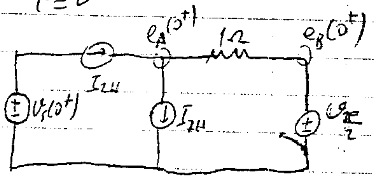
$$A G A^T \begin{bmatrix} e_A \\ e_B \end{bmatrix} = - \begin{bmatrix} i_{7H}(0) - i_{2H}(0) - 2D^{-1} V_s(t) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3D^{-1} + 1 & -1 \\ -1 & 1 + \frac{D}{2} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} i_{2H}(0) - i_{7H}(0) + 2D^{-1} V_s(t) \\ 0 \end{bmatrix} \leftarrow \text{integro-differential equation}$$

Initial Conditions for the solution of Dif. Eqn

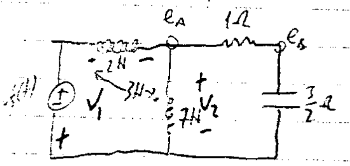
$I_{2H}(0^-) = I_{2H}$; $I_{7H}(0^-) = I_{7H}$; $V_{\frac{3}{2}F}(0^-) = V_{\frac{3}{2}F}$

At $t = 0^+$



$e_B(0^+) = V_{\frac{3}{2}F}(0^-)$

$e_A(0^+) = e_B(0^+) + i_{2H}(0^-) - i_{7H}(0^-)$



KCL at e_B:

$$\frac{e_B - e_A}{1} + \frac{3}{2}\frac{d}{dt}e_B(t) = 0 \checkmark$$

KCL at e_A:

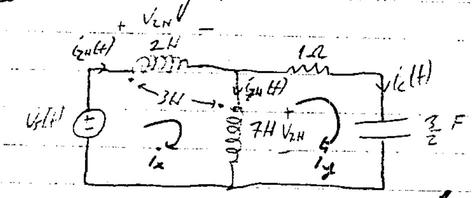
$$\frac{e_A - e_B}{1} + \underbrace{i_{7H}(t) - i_{2H}(t)} = 0$$

$$\begin{bmatrix} i_{2H}(t) \\ i_{7H}(t) \end{bmatrix} = \begin{bmatrix} 7/5\,D^{-1} & -3/5\,D^{-1} \\ -\frac{3}{5}D^{-1} & 2/5\,D^{-1} \end{bmatrix} \begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} + \begin{bmatrix} i_{2H}(0) \\ i_{7H}(0) \end{bmatrix}$$

$$\Rightarrow i_{7H}(t) - i_{2H}(t) = -2D^{-1}\underset{[V_s(t) - e_A(t)]}{\boxed{V_{2H}(t)}} + D^{-1}\underset{e_A}{\boxed{V_{7H}(t)}} - i_{2H}(0^-) + i_{7H}(0^-)$$

$$= \boxed{3D^{-1}e_A - 2D^{-1}V_s(t) - i_{2H}(0) + i_{7H}(0)}$$

Ex (Mesh Analysis)



$$\begin{bmatrix} V_{2H}(t) \\ V_{7H}(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} d/dt\,\boxed{i_{2H}(t)} \\ d/dt\,\boxed{i_{7H}(t)} \end{bmatrix}$$

Branch currents

$i_{2H}(t) = i_x(t)$

$i_{7H}(t) \rightarrow [i_x(t) - i_y(t)]$

KVL around i_x

$$-V_s(t) + V_{2H}(t) + V_{7H}(t) = 0$$

$$-V_s(t) + 5\frac{d}{dt}i_x(t) - 3\frac{d}{dt}i_y(t) + 10\frac{d}{dt}i_x(t) - 7\frac{d}{dt}i_y(t) = 0$$

$$\Rightarrow \boxed{15Di_x(t) - 10Di_y(t) = V_s(t)}$$

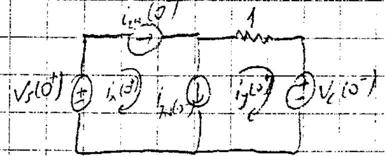
KVL around i_y

$$-V_{7H}(t) + 1\,i_y(t) + \overbrace{V_c(t)}^{V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_c(\tau)d\tau} = 0$$

$i_c = i_y(t)$

$$\Rightarrow -10Di_x(t) + 7Di_y(t) + i_y(t) + V_c(0^-) + \frac{2}{3}D^{-1}i_y(t) = 0$$

$$\begin{bmatrix} 15D & -10D \\ -10D & 7D+1+\frac{2}{3}D^{-1} \end{bmatrix} \begin{bmatrix} i_x(t) \\ i_y(t) \end{bmatrix} = \begin{bmatrix} V_s(t) \\ -V_c(0) \end{bmatrix}$$



$$i_x(0^+) = i_{2H}(0^-)$$

$$i_y(0^+) = i_{2H}(0^-) - i_{7H}(0^-)$$

Modified Node Analysis (MNA):

In MNA, we write eqn.'s s.t. we end up with a differential eqn. instead of a integro-differential equation.

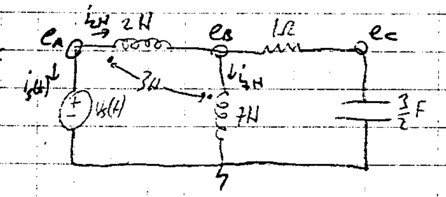
To do that

① Assign auxillary variables (other than node voltage variables) for inductors, voltage sources

② Write the KCL equations as in node analysis, use auxillary variables where necessary.

Write additional equations for auxillary variables.

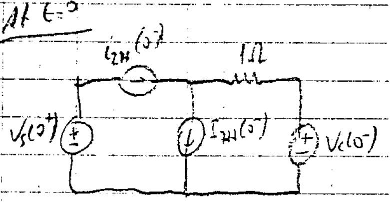
Ex



$$\begin{bmatrix} V_{2H} \\ V_{7H} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} D\, i_{2H}(t) \\ D\, i_{7H}(t) \end{bmatrix}$$

$$\begin{array}{r} \text{KCL at } e_A \\ \text{KCL at } e_B \\ \text{KCL at } e_C \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 1+\frac{3}{2}D & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -2D & -3D \\ 0 & 1 & 0 & 0 & -3D & -7D \end{bmatrix} \begin{bmatrix} e_A(t) \\ e_B(t) \\ e_C(t) \\ i_s(t) \\ i_{2H}(t) \\ i_{7H}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s(t) \\ 0 \\ 0 \end{bmatrix}$$

At $t = 0^+$



$$e_A(0^+) = V_s(0^+)$$

$$e_C(0^+) = V_C(0^-)$$

$$e_B(0^+) = V_C(0^-) + 1.[i_{2H}(0^-) - i_{7H}(0^-)]$$

$$i_{2H}(0^+) = i_{2H}(0^-)$$

$$i_{7H}(0^+) = i_{7H}(0^-)$$

$$i_s(0^+) = -I_{2H}(0^-)$$

State Equations

$$\underline{\dot{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

Ex

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & 2\\ 5 & 6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix} V_s(t) + \begin{bmatrix}0\\ 2\end{bmatrix} i_s(t)}_{\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}V_s(t)\\ i_s(t)\end{bmatrix}}$$

1st order matrix diff. eqn.

$$\left.\begin{aligned} x_1(0^-) &= \text{-----}\\ x_2(0^+) &= \text{-----}\end{aligned}\right\}\ \text{given}$$

Solution is by Laplace Transform or
eigen-decomposition of $\underline{A}$
through state-transition matrices
series expansion.

State variables: {Cap. voltages, Inductor currents}

Steps: 1) Include all voltage sources in tree
all current sources in co-tree

2) Put max. number of possible capacitors in tree (if possible all)
min. number of possible inductors in tree (if possible none)

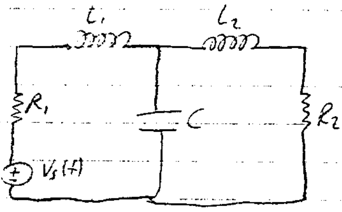
3) If there is a transformer → put only 1 (but 1) branch of transformer in tree.

Writing the equations

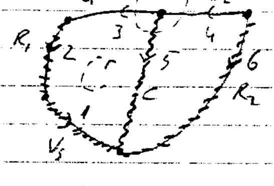
State variables: All cap. voltages in the tree and all inductor currents in the co-tree are state variables.

i) Write fun-loop for each inductor current in the co-tree (KVL)

ii) Write fun-cutset for each cap voltage in the tree (KCL)



$\left.\begin{aligned} &V_c(0^-)\\ &I_{L_1}(0^-),\ I_{L_2}(0^-)\end{aligned}\right\}$ given



State variables = $\{V_c(t), i_{L_1}(t), i_{L_2}(t)\}$

Order of the circuit = # state-variables = 3

For L_1

$$V_3+V_2+V_1-V_5=0$$

$$L_1\dot{i}_{L_1}+R_1 i_{R_1}+V_s-\underbrace{V_c}_{\text{state variable}}=0 \longrightarrow \boxed{\dot{i}_{L_1}(t)=-\frac{R_1}{L_1}i_{L_1}(t)+\frac{1}{L_1}V_c(t)-\frac{V_s(t)}{L_1}}$$

↳ Not a state variable

↓

express i_{R_1} in terms of state variables

↓

write fun-cutset for i_{R_1} branch

↓

$i_{R_1}=\underline{i_{L_1}}$ ✓

For L_2

$$V_{L_2}+V_c-V_{R_2}=0$$

$$L_2\dot{i}_{L_2}+V_c-R_2\underbrace{i_{R_2}}_{=i_{L_2}\text{ (Fun-cutset)}}=0 \longrightarrow \dot{i}_{L_2}(t)=-\frac{V_c(t)}{L_2}-\frac{R_2}{L_2}i_{L_2}(t)$$

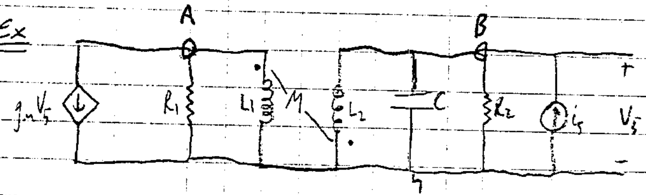
For C

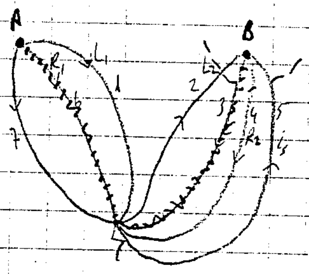
$$i_c-i_{L_2}+i_{L_1}=0 \longrightarrow \dot{V}_c(t)=\frac{-i_{L_1}(t)+i_{L_2}(t)}{C}$$

$$\begin{bmatrix}\dot{V}_c(t)\\ \dot{i}_{L_1}(t)\\ \dot{i}_{L_2}(t)\end{bmatrix}=\begin{bmatrix}0 & -1/C & 1/C\\ \frac{1}{L_1} & -R_1/L_1 & 0\\ -1/L_2 & 0 & -R_2/L_2\end{bmatrix}\begin{bmatrix}V_c(t)\\ i_{L_1}(t)\\ i_{L_2}(t)\end{bmatrix}+\begin{bmatrix}0\\ -1/L_1\\ 0\end{bmatrix}V_s(t)$$

25/02/2010
Perşembe

Ex





$V_c(0^-)=V_0$
$I_4(0^-)=I_0^4$
$I_{L_2}(0^-)=I_0^{L_2}$

State eqn.

Fun-cutset of Cap Branch

$$i_3+i_4-i_5-i_2=0$$

$$C\dot{V}_c+\frac{\overset{V_c}{V_4}}{R_2}-i_s-i_{L_2}=0 \longrightarrow \dot{V}_c(t)=\frac{i_{L_2}(t)}{C}-\frac{V_c(t)}{R_2C}+\frac{i_s(t)}{C}$$

Fun-loop → $V_c=V_4$

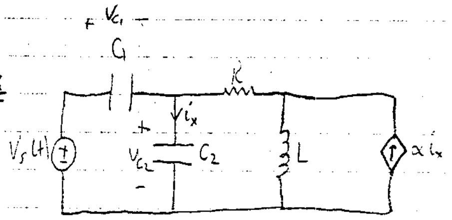
$$\begin{bmatrix} V_{L_1}(t) \\ V_{L_2}(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} d/dt\, i_{L_1}(t) \\ d/dt\, i_{L_2}(t) \end{bmatrix} \longrightarrow \boxed{\begin{bmatrix} d/dt\, i_{L_1}(t) \\ d/dt\, i_{L_2}(t) \end{bmatrix} = \frac{1}{L_1L_2-M^2} \begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix} \begin{bmatrix} V_{L_1} \\ V_{L_2} \end{bmatrix}}$$

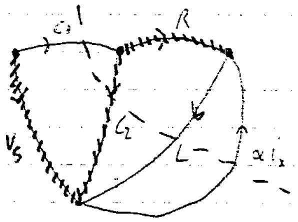
$$V_{L_1} \overset{F.L.}{=} V_{R_1} \overset{T.E.}{=} i_{R_1}R_1 \overset{F.L.}{=} R_1(-g_m V_s - i_4)$$

$$\overset{F.L.}{=} R_1(-g_m V_c - i_{L_1})$$

$$V_{L_2} \overset{F.L.}{=} -V_c$$

Ex





$$-i_{c_1} + i_{c_2} + i_L - \alpha i_x = 0$$

T.E. T.E. Definition

$$-C_1\dot{V}_{c_1} + C_2\dot{V}_{c_2} + i_L - \alpha C_2\dot{V}_{c_2} = 0$$

F.L.

$\dot{V}_s - \dot{V}_{c_2}$

State variables: $\{V_{c_2},\ i_L\}$

Cap's in tree — ind's in co-tree

$$-C_1(\dot{V}_s - \dot{V}_{c_2}) + C_2\dot{V}_{c_2} + i_L - \alpha C_2\dot{V}_{c_2} = 0$$

$$[C_1 + C_2(1-\alpha)]\dot{V}_{c_2} = -i_L + C_1\dot{V}_s$$

$$\boxed{\dot{V}_{c_2}(t) = \frac{-i_L}{C_1+C_2(1-\alpha)} + \frac{C_1}{C_1+C_2(1-\alpha)}\dot{V}_s(t)}$$

Fun loop for L branch

$$V_L = -V_R + V_{c_2}$$

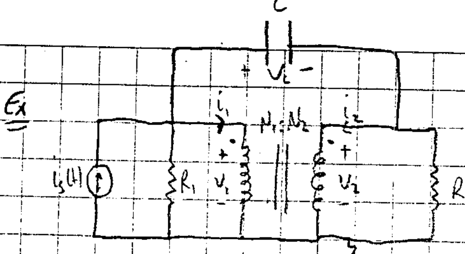
$$i_R R \overset{F.C.}{=} (i_L - \alpha i_x)R = (i_L - \alpha C_2\dot{V}_{c_2})R$$

From 1st state eqn

$$\boxed{L\dot{i}_L(t) = -Ri_L(t) + RC_2\alpha\left(\frac{-i_L(t)}{C_1+C_2(1-\alpha)} + \frac{C_1}{C_1+C_2(1-\alpha)}\dot{V}_s(t)\right) + V_{c_2}}$$

Note: 3 dynamic elements but 2-state variables!!! (Capacitive Loop)
(Inductive Loop)

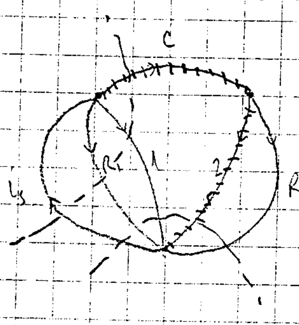
Ex

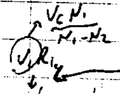


$V_c(0^-) = V_0$

$\frac{V_1}{V_2} = \frac{N_1}{N_2}$; $\frac{i_1}{i_2} = -\frac{N_2}{N_1}$

state variable: $\{V_c\}$





$C\dot{V}_c = i_s - i_{R_1} - i_1$

$\frac{V_1}{V_2} = \frac{N_1}{N_2} \rightarrow \frac{V_c}{V_2} + 1 = \frac{N_1}{N_2}$

$$\boxed{V_2 = \left(\frac{N_1}{N_2} - 1\right)^{-1} V_c = \frac{V_c N_2}{N_1 - N_2}}$$

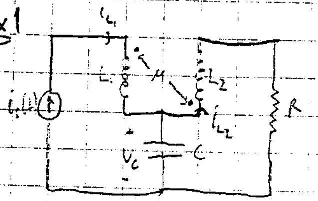
$$\boxed{V_1 = \frac{V_c N_1}{N_1 - N_2}}$$

Fun. Cutset for 2

$i_2 = i_s - i_{R_1} - i_1 - i_{R_2}$

$i_2 = i_s - \frac{V_1}{R_1} - i_1 - \frac{V_2}{R_2} \longrightarrow -\frac{N_1}{N_2} i_1 = i_s - \frac{V_c}{R_1} \frac{N_1}{(N_1 - N_2)} - i_1 - \frac{V_c N_2}{R_2(N_1 - N_2)} \rightarrow \boxed{i_1 = \frac{i_s - \frac{V_c}{(N_1 - N_2)}\left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right)}{1 - \frac{N_1}{N_2}}}$

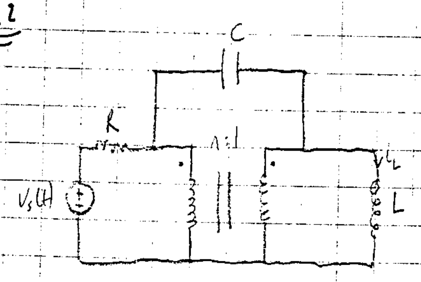
Ex1



$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_{L_2} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L_2 & -R/L_2 \end{bmatrix} \begin{bmatrix} V_c \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i_s + \begin{bmatrix} 0 \\ M/L_2 \end{bmatrix} \dot{i}_s(t)$$

Verify the state equation

Ex2

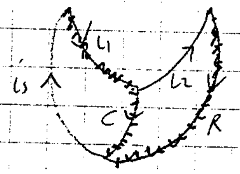


$$\begin{bmatrix} \dot{V}_c \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} \frac{-n^2}{(n-1)^2 RC} & \frac{-1}{C(n-1)} \\ \frac{1}{L(n-1)} & 0 \end{bmatrix} \begin{bmatrix} V_c \\ I_L \end{bmatrix} + \begin{bmatrix} \frac{1}{RC(n-1)} + \frac{1}{RC} \\ 0 \end{bmatrix} V_s(t)$$

↓

1st order matrix dif. eqn.
with constant coef.

Ex1



State variables: $\{V_c, i_{L_2}\}$

$i_C = i_s - i_{L_2}$

$C\dot{V}_c = i_s - i_{L_2}$

$$\boxed{\dot{V}_c = \frac{i_s}{C} - \frac{i_{L_2}}{C}}$$

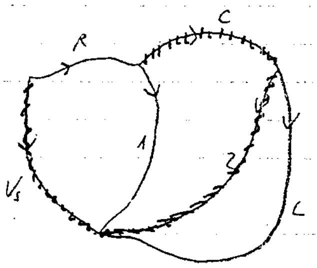
$V_{L_2} = V_c - V_R = M\dot{i}_{L_1} + L_2 \dot{i}_{L_2}$

$\Rightarrow L_2 \dot{i}_{L_2} = V_c - V_R - M\dot{i}_{L_1}$

$$\boxed{\dot{i}_{L_2} = \frac{V_c}{L_2} - \frac{R}{L_2} i_{L_2} - \frac{M}{L_2} \dot{i}_s}$$

$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_{L_2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L_2} & -\frac{R}{L_2} \end{bmatrix} \begin{bmatrix} V_c \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i_s + \begin{bmatrix} 0 \\ -\frac{M}{L_2} \end{bmatrix} \dot{i}_s$$

Ex 2



State variables: $\{V_c, i_L\}$

$\frac{V_1}{V_2} = n \qquad \frac{i_1}{i_2} = -\frac{1}{n}$

$V_2 = V_L = L\dot{i}_L = V_S - V_R - V_C = nL\dot{i}_L - V_c$

$V_1 = nL\dot{i}_L$

$V_R = V_S - nL\dot{i}_L$

$i_2 = C\dot{V}_c - i_L \Rightarrow i_1 = \frac{i_L - C\dot{V}_c}{n}$

$i_c = i_R - i_1$

$C\dot{V}_c = \frac{V_S - nL\dot{i}_L}{R} - \frac{i_L - C\dot{V}_c}{n}$

$C\dot{V}_c\left(\frac{n-1}{n}\right) = \frac{V_S}{R} - \frac{nL\dot{i}_L}{R} - \frac{i_L}{n}$

$\dot{V}_c = \frac{n V_S}{(n-1)RC} - \frac{n^2 L\dot{i}_L}{RC(n-1)} - \frac{i_L}{C(n-1)}$

$V_L = V_2$

$L\dot{i}_L = nL\dot{i}_L - V_E$

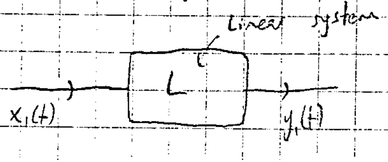
$$\boxed{\dot{i}_L = \frac{V_c}{L(n-1)}}$$

$$\boxed{\dot{V}_c = \frac{n}{(n-1)RC} - \frac{n^2 V_c}{RC(n-1)^2} - \frac{i_L}{C(n-1)}}$$

$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} \frac{-n^2}{(n-1)^2 RC} & \frac{-1}{C(n-1)} \\ \frac{1}{(n-1)L} & 0 \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{n}{(n-1)RC} \\ 0 \end{bmatrix} V_s$$

Linearity, Time-Invarying

A dynamic system is linear if



$y_1(t) = \mathcal{L}\{x_1(t)\}$; $y_2(t) = \mathcal{L}\{x_2(t)\}$

Then;

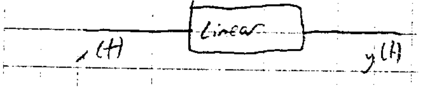
① $\mathcal{L}\{\alpha x_1(t)\} = \alpha y_1(t)$

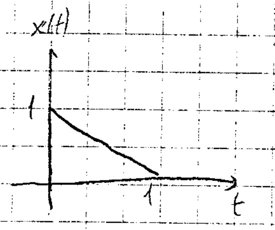
② $\mathcal{L}\{x_1(t) + x_2(t)\} = \mathcal{L}\{x_1(t)\} + \mathcal{L}\{x_2(t)\} = y_1(t) + y_2(t)$

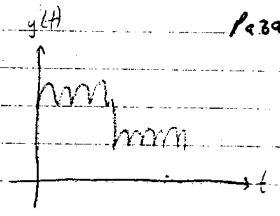
} $\mathcal{L}$ is a linear system (superposition principal applies)

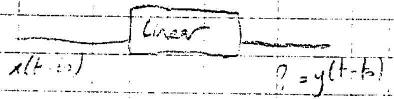
01/03/2010
Pazartesi

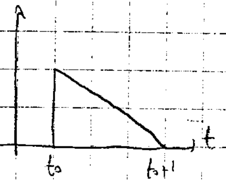
Time Invariance

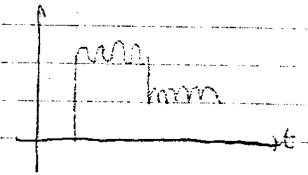








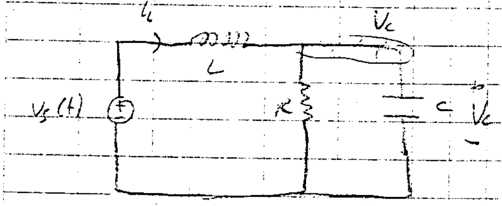




↓

If output is also shifted → system is time-invariant.

Application: R, L, C → LTI components



$V_c(0^-) = V_0$ $I_L(0^-) = I_0$

Node analysis

$(I_L(0^-) + \frac{1}{L}\int_{0^-}^{t}[V_s(\tau) - V_c(\tau)]d\tau$

$C\dot{V}_c + \frac{V_c}{R} - I_L = 0$

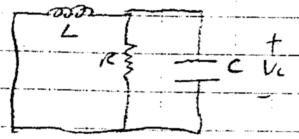
$\frac{d}{dt}$

$C\ddot{V}_c + \frac{\dot{V}_c}{R} - \frac{V_s(t) - V_c(t)}{L} = 0$

$$\boxed{\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_c(t) = \frac{V_s(t)}{LC}}$$

$V_c(0^-) = V_0$ $\dot{V}_c(0^-) = ?$

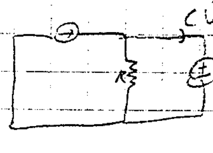
Let's focus on $V_s(t) = 0$ (zero-input solution)



$V_c(0^-) = V_0$
$I_L(0^-) = I_0$

$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_c(t) = 0$

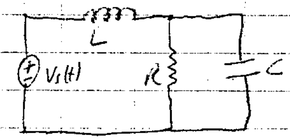
$V_c(0^+) = V_0$
$\dot{V}_c(0^+) = ?$

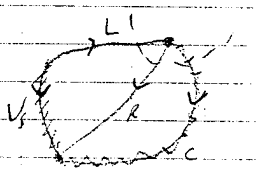


$i_c(0^+) = I_0 - \frac{V_0}{R} = C\dot{V}_c(0^+)$

$\dot{V}_c(0^+) = \frac{I_0 - V_0/R}{C}$

State eqn:





$C\dot{V}_c = \frac{-V_c}{R} + I_L$

$L\dot{I}_L = V_s(t) - V_c(t)$

$$\begin{bmatrix}\dot{V}_c(t)\\ \dot{I}_L(t)\end{bmatrix} = \begin{bmatrix}-1/RC & \frac{1}{C}\\ -1/L & 0\end{bmatrix}\begin{bmatrix}V_c(t)\\ I_L(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix} V_s(t)$$

1^{st} order matrix diff eqn. is equivalent to 2^{nd} order scalar diff. eqn?

Take another derivative of $\dot{V}_c(t)$ equation:

$$\ddot{V}_c(t) = -\frac{1}{RC}\dot{V}_c + \frac{1}{C}\dot{I}_L \leftarrow \frac{V_s(t)}{L} - \frac{V_c(t)}{L}$$

$$D^2 V_c = -\frac{1}{RC} DV_c - \frac{1}{LC} V_c + \frac{1}{LC} V_s(t)$$

$$\boxed{\left(D^2 + \frac{1D}{RC} + \frac{1}{LC}\right) V_c = \frac{1}{LC} V_s(t)}$$ → Same equation, we get from node analysis

Note: when use state equations initial conditions are already given! No need to calculate them from 0^+ analysis.

Solution of 2^{nd} order scale Diff eqn.

$R = \frac{1}{3}$; $C = 1$; $L = \frac{1}{2}$

$$\boxed{\begin{array}{l}(D^2+3D+2)\,V_c(t) = 0\\ V_c(0) = V_0\\ \dot{V}_c(0) = I_0 - 3V_0\end{array}}$$

(still zero input)

$V_c(t) = c\,e^{\lambda t}$ c: constant ← I don't know them! λ: constant

substitute the guess into diff eqn.

$(D^2+3D+2)\,c\,e^{\lambda t} = 0$ ← dif terms

$(\lambda^2+3\lambda+2)\,c\,e^{\lambda t} = 0$

$c = 0$ → trivial solution

$c \neq 0$

$\lambda^2+3\lambda+2 = 0$ ⟹ $V_c(t) = \{e^{-t}, e^{-2t}\} = \alpha e^{-t} + \beta e^{-2t}$

$\lambda = \{-1, -2\}$ modes of the system

↑ zero input response ⇓ superposition of two modes

αe^{-t}: a mode of the circuit.

$$V_c(t) = \underbrace{\alpha e^{-t} + \beta e^{-2t}}_{\text{homogenous}} + \underbrace{\gamma e^{-5t}}_{\text{particular}}$$ olsa one mode excitation'ı nasıl olurdu?

(08/03/2010) *

How do you excite only 1 mode?

$V_c(0) = V_0$

$\dot{V}_c(0) = I_0 - 3V_0$

$V_c(t)\Big|_{t=0} = \alpha e^{-t} + \beta e^{-2t}\Big|_{t=0} \longrightarrow V_0 = \alpha + \beta$

$\dot{V}_c(t) = -\alpha e^{-t} - 2\beta e^{-2t}\Big|_{t=0} \longrightarrow I_0 - 3V_0 = -\alpha - 2\beta$

Two single modes $\alpha \neq 0$, $\beta = 0 \longrightarrow V_0 = \alpha$

$I_0 - 3V_0 = -\alpha \Rightarrow I_0 = 2\alpha$

$V_0 = 1\ V$ $\quad V_c(t) = e^{-t}\ (\alpha = 1, \beta = 0)$

$I_0 = 2A$

To excite the other mode $\alpha = 0$, $\beta \neq 0$ $\quad \left.\begin{matrix} V_0 = \beta \\ I_0 = \beta \end{matrix}\right\} V_c(t) = \beta e^{-2t}$

Natural frequencies Roots of characteristic eqn. are called natural freq (λ_1, λ_2)

Natural Response When we have zero-input response, → natural response.

Mode: Individual components of natural response

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} V_c(t) \\ I_L(t) \end{bmatrix} \qquad V_c(0) = V_0 \ ; \ I_L(0) = I_0$$

$\dot{\underline{X}}(t) = \underline{\underline{A}}\,\underline{X}(t) \ ; \ \underline{X}(0) = \underline{X}_0$

$\underline{X}(t) \triangleq \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t} \Rightarrow \lambda \underline{\underline{I}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t} = \underline{\underline{A}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$

$(\lambda \underline{\underline{I}} - \underline{\underline{A}}) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{0}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ trivial $\qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq 0 \rightarrow \underset{\text{is not invertable}}{(\lambda \underline{\underline{I}} - \underline{\underline{A}})} \Rightarrow \boxed{\det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = 0}$

Remember $\det(\lambda \underline{\underline{I}} - \underline{\underline{A}})$ is called characteristic polynomial

Roots of $|\lambda \underline{\underline{I}} - \underline{\underline{A}}| = 0$ are the eigenvalues of $\underline{\underline{A}}$ matrix

$$|\lambda I - A| = \begin{vmatrix} \lambda+3 & -1 \\ 2 & \lambda \end{vmatrix} = \lambda(\lambda+3) + 2 = \underbrace{\lambda^2 + 3\lambda + 2}_{\text{char eqn.}}$$

then natural freq. are the roots of

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = \{-1, -2\}$

$\underline{X}(t) = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-t}, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{-2t} \right\}$

$\underline{x}(t) = \underline{c}e^{\lambda_1 t} + \underline{d}e^{\lambda_2 t}$; $\lambda_1 = -1$ $\lambda_2 = -2$

$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t)$

$\Rightarrow \dot{\underline{x}}(t) = \underline{c}\lambda_1 e^{\lambda_1 t} + \underline{d}\lambda_2 e^{\lambda_2 t}$

$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) \longrightarrow \underline{c}\lambda_1 e^{\lambda_1 t} + \underline{d}\lambda_2 e^{\lambda_2 t} = \underline{\underline{A}}(\underline{c}e^{\lambda_1 t} + \underline{d}e^{\lambda_2 t})$

to excite $e^{\lambda_1 t}$ mode $\underline{d} = \underline{0}$

$$\underline{c}\lambda_1 \cancel{e^{\lambda_1 t}} + 0 = \underline{\underline{A}}\,\underline{c}\cancel{e^{\lambda_1 t}} + 0$$

$$\Rightarrow \boxed{\underline{\underline{A}}\,\underline{c} = \lambda_1 \underline{c}}$$

To excite a single mode or mode with nat. freq. λ_1, $V_c(0^-)$, $I_L(0^-)$ should be the eigenvector of A corresponding to λ_1.

$\underline{\underline{A}}$ matrix of state eq. is fundamental for analysis; since its eigenvalues are <u>the natural frequencies</u> the eigenvectors of $\underline{\underline{A}}$ are the required initial conditions to excite a mode.

—o—

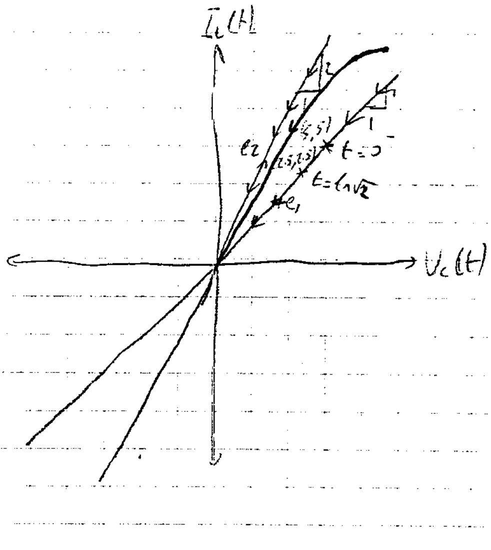
To excite $\lambda_1 = -1$ natural freq;

$$\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \underline{e_1} = \lambda_1 \underline{e_1}$$

$$\Rightarrow (\lambda \underline{\underline{I}} - \underline{\underline{A}})\underline{e_1} = 0 \Rightarrow \begin{bmatrix} \lambda+3 & -1 \\ 2 & \lambda \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \Rightarrow \lambda = -1 \rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k$ ← any number $\Rightarrow \begin{bmatrix} V_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k$ → excites e^{-t} mode.

$\lambda = -2 \rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ eigen vector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} V_c(0^-) \\ I_L(0^-) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k$ excites e^{-2t} mode



$\underline{x}(t) = \begin{bmatrix} 5 \\ 5 \end{bmatrix} e^{-2t}$

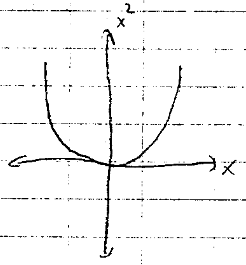
04/03/2010
Perşembe

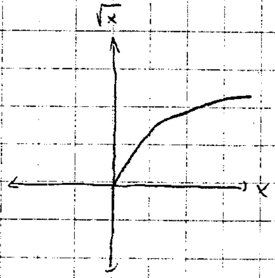
COMPLEX EXPONENTIAL FUNCTION

$f(x)$: Domain $\longrightarrow$ Range $\leftarrow$

an interval
$(a,b), (-\infty,\infty)$

real numbers



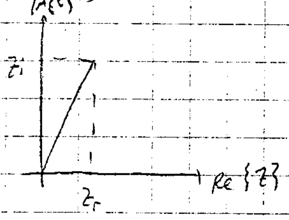


$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad x \in R$$

$z \Rightarrow z \in ¢ \leftarrow$ complex field

$z = \underset{\in R}{z_R} + j\, z_i \quad \nwarrow \in R \qquad j^2 = -1$
$j = \sqrt{-1}$

$\underline{z} = \begin{bmatrix} z_r \\ z_i \end{bmatrix} \Rightarrow$

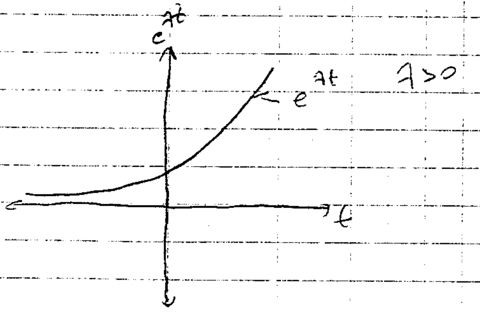


$$f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad , z \in ¢ \leftarrow \text{whole complex plane}$$

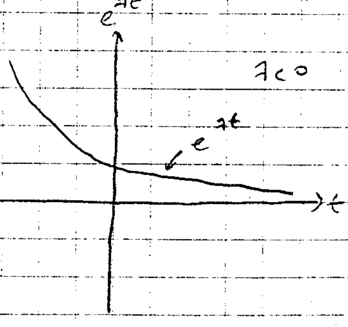
$f(z) : z \in ¢ \rightarrow ¢$

$\hookrightarrow f(z)$:
- z : purely real and positive $(z=2)$
- z : purely real and negative $(z=-2)$
- z : purely imaginary $(z=2j)$
- z : complex valued $(z=1+j2)$

① Purely Real, positive valued.



② Purely Real and negative valued



③ Purely Imaginary

$$f(t) = e^{j\underbrace{\omega t}} \quad \leftarrow t\in\mathbb{R} \quad \leftarrow \omega\in\mathbb{R}$$

$$e^{j\phi} = \sum_{k=0}^{\infty}\frac{(j\phi)^k}{k!} = \sum_{\substack{k\in\text{even}\\ \text{integers}\\ (k>0)}}\frac{(j\phi)^k}{k!} + \sum_{\substack{k\in\text{odd}\\ \text{integers}\\ (k>0)}}\frac{(j\phi)^k}{k!} = \sum_{\ell=0}^{\infty}\frac{(j\phi)^{2\ell}}{(2\ell)!} + \sum_{\ell=0}^{\infty}\frac{(j\phi)^{2\ell+1}}{(2\ell+1)!}$$

$$= \sum_{\ell=0}^{\infty}\frac{(j^2)^{\ell}\phi^{2\ell}}{(2\ell)!} + j\sum_{\ell=0}^{\infty}\frac{(j)^{2\ell}\phi^{2\ell+1}}{(2\ell+1)!} = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\phi^{2\ell}}{(2\ell)!} + j\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}\phi^{2\ell+1}}{(2\ell+1)!} = \cos\phi + j\sin\phi$$

$$\Rightarrow \boxed{e^{j\phi} = \cos\phi + j\sin\phi} \quad \text{Euler's Formula}$$

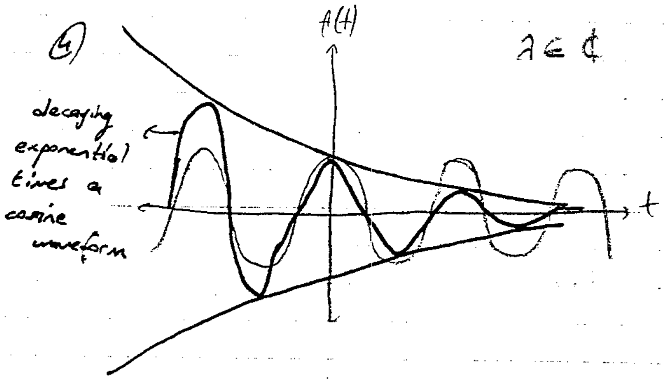
$$e^{jA}\cdot e^{jB} = e^{j(A+B)} = \cos(A+B) + j\sin(A+B)$$

$$(\cos A + j\sin A)(\cos B + j\sin B)$$

$$= (\cos A\cos B - \sin A\sin B) + j(\cos A\sin B + \sin A\cos B)$$

Ex $\sum_{k=0}^{N-1}\cos(\omega k) = \cos(\omega N)$ (Hint: $\cos(\omega k) = Re\{e^{j\omega k}\}$

④



$\lambda\in\mathbb{C}$

$$f(t) = e^{\lambda t} = e^{\lambda_r t}\cdot e^{j\lambda_i t}$$

$\lambda_r < 0$

$$Re\{e^{\lambda t}\} = Re\{e^{j\lambda_i t}\} = \cos(\lambda_i t]$$

STABILITY OF LTI SYSTEMS

Stability refers to keeping current, voltage of each branch bounded (not infinite, finite) through circuit components.

$$|V_c(t)| < M$$
$$\forall t$$
$$|I_L(t)| < N$$

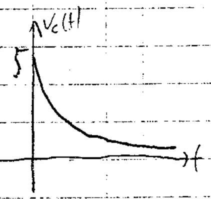
① Zero-Input (No input)

② Stability concept with inputs (later)

① Stability for zero input case

$(D^2+3D+2)\,V_c(t)=0 \qquad V_c(0)=5$

Since natural freq. $(\lambda^2+3\lambda+2=0,\ \lambda_{1,2}=\{-1,-2\}$ are negative valued term.



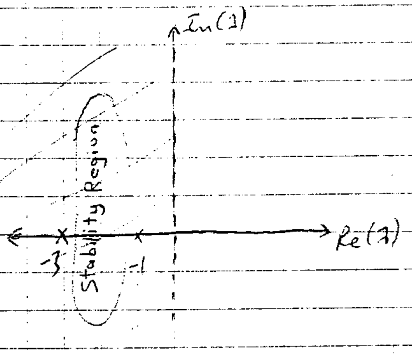
Then given any initial condition if each state goes to zero as $t\to\infty$;
then such a system is called asymptotically stable, (stable)

Note: There are more than one, stability definitions such as exponentially stable, asymptotically stable, Lyapunov stability, etc.

For LTI systems all stability definitions converge to the same condition.

Then LTI stable systems $\longleftrightarrow$ Natural freq. have Negative valued real parts.

Natural freq: roots of char eq. (eigenvalues of $\underline{\underline{A}}$)



λ: natural freq of a system
(x: show a natural freq)

SOLUTION OF DYNAMIC SYSTEM WITH INPUTS

Particular Solution of DE's:

$$(D^2+3D+2)\,x(t)=f(t) \leftarrow \text{forcing term (external input)}$$

$f(t)=e^{st}$ (We limit our interest to exponential inputs. Later we will use more powerful techniques covering more function types

$\underline{\underline{Ex}}$ $(D^2+3D+2)\,x_p(t)=e^{st}$

Guessing method

$x_p(t)=Ae^{st}$

$A(s^2+3s+2)e^{st}=e^{st} \Rightarrow A=\dfrac{1}{s^2+3s+2}$ provided that $s^2+3s+2\neq 0$
($s\notin$ natural freq.)

Ex $s=-3$

$$(D^2+3D+2)x_p(t) = e^{-3t} \Rightarrow x_p(t) = Ae^{-3t} = \frac{1}{2}e^{-3t} \leftarrow \text{particular solution.}$$

$$\Rightarrow X_{complete}(t) = x_p(t) + x_h(t)$$

$$= \frac{1}{2}e^{-3t} + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \{\lambda_1, \lambda_2 : \text{natural freq}\}$$

General → for input e^{st}

$\downarrow$

$$X_{complete}(t) = \frac{1}{s^2+3s+2}e^{st} + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

for DC input $f(t)=1 \rightarrow s=0$

for AC input $f(t)=\cos t \rightarrow s=j \rightarrow Re\{X_{complete}(t)|_{s=-3}\}$

Important Remark

For stable systems as $t \to \infty$; the zero-input solution goes to zero and in the complete response we are left with zero-state solution.

Then if input is in the form e^{st}

$\downarrow$

for LTI systems the particular solution is also an complex exponential with the same exponent

For N^{th} order systems, the changes are the followings

$$(D^N + a_1 D^{N-1} + a_2 D^{N-2} + \dots + a_N)\, x(t) = f(t)$$

① Characteristic Polynomial $= \lambda^N + a_1 \lambda^{N-1} + \dots + a_N = 0$

(N natural freq. (Can be complex valued))

② Particular solution for $f(t) = e^{st}$;

$$\text{The } X_p(t) = \frac{1}{s^N + a_1 s^{N-1} + \dots a_N} e^{st} \qquad s \notin \{\text{natural freq}\}$$

<u>State Eqn's</u> are favorite tool for analysis.

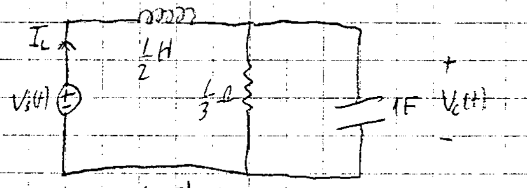
MNA, NA, Mesh Analysis : (Require more number of equation and initial conditions of the equation has to be found and integra-differential

$$\dot{x}(t) = Ax(t) + u(t)$$

08/03/2010
Pazartesi

Particular Solution of N^{th} Diff. Eqn. (Cont'd)

$(D^2+3D+2)\,V_c(t) = 2V_s(t)$



$V_c(0^-) = V_0$
$I_L(0^-) = I_0$

$V^{complete}(t) = \underset{\substack{\uparrow \\ \text{system determined (natural frequencies)}}}{V^h(t)} + V^p(t) \leftarrow$ (input determined depends on $V_s(t)$)

Homogenous: $(D^2+3D+2)V_c^h(t)=0 \rightarrow \lambda^2+3\lambda+2=0 \longrightarrow V_c^h(t) = c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$

$\lambda_{1,2} = \{-1,-2\}$
$\nwarrow$ natural freq.

Particular: Assume exponential input, $V_s(t) = e^{st} \longrightarrow s \in \mathbb{C}$

Guess: $V_c^p(t) = \underset{\nwarrow \text{unknown}}{A}e^{st}$

Substitute the guess into diff. eqn. to find A.

$(D^2+3D+2)\,V_c^p(t) = 2e^{st}$

$(s^2+3s+2)A\cancel{e^{st}} = 2\cancel{e^{st}} \Rightarrow A = \dfrac{2}{s^2+3s+2} \quad s \neq \{-1,-2\}$

<u>Case 1</u> $V_s(t) = e^{-5t}$, $V_c^p(t) = Ae^{-5t}$
$\hookrightarrow$ A is unknown real variable

Substitute the guess and proceed similarly.

$(D^2+3D+2)\,Ae^{-5t} = 2e^{-5t}$

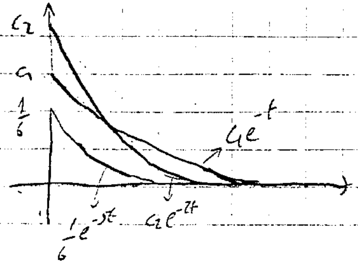
$A(D^2e^{-5t}+3De^{-5t}+2e^{-5t}) = 2e^{-5t}$
$A(25+(-15)+2)e^{-5t} = 2e^{-5t} \Rightarrow A = \frac{2}{12} = \frac{1}{6}$

01.03.2010
Pazartesi

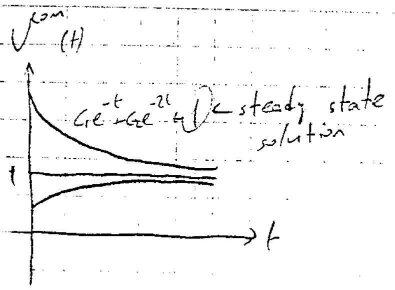
$* \quad V^{complete}(t) = \underbrace{c_1e^{-t}+c_2e^{-2t}}_{\substack{\text{homogenous} \\ \text{(natural response)}}} + \underbrace{\frac{1}{6}e^{-5t}}_{\text{particular}}$

<u>transient</u>: the remaining part in the complete response apart from steady state.

Steady state: part of complete response as $t \to \infty$.



Case ② $V_s(t) = 1V = e^{st}\Big|_{s=0}$



$$V_c^p(t) = Ae^{st}\Big|_{s=0} = A$$

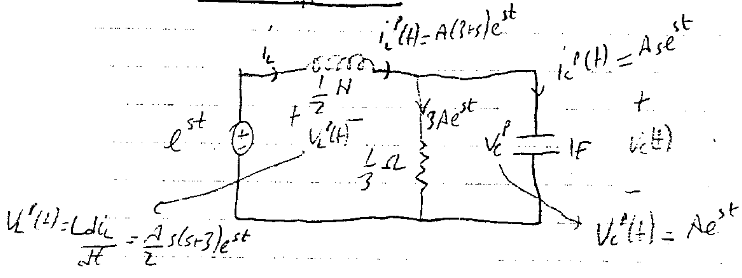
$$(D^2+3D+2)V_c^p(t) = 2 \Rightarrow 2A = 2 \Rightarrow A = 1$$

$$\boxed{V_c^p(t) = 1}$$

Method

For exponential inputs to find particular solution, we make the guess of Ae^{st} then find A.

A 2nd approach



To find A from circuit write KVL

$$e^{st} = V_L^p(t) + V_c^p(t)$$

$$e^{st} = \frac{A}{2}s(s+3)e^{st} + Ae^{st}$$

$$e^{st} = \frac{A}{2}(s^2+3s+2)e^{st} \Rightarrow \boxed{A = \frac{2}{s^2+3s+2}}$$

Case ③ $V_s(t) = \cos(5t)$

$$(D^2+3D+2)\,V_c^p(t) = 2\cos(5t) \qquad V_c^p(t) = A\cos(5t) + B\sin(5t)$$

$$\cos(5t)\underbrace{[2A+15B-25A]}_{2} + \sin(5t)\underbrace{[2B-15A-25B]}_{0} = 2\cos 5t$$

$$\begin{bmatrix} -23 & 15 \\ -15 & -23 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{23^2+15^2}\begin{bmatrix} -23 & -15 \\ 15 & -23 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{-46}{23^2+15^2} \\ \frac{30}{23^2+15^2} \end{bmatrix}$$

$$\boxed{V^p(t) = \frac{-46}{23^2+15^2}\cos(5t) + \frac{30}{23^2+15^2}\sin(5t)}$$

Case ③ : A better Approach

$$V_s(t) = \cos 5t = Re\{e^{j5t}\}$$

$$(D^2+3D+2)\, V_c^p(t) = 2\, Re\{e^{j5t}\}$$

$$V_c^p(t) = Re\{\boxed{A_c} e^{j5t}\}$$ → A_c : complex-valued unknown

↳

$$\frac{d}{dt} V_c^p(t) = \frac{d}{dt} Re\{A_c e^{j5t}\} = Re\left\{\frac{d}{dt} A_c e^{j5t}\right\} = Re\{j5 A_c e^{j5t}\}$$

$$\hookrightarrow \frac{d^2}{dt^2} V_c^p(t) = Re\{(j5)^2 A_c e^{j5t}\}$$

Do the substitution;

$$D^2 V_c^p(t) + 3D\, V_c^p(t) + 2\, V_c^p(t) = Re\{2e^{j5t}\}$$

$$Re\{(j5)^2 A_c e^{j5t}\} + Re\{j15 A_c e^{j5t}\} + Re\{2A_c e^{j5t}\} = Re\{2e^{j5t}\}$$

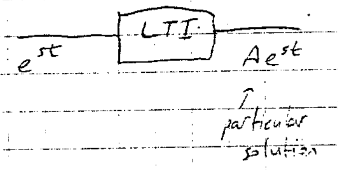
$$Re\{A_c((j5)^2 + 3(j5) + 2)\, e^{j5t}\} = Re\{2e^{j5t}\}$$

$$A_c = \frac{2}{(j5)^2+3(j5)+2} = \frac{2}{-25+j15+2} = \frac{2}{\underset{(-23-j15)}{-23+j15}} = \frac{2(-23-j15)}{23^2+15^2}$$

$$V_c^p(t) = Re\left\{\frac{2(-23-j15)}{23^2+15^2} e^{j5t}\right\} = \frac{-2}{23^2+15^2} Re\{(23+j15)(\cos(5t) + j\sin(5t))\}$$

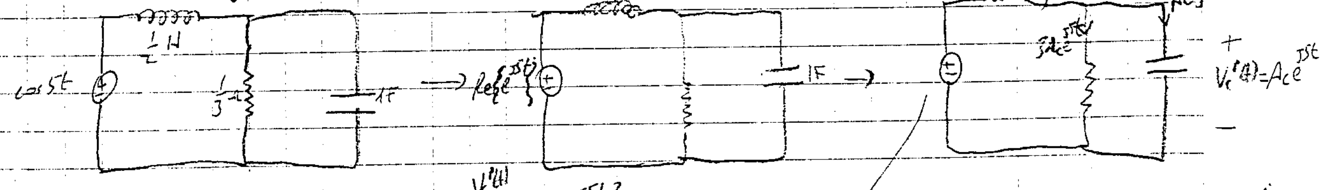
$$= \frac{-2}{23^2+15^2}(23\cos 5t - 15\sin 5t)$$

Exponential functions are the "eigenfunction" of an LTI dynamic system



More on this at other courses

Without using diff. eqn. case ③ can be attempted by following.



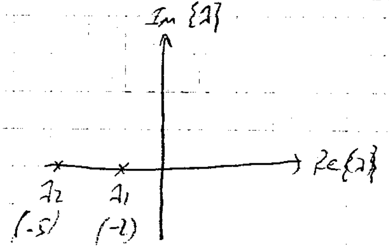
solution is $\overset{V_c^p(t)}{=} Re\{A_c e^{j5t}\}$

A_c - as before

Stability

$(D^2-(\lambda_1+\lambda_2)D+\lambda_1\lambda_2)V_c(t)=V_s(t)$ λ_1,λ_2: natural freq.

① Asymptotically stable ($Re\{\lambda_1\}<0$ and $Re\{\lambda_2\}<0$)



$V_s(t)=e^{st}$

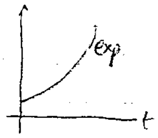
$s=0$ (DC)

$\downarrow$

complete response $\Rightarrow$ $V_c(t)=c_1e^{-2t}+c_2e^{-5t}+A$

BIBO stability seems to be ok

$s=1$



$\downarrow$

$V_c(t)=$ natural response $+\,Ae^{t}$

$s=j2$ (AC)

$\downarrow$

$V_c(t)=$ natural response $+\,Ae^{j2t}$

BIBO is ok for real part of the solution. (imaginary)

BIBO Stability

All Bounded inputs

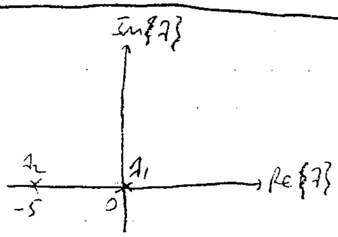
$\downarrow$

Bounded outputs

Bounded functions:

$|x(t)|<M \quad \exists M$

② Stable ($Re\{\lambda_1\}\leq 0$ and $Re\{\lambda_2\}\leq 0$)



$V_s(t)=e^{st}$

$s=0$ (DC)

$\downarrow$

$V_{complete}=c_1e^{\overset{0}{\lambda_1} t}+c_2e^{\lambda_2 t}+$ particular $\rightarrow A+Bt$

Not BIBO stable

$s=-5$

$\downarrow$

$V_{complete}=c_1+c_2e^{-5t}+Ae^{-5t}+Bte^{-5t}$

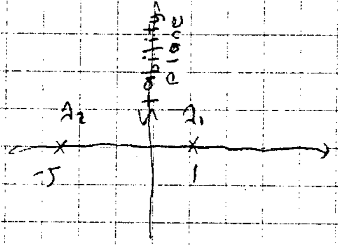
Seems ok for BIBO stability

$s=-3$

$\downarrow$

$V_{complete}=c_1+c_2e^{-5t}+Ae^{-3t}$

Unstable $(Re\{\lambda_1\} > 0$ or $Re\{\lambda_2\} > 0)$



$$V_{complete} = V^h(t) + V^p(t)$$

$$= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \text{particular}$$

11/03/2010
Persembe

About HW#1

A circuit is described by the following dif. eqn.

$$\begin{bmatrix} D+1 & 2 & 0 \\ -1 & D-1 & 1 \\ 0 & 0 & D+2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_s(t)$$; initial cond's are also given.

Find natural freq. of this circuit.

$$\left(\begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{-A} \right) \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_s(t)$$

Bunu da aşağıdaki gibi yapamaz mıyız? Öyle yapmak için zero-input mu olması gerekiyor?

$$\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_s(t)$$

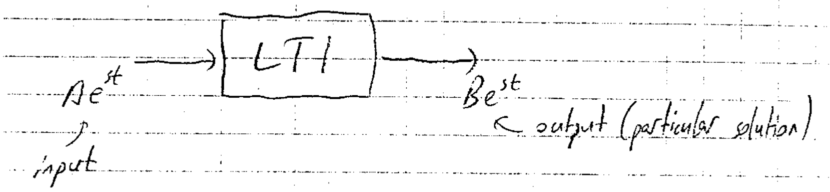
Yoksa naturalfreq. sorduğu için ne olursa olsun bız inputu kendimiz mi sıfırlayacaz?

Natural freq: $V_s(t) = 0$ (zero-input) → find the homogenous of the system

$$\begin{bmatrix} D+1 & 2 & 0 \\ -1 & D-1 & D \\ 0 & 0 & D+2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \hookrightarrow \begin{bmatrix} s+1 & 2 & 0 \\ -1 & s-1 & s \\ 0 & 0 & s+2 \end{bmatrix} e^{st} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} s+1 & 2 & 0 \\ -1 & s-1 & s \\ 0 & 0 & s+2 \end{vmatrix} = 0$$ ⇒ roots of this polynomial is natural frequencies.

PHASORS



We know that exponential family of signals (Ae^{st}) at the input of a N^{th} order dynamic system produces an output (particular solution, solution due to forcing term, i.e. input) in the form Be^{st}. So exponential family, the output is determined by finding B.

Then the coefficient of the exponent is called the phasor. In general A, B are complex numbers.

Specific for AC inputs that is $f(t) = \overset{\text{real}}{M}\cos(wt+\overset{\text{variables}}{\phi})$; we have the following phasor definition

$$f(t) = M\cos(wt+\phi) = Re\{Me^{J(wt+\phi)}\} = Re\{\underbrace{Me^{J\phi}}_{A}e^{Jwt}\} = Re\{Ae^{Jwt}\}$$

complex valued

A: phaser, the coefficient of e^{Jwt}

Then

$$2\cos(wt+30^\circ) \xrightarrow[\text{form}]{\text{phasor}} 2e^{J\pi/6}$$

$$2\angle 30^\circ$$

$$\underset{w=2}{1\angle 45^\circ} \xrightarrow{\text{time}} Re\{\underbrace{1\angle 45^\circ}_{1e^{\frac{J\pi}{4}}} e^{J2t}\} = \cos(2t+\frac{\pi}{4})$$

Ex $\cos(\underbrace{4t+30^\circ}_{A}) + \cos(\underbrace{4t+60^\circ}_{B}) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}) = 2\cos(4t+45^\circ)\cos(15^\circ)$

$\downarrow$

$$Re\{e^{J30^\circ}e^{J4t}\} + Re\{e^{J60^\circ}e^{J4t}\} = Re\{e^{J4t}(e^{J30^\circ}+e^{J60^\circ})\}$$

$$Re\{e^{J4t}[(\cos 30+J\sin 30)+(\cos 60+J\sin 60)]\} = Re\{e^{J4t}[(\frac{\sqrt{3}}{2}+\frac{1}{2})+J(\frac{\sqrt{3}}{2}+\frac{1}{2})]\} = Re\{e^{J4t}(\frac{1+\sqrt{3}}{2})(1+J)\}$$

$$= Re\{e^{J4t}(\frac{1+\sqrt{3}}{2})\sqrt{2}e^{J\pi/4}\} = \frac{1+\sqrt{3}}{\sqrt{2}}\cos(4t+45^\circ)$$

* * *

In the previous example, we have repeatedly used $Re\{e^{J4t}(\ldots)\}$ at every step. We do not write $Re\{e^{J4t}(\ldots)\}$ at every step; but prefer to write the phasor instead

That's

$$\cos(4t+30^\circ)+\cos(4t+60^\circ)\xrightarrow[\text{phasor}]{} 1\angle 30^\circ + 1\angle 60^\circ = \left(\frac{\sqrt{3}+J}{2}\right)+\left(\frac{1+J\sqrt{3}}{2}\right) \leftarrow \text{resulting phasor}$$

$$= \frac{1+\sqrt{3}}{\sqrt{2}} e^{J\pi/4} \xrightarrow[w=4]{\text{time domain}} \frac{1+\sqrt{3}}{\sqrt{2}}\cos(4t+45^\circ)$$

About EPS-I

Problem 1)

$$\begin{bmatrix} D+1 & D \\ (1+2D) & D^{-1}+2 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} V_s(t) = 0$$

$$\Downarrow$$

$$\begin{vmatrix} \lambda+1 & \lambda \\ 1+2\lambda & \frac{1}{\lambda}+2 \end{vmatrix} \Rightarrow \text{characteristic eqn}$$

MATLAB

```
>> syms D;
>> A = [D+1 -D; 1 D]
>> det(A)
     =D^2 + 2D
>> roots(det(A))
     [0 -2]
```

$$\begin{bmatrix} D+1 & -D \\ 1 & D \end{bmatrix}$$

NA → Some of natural frequencies at zero can be missing due to D, D^{-1} cancellation

MNA → used to find natural freq. (has only D's)

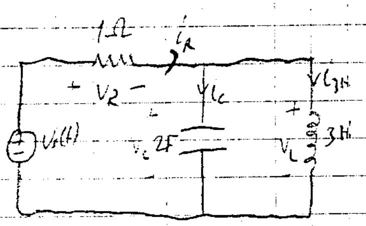
State eqn → used for natural frequencies (has only D's)

Ex $\frac{d}{dt} A\cos(wt+\phi) = -Aw\sin(wt+\phi)$

$$= Aw\cos(wt+\phi+90^\circ)$$

$$\Rightarrow A\angle\phi \xrightarrow[d/dt]{} A\angle\phi+90^\circ$$

Ex

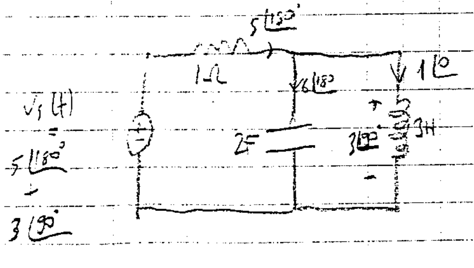


$V_s(t) = ?$ $\quad i_{3H}(t) = \cos(t)$

$V_{3H}(t) = 3\dot{i}_{3H}(t) = -3\sin(t) \rightarrow i_C(t) = C\dot{V}_c(t) = -6\cos(t)$

$i_R = i_C + i_{3H} = -5\cos(t) \rightarrow V_R(t) = -5\cos(t)$

$V_s(t) = -5\cos(t) - 3\sin(t)$



$$V_s \rightarrow 5\angle 180^\circ + 3\angle 90^\circ = -5 + J3 = \boxed{\sqrt{25+9}\; e^{J(\tan^{-1} 3/-5)}}$$

$$\boxed{V_s(t) = \sqrt{25+9}\cos\left(t - \tan^{-1}\frac{3}{5}\right)}$$

Phasors allow us

① Add cosines (with the same frequency)

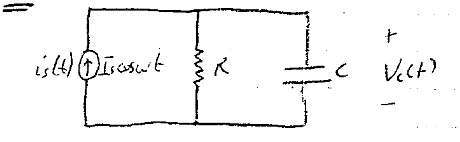
② Differentiate cosines

so adding differentiated and scaled many cosine terms is possible with phasors.

15/03/2010
Pazartesi

Ex

I)



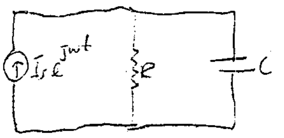
$V_c(0^-) = V_0$

$$\left(D+\frac{1}{RC}\right)V_c(t) = \frac{i_s(t)}{C} = \frac{I_s\cos wt}{C}$$

$$V_c^{complete}(t) = C_1 e^{-\frac{t}{RC}} + A\cos(wt) + B\sin(wt)$$

$$\downarrow$$

II)



$V_c(0^-) = V_0$

$$\left(D+\frac{1}{RC}\right)v_c(t) = \frac{I_s e^{Jwt}}{C}$$

$$_{II}V_c^{complete} = \underbrace{d_1 e^{-\frac{t}{RC}} + D e^{Jwt}}_{\text{can be complex}} \Rightarrow {}_{I}V_c^{complete}(t) = Re\{{}_{II}V_c^{complete}(t)\}$$

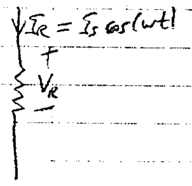
$$_{I}V^{complete}(t) = d_1 e^{-\frac{t}{RC}} + \frac{I_s R}{1+w^2R^2C^2}\cos(wt) + \frac{I_s wR^2C}{1+w^2R^2C^2}\sin wt$$

$$_{I}V^{complete}(0) = d_1 + \frac{I_s R}{1+w^2R^2C^2} = V_0 \Rightarrow d_1 = V_0 - \frac{I_s R}{1+w^2R^2C^2}$$

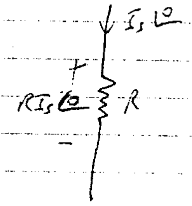
If we're only interested in particular solution for A.C. excitation; we have some simplifications!

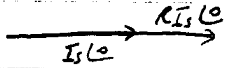
Phasor Circuit Analysis

R:

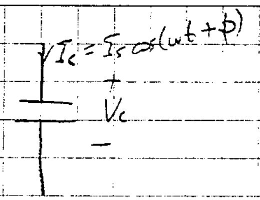


Time domain:	Phasor
$I_R(t) = I_s\cos(wt)$	$I_R = I_s\angle 0$
$V_R(t) = RI_s\cos(wt)$	$V_R = RI_s\angle 0$





C :



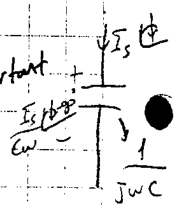
$$I_c(t) = I_s \cos(wt+\phi)$$

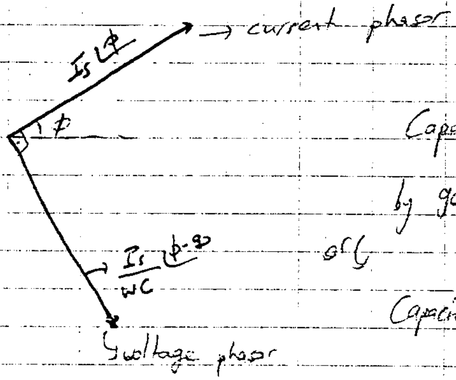
$$V_c(t) = \frac{I_s \sin(wt+\phi)}{Cw}$$

$$= \frac{I_s}{Cw} \cos(wt+\phi-\frac{\pi}{2})$$

$$I_c = I_s \angle\phi$$

$$V_c = \frac{I_s}{Cw} \angle \phi-9\overset{\circ}{0}$$ → important



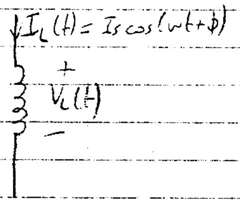


Capacitor voltage (phasor) lags the current (phasor) by 90°.

or,

Capacitor current leads the voltage by 90°

L :



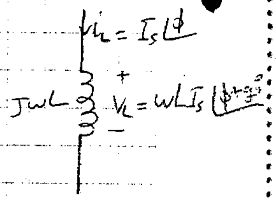
$$I_L(t) = I_s \cos(wt+\phi)$$

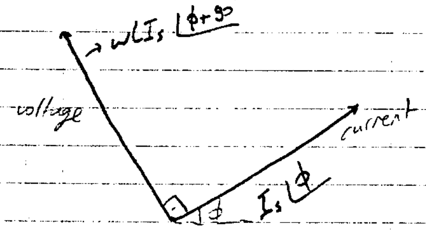
$$V_L(t) = -wLI_s \sin(wt+\phi)$$

$$= wLI_s \cos(wt+\phi+\frac{\pi}{2})$$

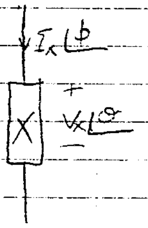
$$I_L = I_s \angle\phi$$

$$V_L = wLI_s \angle \phi+90^\circ$$





Impedence and Admittance



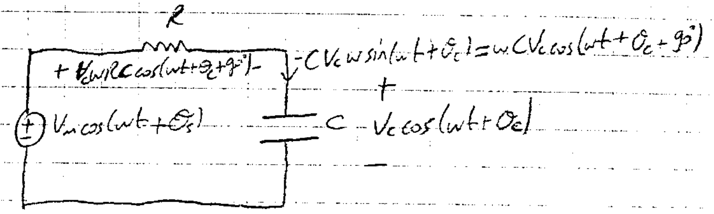
$$\text{Impedance} \Rightarrow Z = \frac{\text{Voltage Phasor}}{\text{Current Phasor}} = \frac{V_x}{I_x} \angle \theta-\phi \quad (\Omega)$$

$$\text{Admittance} \Rightarrow Y = \frac{1}{Z} = \frac{\text{Current Phasor}}{\text{Voltage Phasor}} = \frac{I_x}{V_x} \angle \phi-\theta \quad (\mho)$$

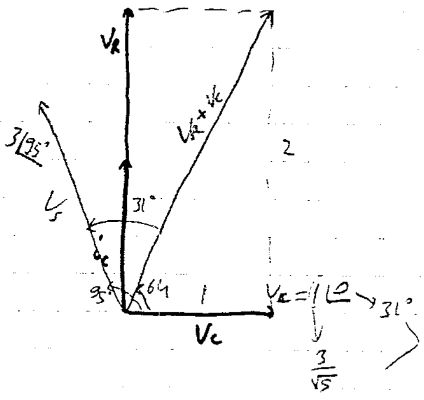
$$Z = \underbrace{Re\{Z\}}_{\text{Resistance}} + J \underbrace{Im\{Z\}}_{\text{Reactance}}$$

$$Y = \underbrace{Re\{Y\}}_{\text{conductance}} + J \underbrace{Im\{Y\}}_{\text{susceptance}}$$

Ex



$$V_m\cos(\omega t+\theta_s) = V_R(t)+V_c(t)$$



$\omega C = 1$

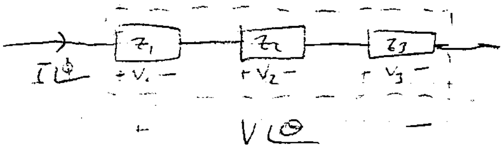
$R = 2$

$V_s = 3\cos(\omega t+95^\circ)$

$$\frac{3\sqrt{5}}{5}\angle 31^\circ \leftarrow \overset{\text{particular}}{V_c(t)} = \frac{3\sqrt{5}}{5}\cos(\omega t+31^\circ)$$

Series and Parallel Combination:

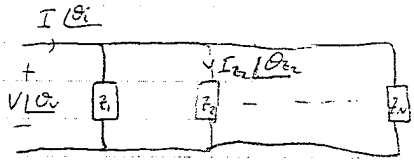
Series



$$V_1+V_2+V_3 = (I\angle\phi\, Z_1)+(I\angle\phi\, Z_2)+(I\angle\phi\, Z_3)$$

$$V\angle\theta = I\angle\phi\,\underbrace{(Z_1+Z_2+Z_3)}_{Z_{comb}}$$

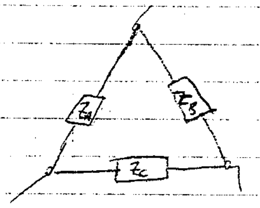
Parallel

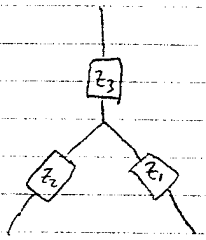


$$I\angle\theta_i = \sum_{k=1}^{N} I_{Z_k}\angle\theta_{Z_k}$$

$$= \sum_k \frac{V\angle\theta_v}{Z_k} = V\angle\theta_v \sum_k \frac{1}{Z_k} \Rightarrow \frac{V\angle\theta_v}{I\angle\theta_i} = \left(\sum_k \frac{1}{Z_k}\right)^{-1}$$

Δ-Y





$$Z_1 = \frac{Z_B Z_C}{Z_A+Z_B+Z_C} \qquad Z_A = \frac{Z_1Z_2+Z_2Z_3+Z_1Z_3}{Z_1}$$

$$Z_2 = \frac{Z_C Z_A}{Z_A+Z_B+Z_C} \qquad Z_B = \frac{Z_1Z_2+Z_2Z_3+Z_1Z_3}{Z_2}$$

$$Z_3 = \frac{Z_A Z_B}{Z_A+Z_B+Z_C} \qquad Z_C = \frac{Z_1Z_2+Z_2Z_3+Z_1Z_3}{Z_3}$$

18/03/2010
Persembe

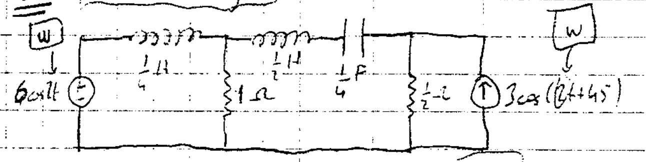
GENERAL AC CIRCUIT ANALYSIS

AC steady-state analysis (Particular solution due to A.C. excitation)

① Node
② Mesh
③ Thevenin-Norton
④ Other simplification methods

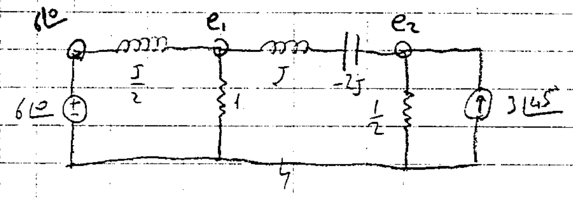
} AC Analysis.

Ex Node Analysis:



Find (A.C) steady state branch current-voltages

Phasor ↓ Domain $\boxed{w=2}$



KCL at e_1;

$$\frac{e_1}{1} + \frac{e_1 - 6\angle 0}{\frac{j}{2}} + \frac{e_1 - e_2}{-j} = 0 \Rightarrow e_1 - 2j(e_1 - 6\angle 0) + j(e_1 - e_2) = 0$$

KCL at e_2;

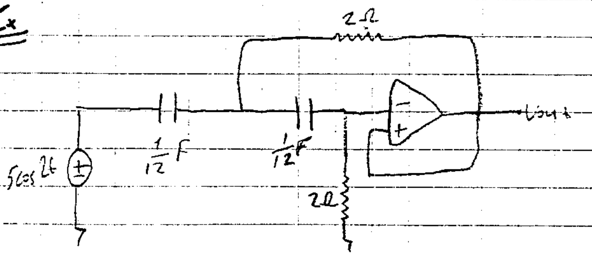
$$2e_2 + j(e_2 - e_1) - 3\angle 45^\circ = 0$$

$$\frac{e_2}{\frac{1}{2}} + \frac{e_2 - e_1}{-j} - 3\angle 45^\circ = 0$$

$$\begin{bmatrix} 1-j & -j \\ -j & 2+j \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -12j \\ 3\angle 45^\circ \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} ? \\ 3.63\angle 14^\circ \end{bmatrix} \Rightarrow e_2(t) = 3.63\cos(2t + 14^\circ) \text{ V}$$

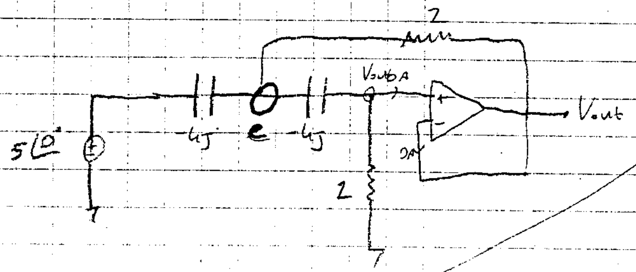
Ex



Assume Op-Amp is in linear region

Find $V_{out}^{ss}(t)$

Phasor domain



KCL at e:

$$\frac{e-5}{-4j}+\frac{e-V_{out}}{2}+\frac{e-V_{out}}{-4j}=0$$

KCL at V_{out};

$$\frac{e-V_{out}}{-4j}=\frac{V_{out}}{2}$$

$$e=(1-2j)V_{out}$$

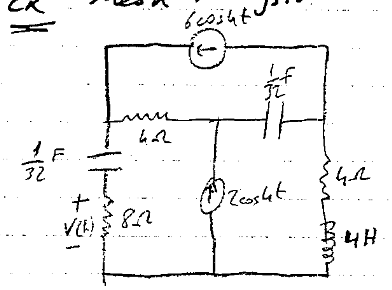
$$\hookrightarrow e(j+2+j)+V_{out}(-2-j)=5j$$

$\nearrow$ $V_{out}(1-2j)$

$$V_{out}\left[(1-2j)(2+2j)-2-j\right]=5j$$

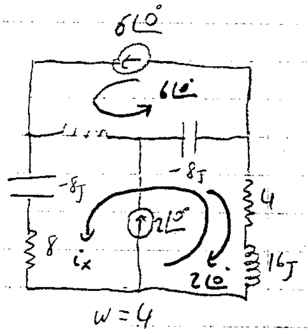
$$V_{out}=\frac{5j}{4-3j}=\frac{5j}{5\angle{-37^\circ}}=1\angle 127^\circ \Rightarrow \boxed{V_{out}=1\cos(2t+127^\circ)}$$

Ex Mesh Analysis



Find V(t)

$\xrightarrow{\text{Phasor Domain}}$



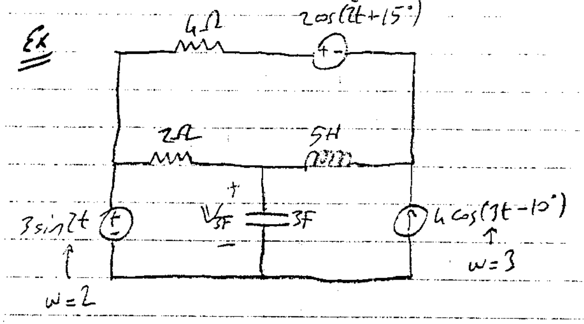
KVL around red loop;

$$4(i_x-6)+(8-8j)i_x+(16j+4)(i_x-2)-8j(i_x-2-6)=0$$

$$i_x=\frac{24+8(4j+1)-64j}{4+8-8j+16j+4-8j}=\frac{8(3-4j)}{16}=\frac{1}{2}5\angle{-53^\circ}\rightarrow V^{ss}(t)=20\cos(4t-53^\circ)$$

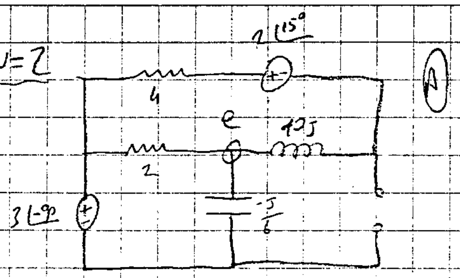
Sources with different frequencies (w):

Ex



$V^{ss}_{3F}(t)=?$

w=2

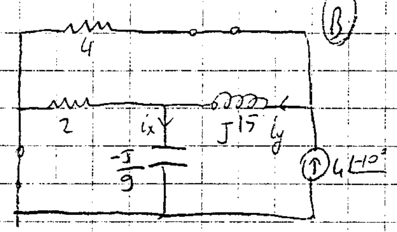


$$\longrightarrow \frac{e}{\frac{-j}{6}} + \frac{e-(3\angle{-90^\circ} - 2\angle{15^\circ})}{4+j10} + \frac{e-3\angle{90^\circ}}{2} = 0$$

$$e = V_{3F}^{Ⓐ}\angle\theta_A$$

$$V_{3F}^{Ⓐ}(t) = V_{3F}^{Ⓐ}\cos(2t+\theta_A)$$

w=3



$$\longrightarrow i_x = 4\angle{-10^\circ}\,\overbrace{\frac{4}{4+(j15+(\frac{-j}{9}\parallel 2))}}^{i_y}\cdot\frac{2}{-j/9+2}$$

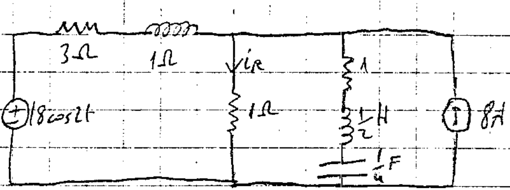
$$V_{3F} = \frac{-j}{9}i_x = V_{3F}^{Ⓑ}\angle\theta_B$$

$$V_{3F}^{Ⓑ}(t) = V_{3F}^{Ⓑ}\cos(3t+\theta_B)$$

Superposition: $V_{3F}(t) = V_{3F}^{Ⓐ}(t) + V_{3F}^{Ⓑ}(t)$

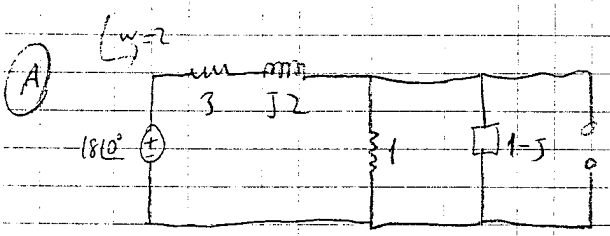
$$\boxed{V_{3F}(t) = V_{3F}^{Ⓐ}\cos(2t+\theta_A) + V_{3F}^{Ⓑ}\cos(3t+\theta_B)\ \ V}$$

Ex



$i_R^{ss}(t) = ?$

$w = \{2, 0\}$



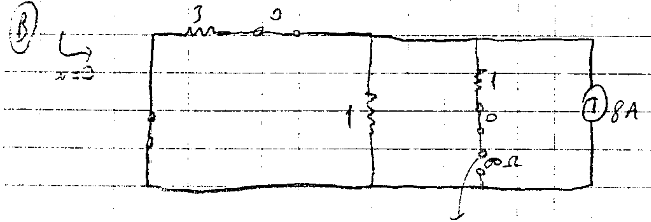
source transformation

$$\Downarrow$$

$$\to i_R^{Ⓐ} = \frac{18}{3+j2}\cdot\frac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{1-j}+\frac{1}{3+j2}} = 2(1-j) = 2\sqrt{2}\angle{-45^\circ}$$

$$\downarrow$$

$$i_R^{Ⓐ}(t) = 2\sqrt{2}\cos(2t-45^\circ)$$



$$i_R^{Ⓑ} = 8\cdot\frac{3}{1+3} = 6\,A$$

$$Z_c = \frac{1}{jwC}$$

$$\uparrow$$

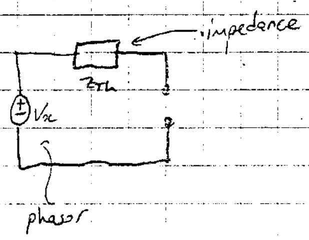
$$0$$

Superposition

$$i_R(t) = 6 + 2\sqrt{2}\cos(2t-45^\circ)\ A$$

GIPTA

Thevenin-Norton (Exactly as in Resistive Circuits with in phasor domain!



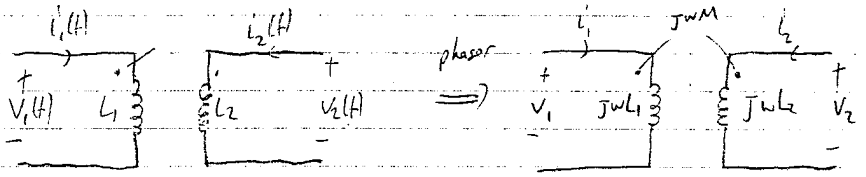
3 Methods for Thv-Nor Eq.

① Z_{th} by inspection (Turn-off sources)

② $Z_{th} = \frac{V_{oc}}{I_{sc}}$

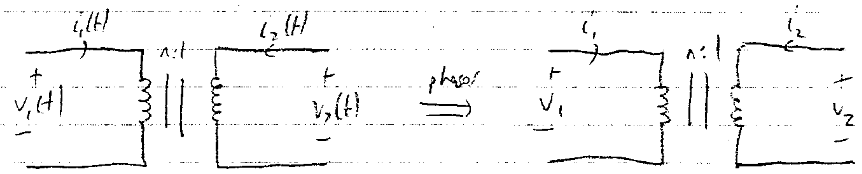
③ Apply V_{test} and check I_{test} . $Z_{Th} = \frac{V_{Test}}{I_{Test}}$

Coupled Inductor



$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} i_1(t) \\ \frac{d}{dt} i_2(t) \end{bmatrix} \Rightarrow \underset{\text{phasor}}{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}} = \begin{bmatrix} J\omega L_1 & J\omega M \\ J\omega M & J\omega L_2 \end{bmatrix} \underset{\text{phasor}}{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}$$

Ideal Transformer



$$\frac{V_1}{V_2} = n \qquad \frac{i_1}{i_2} = -\frac{1}{n}$$

phasor

22/03/2010
Pazartesi

Ex $(D^2+4)V_c(t) = \cos(2t)$

$(D^2+4)V_c(t) = 0$

$\lambda^2+4=0$

$\lambda_{1,2} = \{\mp 2j\}$

$$V_c^{comp}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + Ae^{j2t} + Bte^{j2t}$$

$$= c_1 e^{j2t} + c_2 e^{-j2t} + Ae^{j2t} + \underbrace{Bte^{j2t}}_{\text{part. soln}}$$

$$\cos(2t+\phi) + (\ldots)\underbrace{t}_{\substack{\text{unbounded}\\ \text{no steady state}}}\cos(2t+\varphi)$$

~~AC POWER ANALYSIS~~

AC POWER ANALYSIS

RMS or Effective Values

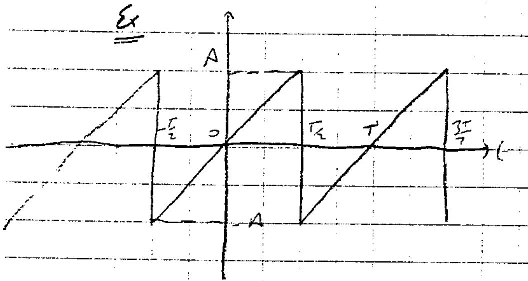
$x(t)$ - Periodic function $\iff \exists T\ (T\neq 0)\ (x(t-T) = x(t)\ \forall t$

Root Mean Square $\Rightarrow$ RMS $\rightarrow \sqrt{\dfrac{1}{T}\underbrace{\int_0^T}_{\substack{\text{Integrate over}\\ \text{a full period}}}(x(t))^2\,dt}$

Ex $x(t) = A\cos(\omega t+\phi)$

$$\text{RMS} \rightarrow \sqrt{\frac{1}{T}\int_0^T A^2\cos^2(\omega t+\phi)\,dt} = \sqrt{\frac{A^2}{T}\int_0^T \frac{1+\cos(2\omega t+2\phi)}{2}\,dt} = \sqrt{\frac{A^2}{T}\left(\frac{T}{2} + \frac{\int_0^T \cos(2\omega t+2\phi)}{2}\right.} = \boxed{\frac{A}{\sqrt{2}}}$$

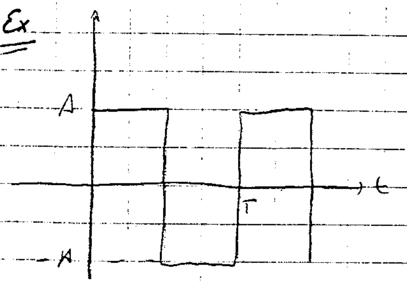
Ex



$$\text{RMS} \rightarrow \sqrt{\frac{1}{T}\int_{-\frac{T}{2}}^{T/2}\left(\frac{2A}{T}t\right)^2 dt} = \sqrt{\frac{1}{T}2\int_0^{T/2}\left(\frac{2A}{T}t\right)^2 dt}$$

$$= \sqrt{\frac{2}{T}\frac{4A^2}{T^2}\int_0^{\frac{T}{2}} t^2\,dt} = \sqrt{\frac{2}{T}\frac{4A^2}{T^2}\frac{(T/2)^3}{3}} = \boxed{\frac{A}{\sqrt{3}}}$$

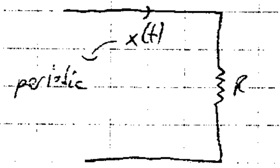
Ex



$\Rightarrow$ RMS$\{x(t)\} = A$

RMS (effective)

WHY RMS?



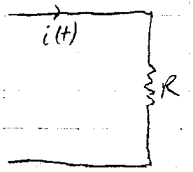
$$p(t) = R(x(t))^2$$

$$E_{\substack{\text{absorbed}\\ \text{by}\\ R}} = \int_0^T p(t)\,dt = \int_0^T Rx^2(t)\,dt = RT\left(\sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}\right)^2 = RT(x_{RMS})^2$$

$$= (R X_{RMS}^2).T$$

↖ energy calculation.

Average Power

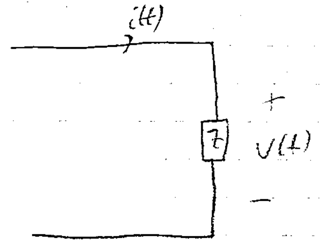


$$P_{AV} = \frac{1}{T}\int_0^T p(t)\,dt \quad \text{or} \quad P_{AV} = \lim_{T\to\infty}\frac{1}{T}\int_0^T p(t)\,dt$$

$$= \left(\frac{1}{T}\int i^2(t)\,dt\right)R \Rightarrow P_{AV} = (i_{RMS})^2 R$$

Energy consumed in 5 periods $= (P_{AV} \cdot \underbrace{5T}_{\text{duration}})$ ← Energy consumed.

Average and Instanteneous Power



$$\left.\begin{array}{l} V(t) = V_m\cos(\omega t+\theta_v) \longrightarrow V = V_m \angle\theta_v \\ i(t) = I_m\cos(\omega t+\theta_i) \longrightarrow i = I_m \angle\theta_i \end{array}\right\} Z = \frac{V_m}{I_m}\overset{\theta_z}{\angle\theta_v-\theta_i}$$

$$p(t) = V(t)\,i(t) = V_m I_m \cos(\omega t+\theta_v)\cos(\omega t+\theta_i)$$

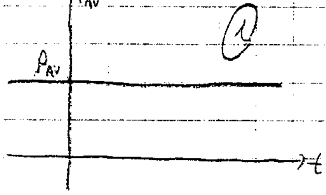
$$= \frac{V_m I_m}{2}\left[\cos(\underbrace{\theta_v-\theta_i}_{\theta_z}) + \cos(2\omega t+\underbrace{\theta_v+\theta_i}_{2\theta_i+\theta_z})\right]$$

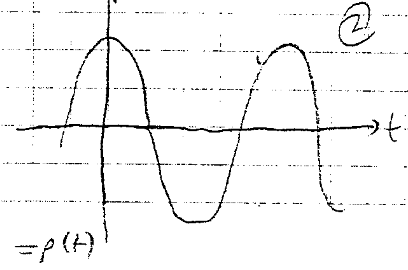
$$= \frac{V_m I_m}{2}\left[\cos\theta_z + \cos(2(\omega t+\theta_i))\cos(\theta_z) - \sin(2(\omega t+\theta_i))\sin(\theta_z)\right]$$

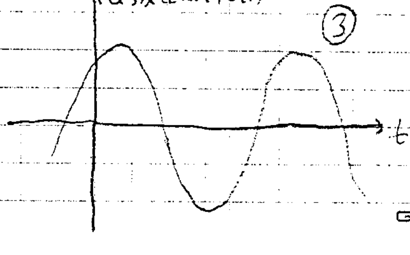
$$= \frac{V_m I_m}{2}\cos(\theta_z)\left\{1+\cos(2(\omega t+\theta_i))\right\} - \frac{V_m I_m}{2}\sin\theta_z\sin(2(\omega t+\theta_i))$$

$$P_{AV} = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_m I_m}{2}\cos(\theta_z) = V_{rms} I_{rms}\cos(\theta_z) = P_{AV}$$

$$p(t) = P_{AV}\left\{1+\cos(2(\omega t+\theta_i))\right\} - Q\sin(2(\omega t+\theta_i)) \qquad Q = \frac{V_m I_m}{2}\sin(\theta_z)$$



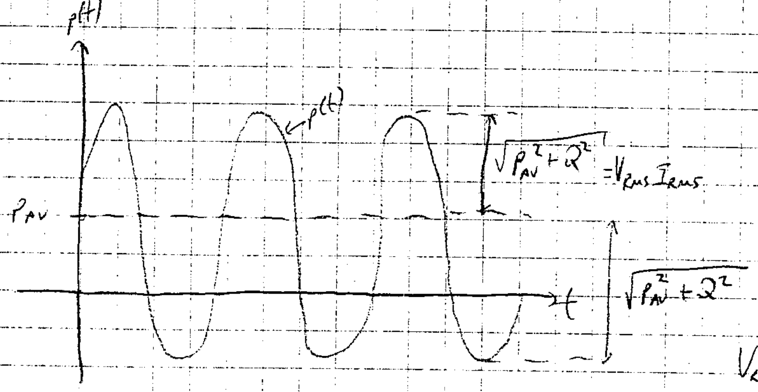




①+②+③ $= p(t)$

$$p(t) = P_{AV} + P_{AV}\cos(2(\omega t+\theta_i)) - Q\sin(2(\omega t+\theta_i))$$

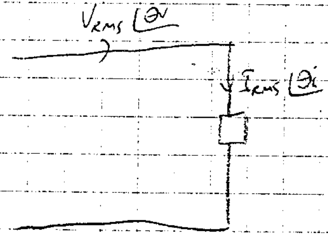
$$= P_{AV} + \sqrt{P_{AV}^2+Q^2}\cos(2\omega t + 2\theta_i + \tan^{-1}\frac{Q}{P_{AV}})$$



$$\sqrt{P_{AV}^2+Q^2} = \sqrt{\left(\frac{V_m I_m}{2}\cos\theta_z\right)^2 + \left(\frac{V_m I_m}{2}\sin\theta_z\right)^2}$$

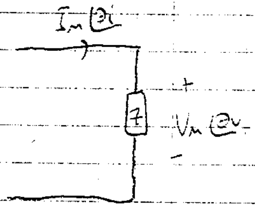
$$= \frac{V_m I_m}{2} = V_{RMS} I_{RMS}$$

$V_{RMS}I_{RMS}$: Apparent Power



Apparent Power $= V_{RMS} I_{RMS}$

Special Cases



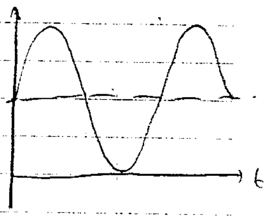
$Z = \frac{V_m}{I_m}\underbrace{\angle\theta_v-\theta_i}$ θ_z

① $Z=R \rightarrow \theta_z=0$

$P_{AV} = \frac{V_m I_m}{2} = V_{RMS} I_{RMS}$ $\qquad Q=0$

$p(t) = P_{AV}\{1+\cos(2(\omega t+\theta_i))\}$

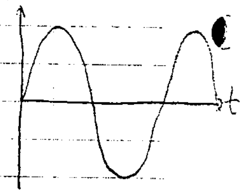
p(t) ; for resistor



② Z = Inductor $\longrightarrow \theta_z = 90°$

$P_{AV}=0 \quad Q = \frac{V_m I_m}{2} \qquad p(t) = -\frac{V_m I_m}{2}\sin(2(\omega t+\theta_i))$

p(t): for inductor



Q : Reactive Power (VAR)

Volt, Ampere, Reactive

$Q = \frac{V_m I_m}{2}\sin(\theta_z)$

25/03/2010
Persembe

$p(t) = P_{AV}\{1+\cos(2(\omega t+\theta_i))\} - Q\sin(2(\omega t+\theta_i))$

$P_{AV} = V_{RMS} I_{RMS}\cos(\theta_z)$
$Q = V_{RMS} I_{RMS}\sin(\theta_z)$

$\frac{1}{T}\int_0^T p(t)dt = P_{AV}$ in Watts (time average of $p(t)$)

work done in "T_s" sec's $= P_{AV} \cdot T_s \Rightarrow$ Joule's

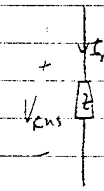
P_{AV}: watts

T_s: Assume $(T_s = kT)$

k: integer

Let's check

$$p(t) = -V_{rms} I_{rms} \sin(2(wt+\theta_i))$$

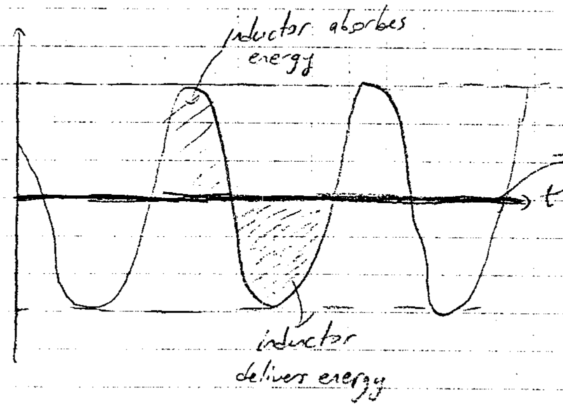


$Z = jwL$

$Z = wL \angle 90°$

$$P_{AV} = V_{rms} I_{rms} \cos(90°) = 0$$

$$Q = V_{rms} I_{rms} \sin(90°) = V_{rms} I_{rms}$$

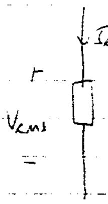


$$\rightarrow \langle p(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t)\,dt$$

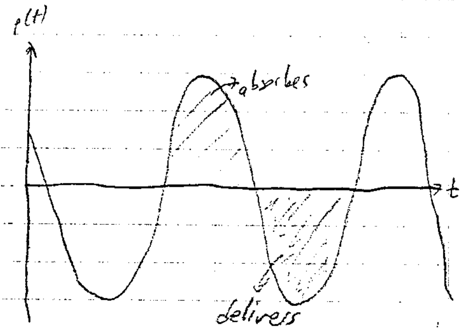
Inductor does not do any work!

at times it collects energy

at other times it delivers (energizes) other component.



$Z = \frac{1}{jwC} = \frac{1}{wC} \angle -90° \nearrow \begin{array}{l} P_{AV} = 0 \\ Q \neq 0 \end{array}$



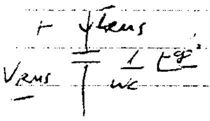
Average Stored Energy;

① Capacitor : $E_c(t) = \frac{1}{2} C V_c^2(t)$

$$E_c^{AVG} = \frac{1}{T}\int_0^T (E_c(t))\,dt \Rightarrow E_c^{AVG} = \frac{1}{2} C (V_c^{RMS})^2$$

② Inductor: $E_L^{AVG} = \frac{1}{2} L \left(I_L^{RMS}\right)^2$

Then;



$Q = V_{rms} I_{rms} \sin(\theta_z) = -V_{rms} I_{rms}$

$P_{AV} = 0$

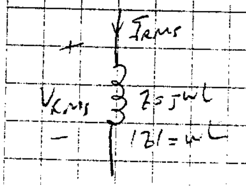
$$\frac{V_{phasor}}{I_{phasor}} = Z = \frac{1}{jwC}$$

$$\frac{V_{rms}}{I_{rms}} = |Z| = \frac{1}{wC}$$

$$Q = -V_{RMS} \frac{V_{RMS}}{\frac{1}{wC}}$$

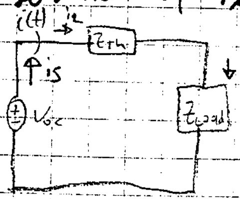
$$Q = -WC V_{RMS}^2$$

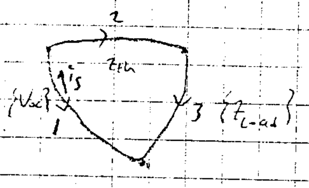
$$Q = -2w E_C^{Avg}$$



$$Q = V_{RMS} I_{RMS} \sin(90) = V_{RMS} I_{RMS}$$
$$= \omega L I_{RMS}^2$$
$$= 2\omega E_L^{AVG}$$

Conservation of P_{AVG} and Q:





$$i_k(t) = I_m^k \cos(\omega t + \theta_i^k)$$

$$V_k(t) = V_m^k \cos(\omega t + \theta_v^k)$$

Tellegen's theorem;

$$\sum_{k=1}^{3} \underset{\overset{\uparrow}{V_k(t)\cdot i_k(t)}}{p_k(t)} = 0 \qquad \Rightarrow p_2(t) + p_3(t) = -p_1(t) = p_s(t) \leftarrow \text{power supplied by } V_{oc} \quad \forall t$$

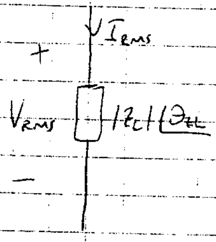
$$P_{AV}^{②}\{1+\cos(2(\omega t+\theta_i^{②}))\} - Q^{②}\sin(2(\omega t+\theta_i^{②})) + P_{AV}^{③}\{1+\cos(2(\omega t+\theta_i^{③}))\} - Q^{③}\sin(2(\omega t+\theta_i^{③})) = P_{AV}^{⑤}\{1+\cos(2(\omega t+\theta_i^{⑤}))\} - Q^{⑤}\sin(2(\omega t+\theta_i^{⑤})), \forall t$$

due to valid $\Big\downarrow$ for $\forall t$

$$\sum P_{AVG}^{absorbed} = P_{AVG}^{supplied}$$

$$\sum Q^{absorbed} = Q^{supplied}$$

Complex Power

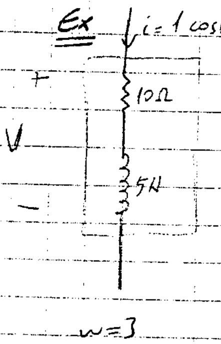


$$\underset{\substack{\text{complex}\\\text{power}}}{\underset{\sim}{S}} = P + jQ = V_{RMS} I_{RMS} \cos(\theta_Z) + j V_{RMS} I_{RMS} \sin(\theta_Z)$$

Note: $p(t) = P_{AVG} + P_{AVG}\cos(2(\omega t+\theta_i)) - Q\sin(2(\omega t+\theta_i))$

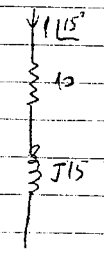
$$= P_{AVG} + Re\{\underbrace{(P_{AVG} + jQ)}_{S} e^{j2(\omega t+\theta_i)}\}$$

Ex



Find P_{AV}, Q, S for the component

Method ①

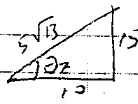


$$\rightarrow i_{RMS} = \frac{1}{\sqrt{2}}$$

$$V_{RMS} = \frac{5\sqrt{13}}{\sqrt{2}}$$

$$V = (1\angle 15°)(10 + j15) = (1\angle 15°)\, 5\sqrt{13}\, \angle \tan^{-1}\tfrac{3}{2}$$
$$= 5\sqrt{13}\, \angle 15° + \tan^{-1}\tfrac{3}{2}$$

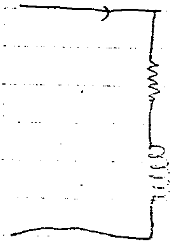
$$P_{AVG} = I_{RMS} V_{RMS} \cos(\theta_Z)$$
$$= \frac{5\sqrt{13}}{2}\,\frac{10}{5\sqrt{13}} = 5 \text{ watts}$$



$$Q = I_{RMS} V_{RMS} \sin(\theta_Z) = \frac{5\sqrt{13}}{2}\,\frac{15}{5\sqrt{13}} = 7.5 \text{ VAR}$$

Method ②

P_{AVG} of $(10+J15)$ is only due to $10\,\Omega$



$$I_{RMS} = \frac{1}{\sqrt{2}}$$

$$P_{AVG}^{10\Omega} = V_{RMS}\, I_{RMS} \cos\underbrace{(\theta_R)}_{1}$$

for a resistor θ

$$= (I_{RMS} R)\, I_{RMS}$$

$$= I_{RMS}^2 . R = \left(\frac{1}{\sqrt{2}}\right)^2 10 = 5 \text{ watts}$$

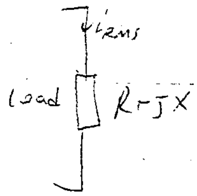
$$Q^{J15} = \overbrace{V_{RMS}\, I_{RMS} \sin(\theta_Z)}^{0}$$

$$= (|Z_{J15}|\, I_{RMS})\,(I_{RMS})$$

$$= 15\,(I_{RMS})^2$$

$$= \frac{15}{2} = 7.5 \text{ VAR}$$

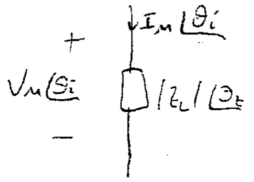
So in general;



$$P_{Load} = I_{RMS}^2 . R$$

$$Q_{Load} = I_{RMS}^2\, X$$

Method ③

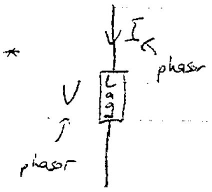


$$S = \frac{1}{2} \overset{\text{phasor}}{V} \overset{\text{phasor}}{I^*}$$

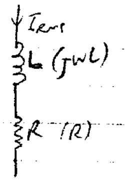
$$S = V_{RMS}\, I_{RMS}\, (\cos(\theta_Z) + J\sin(\theta_Z))$$

$$= \frac{|V|}{\sqrt{2}} \frac{|I|}{\sqrt{2}} e^{J\theta_Z}{}_{(\theta_v - \theta_i)} = \frac{1}{2}\left(|V|e^{J\theta_v}\right)\left(|I|e^{-J\theta_i}\right) = \frac{1}{2} \underbrace{V\, I^*}_{\text{phasor}}$$

$$\Rightarrow S_{Load} = \frac{1}{2} V I^*$$

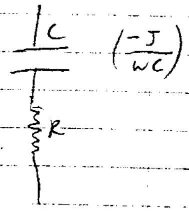


Note: → Inductive loads



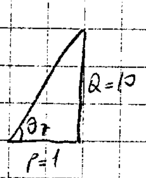
$$S_{Load} = P + J\,Q_{ind} \begin{pmatrix} P > 0 \\ Q > 0 \end{pmatrix} \quad P = I_{RMS}^2 R \quad Q_{ind} = I_{RMS}^2\, \omega L$$

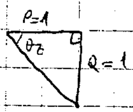
→ Capacitive loads



$$S_{Load} = P + J\,Q_{cap} \quad P > 0 \quad Q < 0 \quad P = I_{RMS}^2 R \quad Q_{cap} = -\,I_{RMS}^2 \frac{1}{\omega C}$$

Ex $S = 1 + j10$ $S = 1 - j1$





Power factor

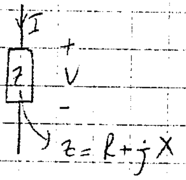
Power factor $= \cos(\theta_z)$

For inductor → lagging (Current lags)

For capacitor → leading (Current leads)

29/03/2010
Pazartesi

Review AC Steady-State Power



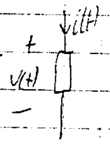
$$S = \frac{1}{2} V I^* = \frac{1}{2}|I|^2 Z = I_{rms}^2 Z = I_{rms}^2 (R + jX)$$

$$\measuredangle S = \measuredangle Z$$

$pf = \cos(\measuredangle Z)$ lagging / leading

$S = P + jQ$
$P = I_{rms}^2 R$
$Q = I_{rms}^2 X$

Ex



$v(t) = 3\cos(4t + 30^\circ)$

$i(t) = 6\cos(4t - 20^\circ)$

① Find real power (watts) and Reactive Power (VAR) of the component.

$$S = \frac{1}{2} V I^* = \frac{1}{2} 3\angle 30^\circ (6\angle -20)^* = 9\angle 50^\circ = 5.7 + j6.8$$

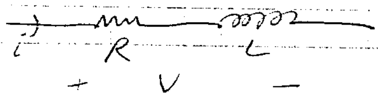
$P = 5.7$ watts $Q = 6.8$ VAR's

② Find p.f. of the component

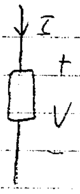
$p.f = \cos(\measuredangle S) = \cos(50^\circ) = 0.64$ (lagging) (inductive load)
(current lags voltage)

③ Assuming that the component is constructed from R, L, C, find suitable R, L, C to realize the component

$$Z = \frac{V}{I} = \frac{3\angle 30^\circ}{6\angle -20} = \frac{1}{2}\angle 50^\circ = \underbrace{0.32}_{R} + \underbrace{j0.38}_{j\omega L} \Rightarrow L = \frac{0.38}{4} = 0.095\,H$$



Ex



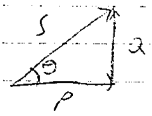
$V = 100\,V$ (RMS) $\leftarrow V(t) = 100\sqrt{2}\cos(\omega t + \theta)$

$P = 5\,kW$

$p.f = 0.8$ (lagging)

a) Find VAR (Reactive power)

$+ \Rightarrow$ lagging



$p.f = 0.8 \rightarrow \cos\theta = 0.8$

$S = P \overset{-}{\oplus} jQ$

$Q = P \cdot \tan(\theta) = 5000 \cdot \frac{3}{4} = 3750$

$S = 5000 + j\,3750$

b) Apperent power = ?

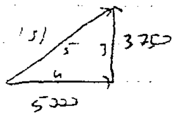
Apparent Power = $V_{rms} I_{rms}$

$$S = 5000 + j\,3750 = I_{rms}^2 (R + jX) = \frac{1}{2}|I|^2 Z = \frac{1}{2}\left|\frac{V}{Z}\right|^2 \cdot Z = \frac{1}{2}\frac{|V|^2}{|Z|^2} Z = \frac{1}{2}\frac{|V|^2}{Z^*} = \frac{V_{rms}^2}{Z^*}$$

$$S = 5000 + j\,3750 = \frac{10000}{Z^*} = Z^* = \frac{1}{0.5 - j\,0.375} \Rightarrow Z = \frac{0.5 + j\,0.375}{0.5^2 + 0.375^2}$$

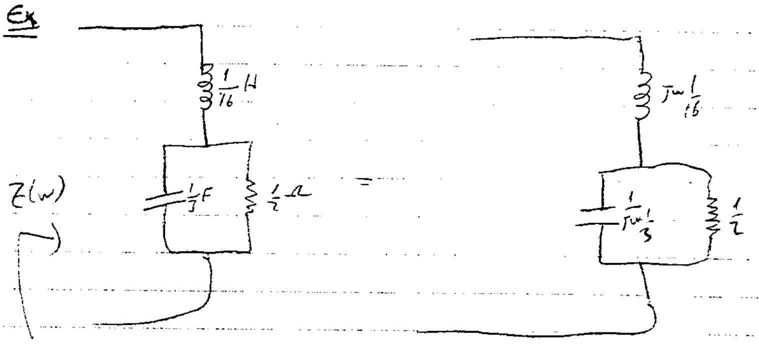
$$\text{Apparant power} = V_{rms} I_{rms} = 100 \cdot \frac{100}{|Z|} = \frac{100 \times 100}{\frac{1}{\sqrt{0.5^2 + 0.375^2}}} = 10000\sqrt{0.5^2 + 0.375^2} = 6250$$

Apparant power = $|S|$ $\left(\begin{array}{l} S = V_{rms} I_{rms} (\cos \measuredangle\theta_z + j \sin \measuredangle\theta_z) \\ |S| = V_{rms} I_{rms} \end{array} \right)$



$|S| = \frac{5}{4} \cdot 5000 = 6250$ VA

Ex



$$Z(\omega) = j\omega\frac{1}{16} + \frac{\frac{3}{j\omega} \cdot \frac{1}{2}}{\frac{3}{j\omega} + \frac{1}{2}}$$

$$Z(\omega) = \frac{j\omega}{16} + \frac{3}{6 + j\omega} = \frac{j\omega}{16} + \frac{3(6 - j\omega)}{36 + \omega^2}$$

$$Z(\omega) = \frac{j\,36\omega + j\omega^3 + 288 - j\,48\omega}{16(36 + \omega^2)}$$

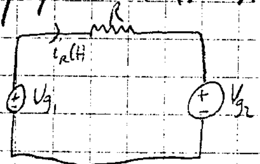
$$Z(\omega) = \frac{288 + j\omega(\omega^2 - 12)}{16(36 + \omega^2)}$$

① $\omega = \sqrt{12} \rightarrow Z(\omega)\Big|_{\omega = \sqrt{12}} = \frac{288}{16(36 + 12)}$

② $\omega > \sqrt{12} \rightarrow$ Inductive load

③ $\omega < \sqrt{12} \rightarrow$ Capacitive load

Superposition in AC Power



$V_{g_1} = V_1 \cos(w_1 t + \theta_1)$

$V_{g_2} = V_2 \cos(w_2 t + \theta_2)$ $\quad w_1 \neq w_2$

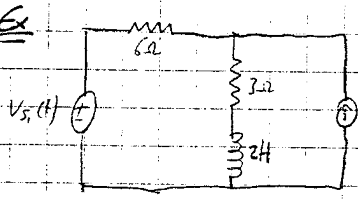
$i_R(t) = I_1 \cos(w_1 t + \phi_1) + I_2 \cos(w_2 t + \phi_2)$ (By superposition principle)

$$P_{av} = \frac{1}{T}\int_0^T (i_R(t))^2 R\,dt \longrightarrow P_{AV} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (i_R(t))^2 R\,dt$$

$$i_R^2(t) = I_1^2 \cos^2(w_1 t + \phi_1) + I_2^2 \cos^2(w_2 t + \phi_2) + 2 I_1 I_2 \cos(w_1 t + \phi_1)\cos(w_2 t + \phi_2)$$

$$P_{AV} = (I_1^{rms})^2 R + (I_2^{rms})^2 R + \underbrace{\frac{2}{T} I_1 I_2 \int_0^T \frac{1}{2}\left(\cos((w_1+w_2)t + \phi_1 + \phi_2) + \cos((w_1 - w_2)t + \phi_1 - \phi_2)\right) dt}_{0}$$

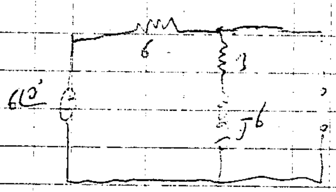
Ex



$V_{s_1}(t) = 6\cos(3t)$ V $\qquad P_{3\Omega}^{AVG} = ?$

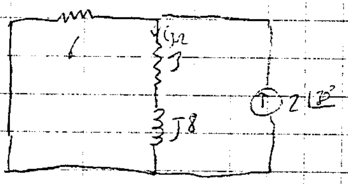
$i_{s_2}(t) = 2\cos(4t + 3°)$ A

w=3



$$i_{3\Omega} = \frac{6\angle 0°}{9 + j6} = \frac{2}{3 + j2} = \frac{2(3 - j2)}{13} = \frac{2\sqrt{13}}{13} \angle \tan^{-1}\left(-\frac{2}{3}\right) \rightarrow I_{3\Omega, rms} = \frac{2}{\sqrt{13}}\frac{1}{\sqrt{2}} = \sqrt{\frac{2}{13}}$$

W=4

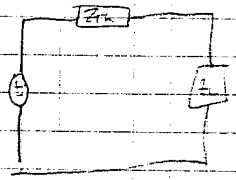


$$i_{3\Omega} = 2\angle 3° \frac{6}{9 + j8} = \frac{12\angle 3°}{\sqrt{145}\angle \tan^{-1} 8/9} = \frac{12}{\sqrt{145}} \angle 3° - \tan^{-1} 8/9 \;,\; I_{3\Omega, rms} = \frac{6\sqrt{2}}{\sqrt{145}}$$

$i_{3\Omega}(t) = 0.555\cos(3t - 33.69) + 0.997\cos(4t - 11.634°)$ A

$$P_{AV} = P_{AV}^{w=3} + P_{AV}^{w=4} = (I_{rms}^{w=3})^2 \cdot 3 + (I_{rms}^{w=4})^2 \cdot 3 = 0.462 + 1.491 = 1.953 \text{ Watt}$$

Maximum Power Transfer



V_{oc}, Z_{th} : fixed

What should be the value for Z_L

such that P_{AVG} of the load (Z_L) is maximized?

$Z_L = Z_{th}^*$ is the max. power transfer solution

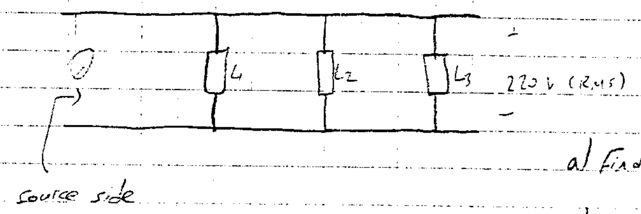
Proof

$$P_{load} = I_{rms}^2 \cdot R_L = \frac{1}{2}\left|\frac{V_{oc}}{(R_{th} + jX_{th}) + (R_L + jX_L)}\right|^2 \cdot R_L$$

$$P_{load}(R_L, X_L) = \frac{1}{2}\frac{(V_{oc})^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2} \cdot R_L \Rightarrow \underbrace{\frac{\partial P_{load}}{\partial R_L} = 0 \rightarrow \frac{\partial P_{load}}{\partial X_L} = 0}_{\text{solve together}}$$

Ex

01/04/2010
Persembe



Load 1 : 16 kW , 18 VAR
Load 2 : 10 kVA at 0.6 lead
Load 3 : 8 kW at unity p.f.

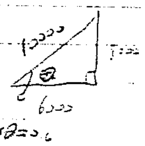
a) Find p.f on the source side

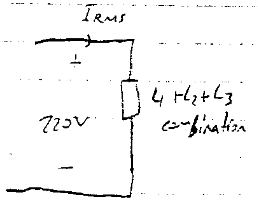
b) Find the impedance seen by the source.

a) $S_1 = 16 + j18$ kVA
$S_2 = 6 - j8$ kVA
$+ \; S_3 = 8$ kVA

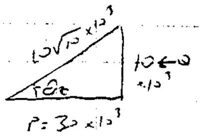
$S_{total} = 30 + j10$ kVA

$S_{source}^{supplied} = 30 + j10$ kVA





$S_{total} = I_{RMS}^2 \underset{(R+jX)}{\overset{\uparrow}{Z_{comb}}} \rightarrow \measuredangle S = \measuredangle X$



$p.f = \cos(\theta_Z) = \frac{30}{10\sqrt{10}} = \boxed{\frac{3\sqrt{10}}{10}}$ lagging

b) $S = I_{RMS}^2 Z_{comb}$; $\underset{10\sqrt{10}\times 10^3}{|S|} = \underset{220V}{V_{RMS}} I_{RMS} \Rightarrow I_{RMS} = \frac{10^4\sqrt{10}}{220} = 163.74$ A (RMS)

$Z_{comb} = \frac{(30+j10) \times 10^3}{(163.74)^2} = 1.45 + j0.48$

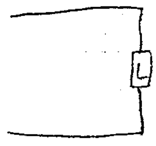
Power Factor Compensation

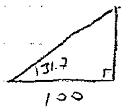
Ex A mill consumes 100 kW from 220 V RMS line, at p.f 0.85 lagging

a) Find the current (RMS), supplied by the source

b) " " " " " " " " if p.f were 0.95 lagging

a)





$\cos(31.7) = 0.85$ $\Rightarrow S_L = 100 + j100\tan(31.7)$

$|S_L| = \frac{100}{0.85} = 117.66$ kVA

$|S_L| = 117.66$ kVA $= V_{RMS} I_{RMS} = 220 \, I_{RMS} \Rightarrow \boxed{\frac{117.66 \times 10^3}{220} = I_{RMS}}$

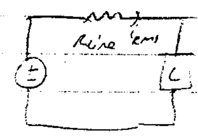
$\Rightarrow \boxed{I_{RMS} = 534.8 \text{ A}}$

b) $\cos(18) = 0.95$

$S_L = 100 + j(\tan 18) 100$ kVA

$|S_L| = \frac{100}{0.95} = 105.3$ kVA $\Rightarrow I_{RMS} = \frac{|S_L|}{220V} = \boxed{478 \text{ A}}$

c) If there is a line connecting generator to the load and $R_{line} = 0.1$; Find the power loss over the line for p.f. = {0.85, 0.95}



i) p.f = 0.85 ⇒ $P_{line} = (0.1) I_{RMS}^2 = 0.1 (534.8)^2 = 28.6$ kW

ii) p.f = 0.95 ⇒ $P_{line} = (0.1) I_{RMS}^2 = 0.1 (478)^2 = 22.9$ kW

d) Define efficiency

$$\text{efficiency} = \frac{\text{Real Power supplied to load}}{\text{Real Power supplied by source}}$$

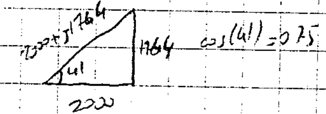
$p.f = 0.85 \quad eff = \frac{100}{100+28.6} = 77\%$

$p.f = 0.95 \quad eff = \frac{100}{100+22} = 82\%$

Ex Load requires 2kW at 0.75 pf lagging at 220V (RMS)

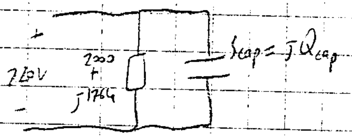
Calculate the reactive power supplied by the compensating capacitor to make p.f 0.9 and find the impedance of the capacitor and assuming 50 Hz, 220V (RMS) system find the capacitor value in terms of Farads.

Before

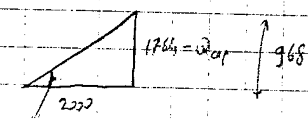


$\cos(41) = 0.75$

After



$S_{before} = 2000 + j1764$



$\Rightarrow Q_{cap} = 796$

$\Rightarrow S_{cap} = -j796$

$pf = 0.9$

$\cos(25) = 0.9$

$\frac{|S_{cap}|}{220} = \frac{796}{220} = 3.61A = I\ (RMS)$

$\Rightarrow S_{cap} = I_{RMS}^2 Z_{cap} = -j796 \Rightarrow Z_{cap} = \frac{-j796}{(3.61)^2} = -j61.07$

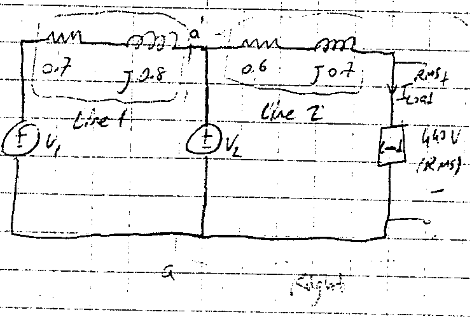
$Z_{cap} = \frac{-j}{wC} = +j61.07 \Rightarrow C = \frac{1}{2\pi \times 50 \times 61.07} = \boxed{52\mu F}$

Volt-Ampere Method:

We make use of conservation of complex power in the Volt-Ampere method

$$\sum_{k=1}^{\#\text{ of generators}} S_k^{\text{supplied}} = \sum_{k=1}^{\#\text{ of absorbing elements}} S_k^{\text{absorbed}}$$

Ex Two generators supply 15kW load at 0.8 p.f lagging. The generator-2 supplies 5kW at 0.6 p.f lagging. Find the voltages of V_1 and V_2 (RMS), the apparent power of generator-1 and p.f of generator-1



$S_{G2}^{sup} = 5000 + j6666$ VA

$S_L = 10000 + j7500$ VA

$$I_{Load}^{RMS} = \frac{|S_{Load}|}{V_{Load}^{RMS}} = \frac{10000/0.8}{440} = \frac{12500}{440} = 28.4 \text{ A} \quad RMS$$

$$S_{Line2} = (I_{Load}^{RMS})^2 (0.6 + j\,0.7) = 484 + j\,565$$

$$S_{Load+Line2} = 10484 + j\,8065$$

$$\hookrightarrow \; |S_{Load+Line2}| = I_{Load}^{RMS} \; V_2^{RMS}$$

$$\sqrt{10484^2 + 8065^2} = (28.4)\, V_2^{RMS}$$

$$\boxed{V_2^{RMS} = 466\text{V} \quad RMS}$$

$$S_{\substack{\text{right of}\\ a-a'}} = S_{Load+Line2} - S_{G2}^{sup} = 5484 + j\,1399$$

$$\Rightarrow |S_{\substack{\text{right of}\\ a-a'}}| = I_{Line1}^{RMS} \; V_{a-a'} \leftarrow V_2^{RMS}$$

$$\sqrt{5484^2 + 1399^2} = I_{Line1}^{RMS} \; 466$$

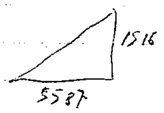
$$I_{Line1}^{RMS} = 12.14 \text{ A}$$

$$\Rightarrow S_{Line1} = (I_{Line}^{RMS})^2 (0.7 + j\,0.8) = 103 + j\,118$$

$$S_{G1}^{supp} = S_{Line1} + S_{\substack{\text{right of}\\ a-a'}} = 5587 + j\,1516$$

$$|S_{G_1}^{supp}| = V_1^{RMS} \underbrace{I_{Line}^{RMS}}_{12.14} \Rightarrow V_1^{RMS} = \frac{\sqrt{5587^2 + 1516^2}}{12.14} \Rightarrow \boxed{V_1^{RMS} = 477\text{V}}$$

$$p.f \text{ of generator } 1 = \frac{5587}{\sqrt{5587^2 + 1516^2}} \text{ logging}$$



$$= \boxed{0.965}$$

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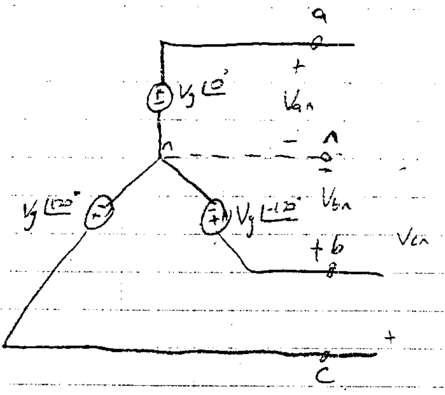
3-PHASE BALANCED CIRCUITS

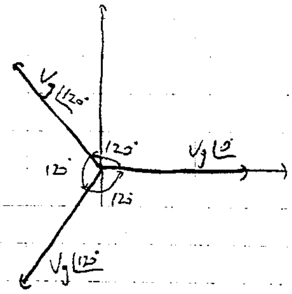
Y Connection

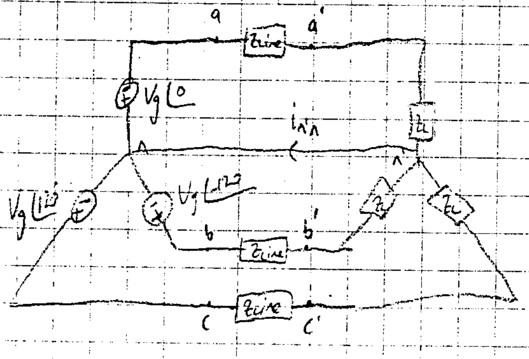


$$V_{an} = V_g \angle 0°$$

$$V_{bn} = V_g \angle -120°$$

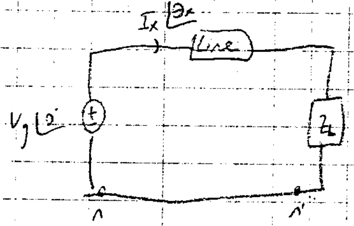
$$V_{cn} = V_g \angle 120°$$

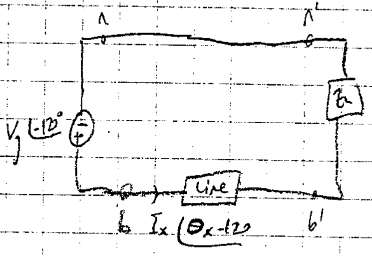




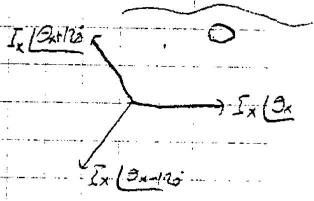
Balanced 3-ϕ system.

Note Solving only phase a is equivalent to solving all phases



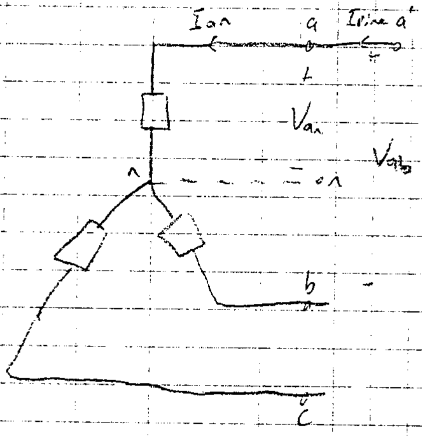


$I_{nn'} = \underbrace{I_x\angle\theta_x + I_x\angle\theta_x-120 + I_x\angle\theta_x+120}_{0}$



Phase-Line Currents / Voltages

Y Connection



V_{an}, V_{bn}, V_{cn} : Phase voltages

V_{ab}, V_{ac}, V_{bc} - Line voltages (Line to Line)

$V_{ab} = V_a - V_b = V_{an} - V_{bn} = V_{an} + V_{nb}$

I_{an} : Phase current

$I_{aa'}$: Line current

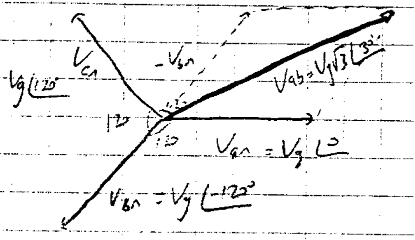
} Y connection ⇒ $I_{aa'} = I_{an}$

Phase voltage ≠ Line voltage

Phase current ≠ Line current

} For Y connection

Line-Voltage, Phase-Voltage Relation for Y-Connection



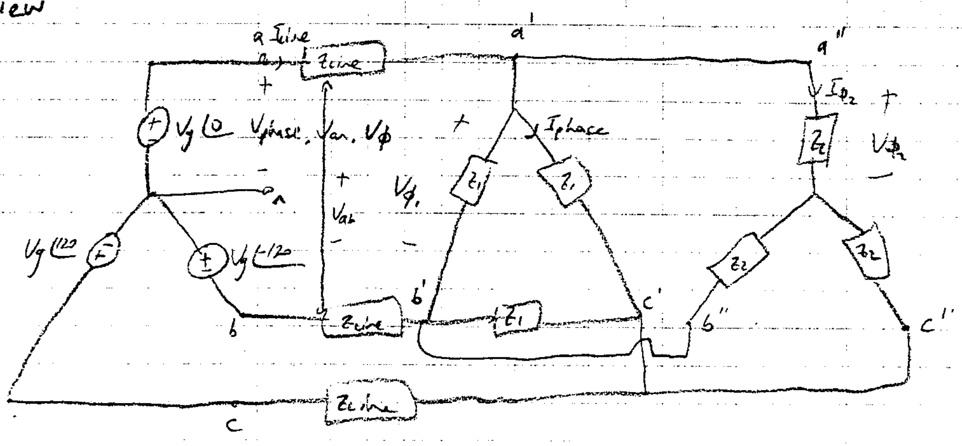
$V_{ab} = \sqrt{3}V_g\angle 30$

$V_{ac} = \sqrt{3}V_g\angle -30$

$\underbrace{V_{bc} = \sqrt{3}V_g\angle -90}_{\text{Line voltages}}$

Review

08/04/2010
Persembe



V_ϕ : Phase voltage

I_ϕ : Phase current
(Current in a single phase of the component)

V_{line} : Line voltage (V_{ab}, V_{bc}, V_{ac}

I_{line} : Current passing through the line (Z_{line})

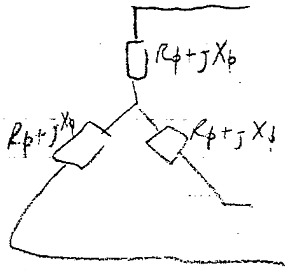
Power Relations for Y Connection

$P_{Tot} = 3 P_\phi$ ~ per phase

$S_{Tot} = 3 S_\phi$

$P_\phi = | I_{\phi,rms} |^2 R_\phi$

$S_\phi = | I_{\phi,rms} |^2 (R_\phi + j X_\phi)$



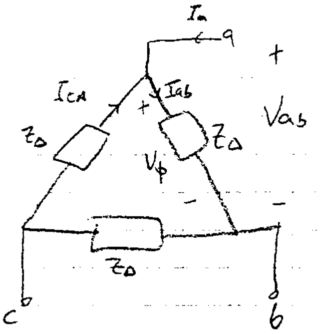
$$P_{Tot} = Re\{S_{Tot}\} = 3 Re\{S_\phi\}$$

$$= 3 Re\{ | I_{\phi,rms} |^2 | Z_\phi | e^{j \angle Z_\phi} \}$$

$$= 3 \, Re\{ | I_{\phi,rms} | \cdot | V_{\phi,rms} | e^{j \angle Z_\phi} \}$$

$$= 3 | I_{\phi,rms} | | V_{\phi,rms} | \cos(\angle Z_\phi) = \sqrt{3} \, | V_{line,rms} | \, | I_{line,rms} | \, \cos(\angle Z_\phi)$$

Δ - Connection.



For Δ - load;

$V_{line,rms} = V_{\phi,rms}$

$I_{line,rms} = \sqrt{3} \, I_{\phi,rms}$

$$I_a = I_{ab} - I_{ca}$$

$$= \frac{V_{ab}}{Z_\Delta} - \frac{V_{ca}}{Z_\Delta}$$

$$= \frac{V_\phi \angle 0^\circ}{Z_\Delta} - \frac{V_\phi \angle -120^\circ}{Z_\Delta}$$

$$= I_\phi - I_\phi \angle -120$$

$$I_a = \sqrt{3} \; I_\phi \angle 30^\circ$$

Generators are Y connected
In general, not Δ connected

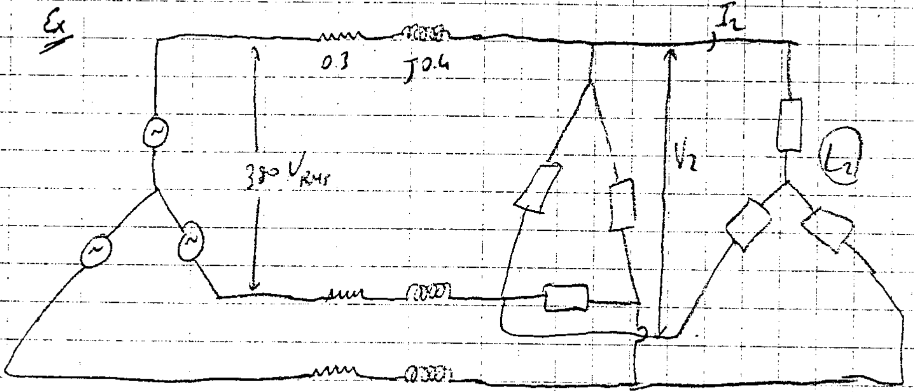
Power Calculation for Δ load:

$$P_{Tot} = 3P_\phi = 3Re\{ V^\Delta_{\phi,rms}\ I^\Delta_{\phi,rms}\ e^{j\angle Z_\Delta} \}$$

$$= 3Re\{ V^\Delta_{Line,rms}\ \frac{I_{Line,rms}}{\sqrt{3}}\ e^{j\angle Z_\Delta} \}$$

$$= \sqrt{3}\ V_{Line,rms}\ I_{Line,rms} \cos(\angle Z_\Delta)$$ ← same formula for both Δ and Y loads.

Ex

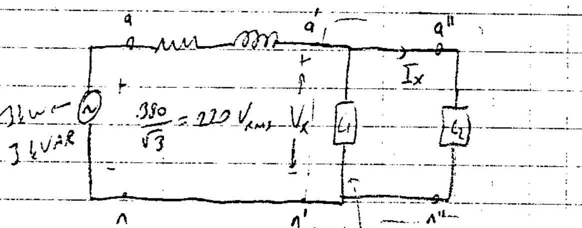


Generator { produces 380 V_{rms} line to line voltage
{ produces 9 kW and 9 kVAR

Load 1 6 kW at 0.6 p.f. lagging → (6000 + j8000)

Find V_2, I_2 and power absorbed by L_2.

① Using single phase equivalent circuit.



converted to an Y-load by Δ-Y connection

$S_G = 3 + j3$ kVA

$|S_G| = 3\sqrt{2}$ kVA $= 220\ I_{RMS} \Rightarrow I_{s,RMS} = 19.3$ A

$S_{Line} = I^2_{s,RMS}\ (0.3 + j0.4) = 112 + j149$

$S_{L_1} + S_{L_2} = 3000 + j3000 - (112 + j149)$

$= 2888 + j2851$

$|S_{L_1+L_2}| = |V_{x_{RMS}}|\ |I_{s,RMS}|$

4.058 kVA $= V_{x,RMS} \cdot 19.3$

$\boxed{V_{x,RMS} = 210.3} \Rightarrow V_2 = \sqrt{3}\ V_x = \boxed{364\ V_{RMS}}$

$S_{L_2} = S_{L_1+L_2} - S_{L_1}$

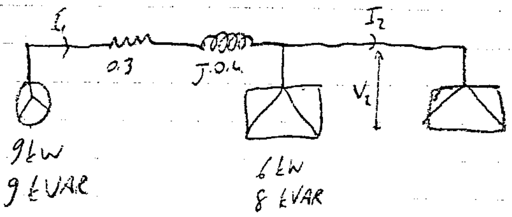
$S_{L_2} = 2888 + j2851 - \frac{(6000 + j8000)}{3} = 888 + j187$

$|S_{L_2}| = |V_x| \cdot |I_{x,RMS}| \Rightarrow I_{x,RMS} = 4.31\ A_{RMS} \Rightarrow I_{2_{RMS}} = I_{x,RMS} = \boxed{4.31\ A_{RMS}}$

Power absorbed by load 2

$S_{L_2}^{tot} = 3 S_{L_2} = \boxed{2664 + j561}$

② Single Line Diagram



For both loads

3Φ Total power quantities

$P_{load} = \sqrt{3}\, V_{line,rms}\, I_{line,rms} \cos(\measuredangle \theta_{load})$

$Q_{load} = \sqrt{3}\, V_{line,rms}\, I_{line,rms} \sin(\measuredangle \theta_{load})$

$S_{load} = P_{load} + jQ_{load}$

① Find $I_1 \rightarrow \underbrace{|S_{generator}^{total}|}_{9000\sqrt{2}} = \sqrt{3}\, \underbrace{V_{line,rms}^{generator}}_{380}\, I_{line}^{generator} \Rightarrow I_{line}^{generator} = 19.3$ Arms

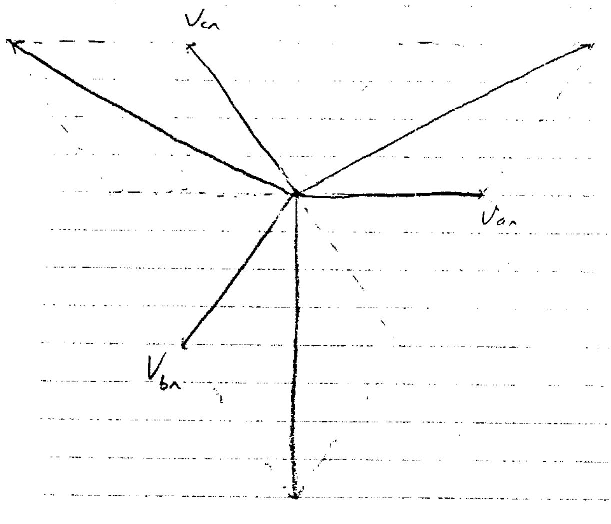
② Line losses $S_{line} = 3(19.3)^2(0.3 + j0.4) = 336 + j447$

③ Power left for L_1 and L_2 → $S_{L_1} + S_{L_2} = 8664 + j8553$

④ $\underbrace{|S_{L_1+L_2}|}_{12174.5} = \sqrt{3}\, |V_{L_1+L_2}^{line}| \cdot \underbrace{|I_{L_1+L_2}^{line}|}_{19.3} \Rightarrow |V_{L_1+L_2}^{line}| = 364$ Vrms

⑤ $S_{L_2} = 8664 - 6000 + j(8553 - 8000) = 2664 + j561$

12/04/2010
Pazartesi

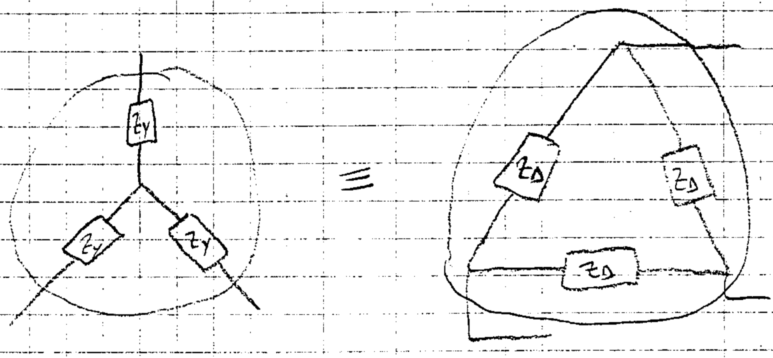


ω ← phasor diagram rotates ω rad/sec

a → b → c (positive sequence) ⇒ (a at 0°, then b at 0°, c at 0°)

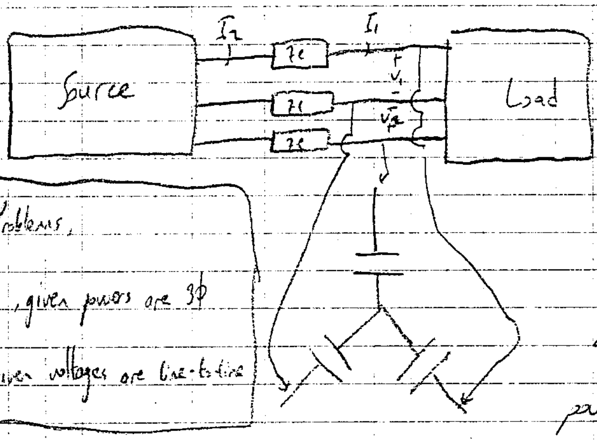
a → c → b (negative sequence)

Δ-Y Transformation



$Z_\Delta = 3 Z_Y$

Ex ZPS IV Problem 6



In 3φ problems,

① By default, given powers are 3φ

② By default, given voltages are line-to-line

A balanced 3φ circuit

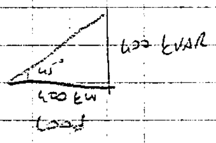
Load: 400 kW $pf = \frac{1}{\sqrt{2}}$ lagging

$V_1 = 400\angle 30°$ $V_2 = 400\angle -30°$ V_{rms}

$Z_\ell = 0.27 + j0.16$

a) Switches are open. Find I_1, $V_{s,eff}$, the complex power supplied by the source and efficiency.

a) $P_{tot} = 400\text{ kW} = \sqrt{3}\, \underbrace{V_{line}}_{400}\, \underbrace{I_{line}}_{I_1}\, \underbrace{\cos(\measuredangle\theta_z)}_{\frac{1}{\sqrt{2}}} \Rightarrow I_1 = 816.5 \text{ A}_{RMS}$



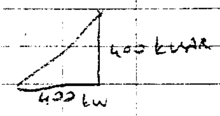
$S^{tot}_{line} = 3\, I^2_{1,RMS}\, (Z_\ell) = 140 + j320$ kVA

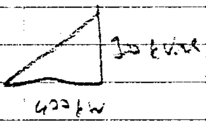
$S^{tot}_{supp} = 540 + j(320 + 400) = 540 + j720$ kVA

$Eff = \frac{400}{540} = 74\%$

b) Switches are closed, and p.f of the compensated load becomes 0.8 lagging Find the susceptance of the capacitors in the bank, I_1, I_2, $V_{s,eff}$, Complex power supplied by the load

b) Before comp   After Comp.





$|S^{Load}_{tot}| = \sqrt{3}\, V_{line} I_{line}$

$500 = \sqrt{3}\, 400\, I_{line} \Rightarrow I_{line} = I_2 = 721.6$ A

$S^{tot}_{line} = 3\, (I_{2,RMS})^2\, Z_\ell = 109 + j250$ kVA

$S^{tot}_{supp} = 509 + j555$ kVA

$Eff = \frac{400}{509} = 78\%$

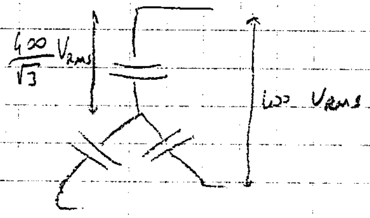
$V^{sup}_{line} = \frac{|S^{tot}_{supp}|}{\sqrt{3}\, |I^{line}_{supp}|} = 600\ V_{RMS}$

↖ 721.6

$S^{cap\text{-}bank} = -j100$ kVA

$S^{cap}_{\phi} = -j\frac{100}{3}$ kVA

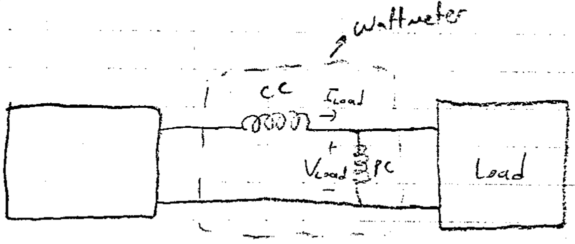
GIPTA



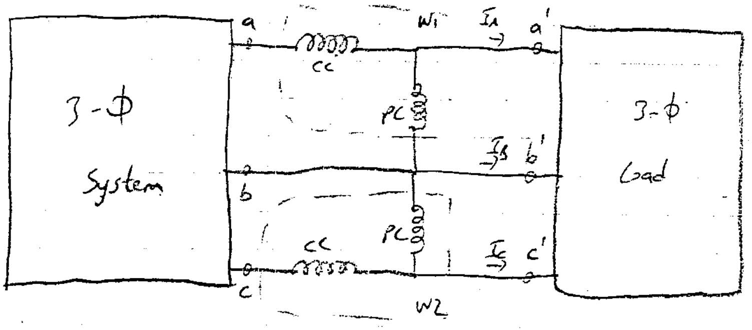
$$-J\frac{100}{3} = \frac{|V_{RMS}^{cap}|}{Z^*}$$

$$\frac{1}{Z_{cap}^*} = \frac{-J\frac{100}{3}}{\left(\frac{400}{\sqrt{3}}\right)^2} = -J\frac{10}{16} \Rightarrow Z_{cap}^* = \frac{16J}{10} \Rightarrow Z_{cap} = -J\frac{16}{10}$$

3-ϕ Power Measurement



CC: Current coil (low impedance)
PC: Potential coil (high impedance)

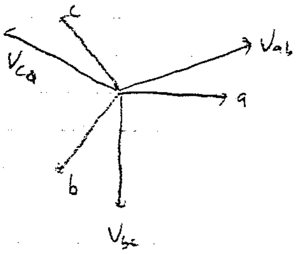


Measurement of W_1: $Re\{V_{a'b'} I_A^*\}$

Measurement of W_2: $Re\{V_{c'b'} I_C^*\}$

$V_{a'b'} = V_{line}\angle 30^\circ$

$V_{b'c'} = V_{line}\angle -90^\circ$



$I_A = I_{line}\angle -\theta$

$I_C = I_{line}\angle -\theta+120^\circ$

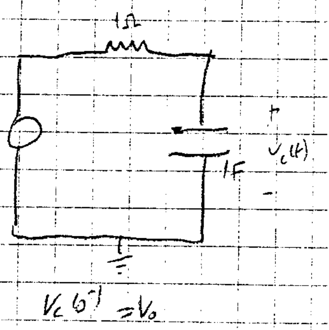
$$W_1 = Re\{V_{line}\angle 30^\circ\ I_{line}\angle \theta\} = V_{line} I_{line} \cos(\theta+30^\circ)$$

$$W_2 = Re\{-V_{line}\angle -90^\circ\ I_{line}\angle \theta-120^\circ\} = V_{line} I_{line} \cos(\theta-30^\circ)$$

$$W_1+W_2 = V_{line} I_{line}\left(\cos(\theta+30) + \cos(\theta-30)\right) = \sqrt{3}\, V_{line} I_{line} \cos\theta \Rightarrow$$ The total power absorbed by the load

Zafer

15/04/2010
Persembe



$V_c(0^-) = V_0$

$\dot{V}_c(t) = -V_c(t) + V_s(t)$

$V_c(0^-) = V_0$

s-domain

$\mathcal{L}\{\dot{V}_c(t)\} = \mathcal{L}\{-V_c(t) + V_s(t)\}$

$sV_c(s) - \underbrace{V_c(0^-)}_{V_0} = -V_c(s) + V_s(s)$

$(s+1)V_c(s) + V_0 = -V_c(s) + V_s(s)$

$(s+1)V_c(s) = V_0 + V_s(s)$

$V_s(t) = \delta(t) \Rightarrow V_s(s) = 1 \Rightarrow V_{c_{zs}}(s) \leftarrow H(s) = \mathcal{L}\{h(t)\}$
$= \frac{1}{s+1}$

$V_c(s) = \underbrace{\frac{1}{s+1} V_s(s)}_{V_{c_{zs}}(s)} + \underbrace{\frac{V_0}{s+1}}_{V_{c_{zi}}(s)}$

$h(t) = e^{-t}u(t)$

$V_c(t) = V_{c_{zs}}(t) + V_0 e^{-t}$

$u(t) \Rightarrow \frac{1}{s}$ $V_{c_{zs}}(s) = \frac{1}{s+1} \cdot \frac{1}{s} = \frac{a}{s+1} + \frac{b}{s}$ $a = -1$ $b = 1$

$= \frac{1}{s+1} + \frac{1}{s} \Rightarrow V_{c_{zs}}(t) = -e^{-t} + 1$ V, $t \geq 0$

$tu(t) \rightarrow \frac{1}{s^2}$

$V_{c_{zs}}(s) = \frac{1}{s+1} \frac{1}{s^2} = \frac{a}{s+1} + \frac{b}{s} + \frac{c}{s^2}$ $a = 1, b = -1, c = 1$

$V_{c_{zs}}(t) = 1 \cdot e^{-t} - 1 + t$

$e^{t}u(t) \rightarrow \frac{1}{s-1}$

$e^{-t}u(t)$ $\frac{1}{s+1}$

$V_{c_{zs}}(s) = \frac{1}{(s+1)(s-1)} = \frac{a}{s+1} + \frac{b}{s-1}$

$V_{c_{zs}}(s) = \frac{1}{(s+1)^2}$

$a = -\frac{1}{2}$ $b = \frac{1}{2}$

$V_{c_{zs}}(t) = te^{-t}$ V, $t \geq 0$

$V_{c_{zs}}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$ V, $t \geq 0$

$\cos(t)u(t) \rightarrow \frac{s}{s^2+1}$

$V_{c_{zs}}(s) = \frac{s}{(s+1)(s^2+1)} = \frac{a}{s+1} + \frac{bs+c}{s^2+1}$

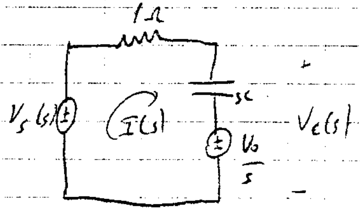
$a = -\frac{1}{2}$ $b = \frac{1}{2}$ $c = \frac{1}{2}$

$V_{c_{zs}}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$

$i_1(t)+i_2(t)-i_3(t)=0$ $\quad$ $v_R(t)=Ri_R(t)$ $\quad$ $v_L(t)=LDi_L(t)$ $\quad$ $i_c(t)=CDV_c(t)$

$I_1(s)+I_2(s)\cdot I_3(s)=0$ $\quad$ $V_R(s)=RI_R(s)$ $\quad$ $V_L(s)=L\{sI_L(s)-I_0\}$ $\quad$ $I_c(s)=sCV_c(s)-CV_0$

$V_L(s)=sLI_L(s)-LI_0$ $\quad$ $V_c(s)=\frac{1}{sC}I_c(s)+\frac{V_0}{s}$

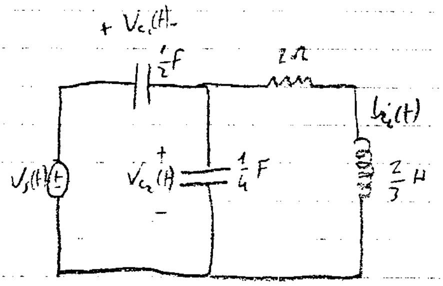


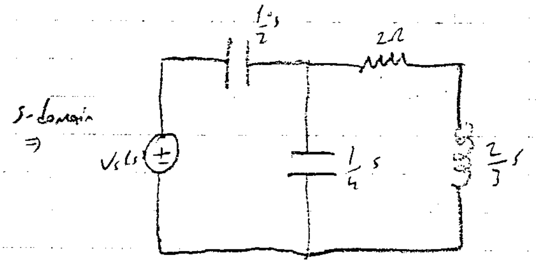
$$V_c(s)=\frac{V_0}{s}+\frac{\frac{1}{s}}{\frac{1}{s}+1}\left(V_s(s)-\frac{V_0}{s}\right)$$

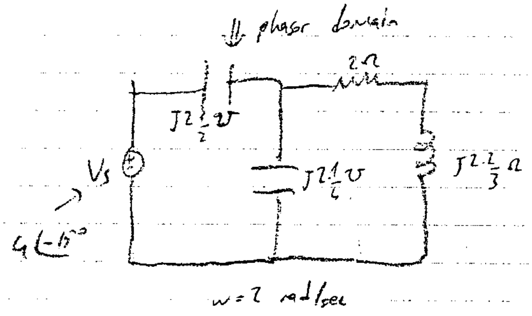
$$=\frac{1}{s+1}V_s(s)+\frac{V_0}{s}\left(1-\frac{1}{s+1}\right)$$

$$V_c(s)=\frac{1}{s+1}V_s(s)+\frac{V_0}{s+1}$$

Ex







$$Z_0=\frac{1}{\frac{1}{4}s}\,\Big|\Big|\left(2+\frac{2}{3}s\right)=\frac{\frac{4}{s}\left(2+\frac{2s}{3}\right)}{\frac{4}{s}+2+\frac{2s}{3}}=\frac{8\left(\frac{s}{3}+1\right)}{\frac{4+2s+2s^2}{3}}$$

$$=\frac{4(s+3)}{s^2+3s+6}$$

$$V_{c_2}(s)=\frac{Z_0}{Z_0+\frac{1}{\frac{1}{2}s}}V_s(s)$$

$$\Rightarrow V_{c_2}(s)=\frac{\frac{4(s+3)}{s^2+3s+6}}{\frac{4(s+3)}{s^2+3s+6}+\frac{2}{s}}V_s(s)=\underbrace{\frac{2}{3}\frac{s(s+3)}{(s+1)(s+2)}}_{H(s)}V_s(s)$$

$$H(s)=\frac{2}{3}-\frac{4}{3}\frac{1}{(s+1)(s+2)}=\frac{2}{3}-\frac{4}{3}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$$

$$h(t)=\frac{2}{3}\delta(t)-\frac{4}{3}e^{-t}+\frac{4}{3}e^{-2t}$$

S-DOMAIN ANALYSIS

19/04/2010
Pazartesi

$$(D^2+3D+2)x(t) = f(t)$$

$$\Downarrow$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}\underline{b}\end{bmatrix} f(t)$$

Char. eqn: $\rightarrow \lambda^2 + (\ldots)\lambda + (\ldots) = 0$

Laplace transform

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt \Rightarrow \mathcal{L}\{f(t)\} = F(s)$$

$s \in \mathbb{C}$

, $s \in$ Region of Convergence (R.O.C)
(for which Laplace Transform integral converges}

Ex. $$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt = \int_{0}^{\infty} 1.e^{-st}\,dt = \frac{e^{-st}}{-s}\Bigg|_{t=0}^{t=\infty} = e^{-s\infty} - \left(\frac{-1}{s}\right) = \frac{1}{s}$$

if real $s>0$

$\underbrace{Re\{s\}>0}_{R.O.C}$

Two Types of Laplace Transforms:

1) Unilateral

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

More interesting, since we have initial value problems.

2) Bilateral

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

Ex. $$\mathcal{L}\{f^{(1)}(t)\} = \int_{0^-}^{\infty} f^{(1)}(t)e^{-st}\,dt = e^{-st}f(t)\Big|_{t=0}^{t=\infty} - \int_{0^-}^{\infty} f(t)(-se^{-st})\,dt$$

$$= \left[e^{-\infty s}f(\infty) - e^{0}f(0)\right] + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt = sF(s) - f(0)$$

Ex. $$\mathcal{L}\{f^{(2)}(t)\} = \mathcal{L}\{g^{(1)}(t)\} = sG(s) - g(0) \qquad g(t) = f^{(1)}(t) = s\mathcal{L}\{f^{(1)}(t)\} - f^{(1)}(0)$$

$$= s(sF(s) - f(0)) - f^{(1)}(0) = s^2F(s) - sf(0) - f^{(1)}(0)$$

Ex. $(D^2+3D+2)V_c(t) = f(t)$ ← $f(t) = u(t)$

$V_c(0^-) = V_0 \quad \dot{V}_c(0^-) = \dot{V}_0$

Let there be a solution;

$2V_c(t) \rightleftarrows 2V_c(s)$

$3\dot{V}_c(t) \rightleftarrows 3sV_c(s) - 3V_0$

$+ \quad \ddot{V}_c(t) \rightleftarrows s^2V_c(s) - sV_0 - \dot{V}_0$

$\rightarrow \underset{u(t)}{\overset{\|}{(D^2+3D+2)V_c(t)}} \rightleftarrows \underset{\frac{1}{s}}{\overset{\|}{V_c(s)[s^2+3s+2] - V_0(s+3) - \dot{V}_0}}$

$$V_c(s)[s^2+3s+2] - V_0(s+3) - \dot{V}_0 = \frac{1}{s}$$

$$V_c(s) = \frac{\frac{1}{s} + \dot{V}_0 + V_0(s+3)}{s^2+3s+2}$$

$$V_c(s)=\frac{1+s\dot{V}_0+s(s+3)V_0}{s(s^2+3s+2)}=\frac{1+sV_0+s(s+3)V_0}{s(s+1)(s+2)}$$

$$=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$A=\frac{1}{2}\quad B=2V_0+\dot{V}_0-1\quad C=\frac{1-2\dot{V}_0-2V_0}{2}$$

$$V_c(s)=\frac{1}{2s}+\frac{2V_0+\dot{V}_0-1}{s+1}+\frac{1-2\dot{V}_0-2V_0}{2(s+2)}$$

$$V_c(t)=\frac{1}{2}u(t)+(2V_0+\dot{V}_0-1)e^{-t}+\frac{(1-2\dot{V}_0-2V_0)}{2}e^{-2t}$$

$\mathcal{L}\{u(t)\}=\frac{1}{s}$

$\mathcal{L}\{\delta(t)\}=1$

$\mathcal{L}\{e^{\alpha t}\}=\frac{1}{s-\alpha}$

$\mathcal{L}\{\cos\beta t\}=\frac{s}{s^2+\beta^2}$

$\mathcal{L}\{\sin\beta t\}=\frac{\beta}{s^2+\beta^2}$

$\mathcal{L}\{r(t)\}=\frac{1}{s^2}$

$\mathcal{L}\{\int_{0^-}^{t}f(\tau)d\tau\}=\frac{F(s)}{s}$

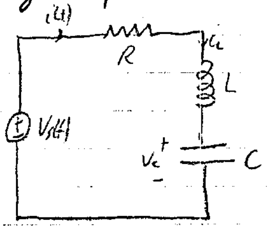
Ex// $F(s)=\frac{(s-2)}{s(s+1)^2}$

$$F(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}\Rightarrow A=-2\quad C=3\quad B=2$$

Circuit Response using S-domain Methods

1) Taking Laplace Transform of Time-Domain Relation

$V_c(0^-)=V_0$
$i_L(0^-)=I_0$



$$KVL: -V_s(t)+Ri(t)+L\frac{d}{dt}i(t)+V_c(0^-)+\frac{1}{C}\int_{0^-}^{t}i(\tau)d\tau=0\quad t>0$$

$\mathcal{L}\{\}$

$$-V_s(s)+RI(s)+L(sI(s)-I_0)+\frac{V_c(0^-)}{s}+\frac{1}{C}\frac{I(s)}{s}=0$$

$$\Rightarrow I(s)\left[sL+R+\frac{1}{Cs}\right]=V_s(s)+LI_0-\frac{V_0}{s}\quad\Rightarrow I(s)=\frac{CsV_s(s)+sLCI_0-CV_0}{s^2(LC)+(RC)s+1}$$

$$I(s)=\underbrace{\frac{sV_s(s)/L}{s^2+\frac{R}{L}s+\frac{1}{LC}}}_{\mathcal{L}^{-1}\{\}}+\underbrace{\frac{sI_0-\frac{1}{L}V_0}{s^2+\frac{R}{L}s+\frac{1}{LC}}}_{\mathcal{L}^{-1}\{\}}$$

$\mathcal{L}^{-1}\{\}$

$$\underset{\text{solution}}{i(t):\text{complete}}=i^{zs}(t)+i^{zi}(t)$$

22/04/2010
Perşembe

a) Let $L=1\ H$; $R=6\Omega$; $C=0.04\ F$; $I_0=5A$; $V_0=1V$ $\quad V_s(t)=12\sin 5t\quad t\geq 0$

$$\Rightarrow I(s)=\frac{sV_s(s)}{s^2+6s+25}+\frac{5s-1}{s^2+6s+25}\quad\propto\quad V_s(s)=\mathcal{L}\{12\sin 5t\}=12\frac{5}{s^2+25}$$

$$\Rightarrow I_s=\frac{60s}{(s^2+6s+25)(s^2+25)}+\frac{5s-1}{\underset{(s+3)^2+16=(s+3+j4)(s+3-j4)}{s^2+6s+25}}$$

$$\Rightarrow I^{zs}(s)=\frac{60s}{(s^2+6s+25)(s^2+25)}=\frac{K_1}{(s+3+j4)}+\frac{K_1^*}{(s+3-j4)}+\frac{K_2}{(s+5j)}+\frac{K_2^*}{(s-5j)}$$

$K_1 = -j1.25 \quad K_2 = j$

$$I^{zs}(s) = \underbrace{\frac{j1.25}{s+3-j4} + \frac{-j1.25}{s+3+j4}}_{\downarrow} + \frac{-j}{s-j5} + \frac{j}{s+j5}$$

$$I^{zs}(s) = \frac{-10}{(s+3)^2+4^2} + \frac{10}{s^2+25}$$

$\mathcal{L}^{-1}\{\}$

$$i^{zs}(t) = \underbrace{\frac{-5}{2}e^{-3t}\sin(4t)}_{\text{transient part for zs solution}} + \overbrace{2\sin(5t)}$$

steady state part of
zero state response
(AC phasor domain analysis gives us this solution)

$$\rightarrow \boxed{2^{nd} \text{ method}} \Rightarrow \mathcal{L}^{-1}\{\} \Rightarrow i^{zs}(t) = \underbrace{j1.25e^{-(3-j4)t} - j1.25e^{-(3+j4)t}}_{\text{conjugate of each other}} - je^{j5t} + je^{-j5t}$$

$$= 2Re\{j1.25e^{-(3-j4)t}\} + 2Re\{-je^{j5t}\} = \frac{5}{2}e^{-3t}\underbrace{Re\{e^{j(4t+90°)}\}}_{\cos(4t+90°)} - 2\underbrace{Re\{e^{j(5t+90°)}\}}_{\cos(5t+90°)}$$

$$= \frac{5}{2}e^{-3t}\sin(4t) + 2\sin(5t)$$

$$I^{zi}(s) = \frac{5s-1}{(s+3)^2+4^2} = \frac{K_1}{s+3+j4} + \frac{K_1^*}{s+3-j4} \Rightarrow K_1 = \frac{5}{2} - j2$$

$\mathcal{L}^{-1}\{\}$

$$i^{zi}(t) = 2Re\{K_1e^{-(3+j4)t}\} = 2e^{-3t}Re\{K_1e^{-(j4)t}\} = 2e^{-3t}\left(\frac{5}{2}\cos 4t - 2\sin 4t\right)$$

$$i^{zi}(t) = 2e^{-3t}\sqrt{\left(\frac{5}{2}\right)^2+2^2}\cos\left(4t - \tan^{-1}\left(\frac{-2}{\frac{5}{2}}\right)\right)$$

b) Step response (zero-state response for u(t) input)

Same values for R, L, C, etc. $\mathcal{L}\{u(t)\} = \frac{1}{s}$

$$I^{zs}(s) = \frac{sV_s(s)}{s^2+6s+25}$$

$$\Rightarrow I^{zs}(s) = \frac{1}{s^2+6s+25} = \frac{K_1}{s+3-j4} + \frac{K_1^*}{s+3+j4} \Rightarrow K_1 = -j/8$$

$$i^{zs}(t) = \frac{1}{4}e^{-3t}\cos(4t-90°)$$

c) Impulse Response

$$\mathcal{L}\{\delta(t)\} = 1$$

$$I^{zs}(s) = \frac{s\, V_s(s)}{s^2+6s+25}$$

$$I^{zs}(s) = \frac{s}{s^2+6s+25} = \frac{s+3}{(s+3)^2+4^2} - \frac{3}{(s+3)^2+4^2}$$

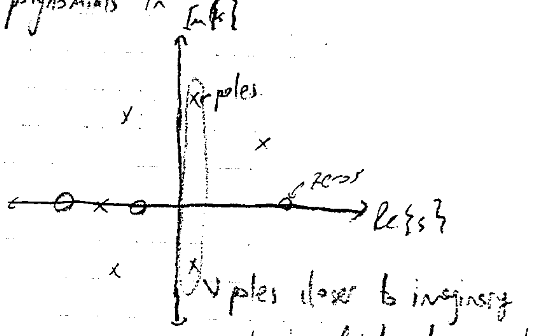
$$i^{zs}(t) = h(t) = e^{-3t}\cos(4t) - \frac{3}{4} e^{-3t}\sin(4t)$$

Poles and Zeros in s-domain

$$I^{zs}(s) = K \frac{\prod_{k=1}^{z} (s-z_k)}{\prod_{k=1}^{p} (s-p_k)}$$ ← a ratio of two polynomials in "s-domain"

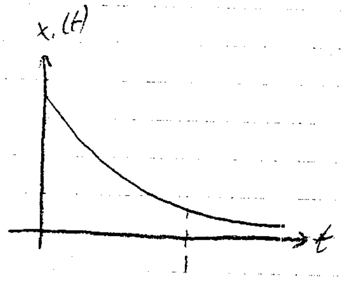
p_k : Poles of $I^{zs}(s)$ (Singularities)

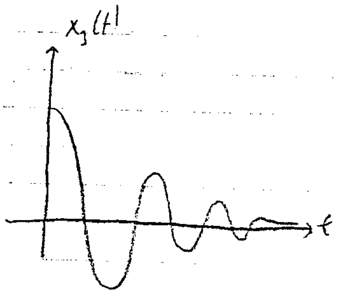
z_k : Zeros of $I^{zs}(s)$ $(I^{zs}(s_k) = 0)$

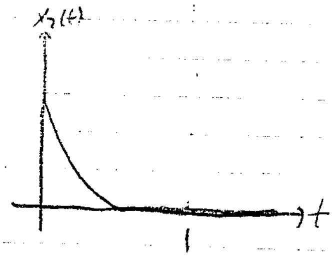


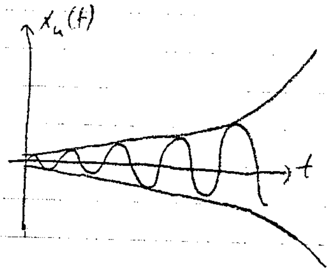
poles closer to imaginary axis is of interest in practice

Ex

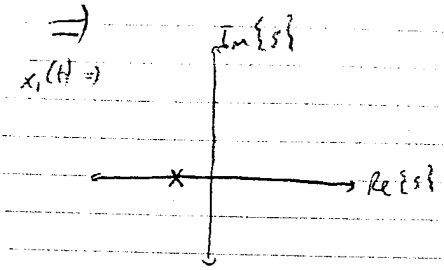


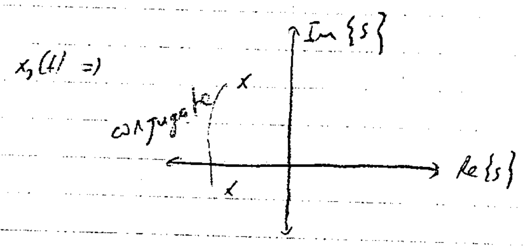


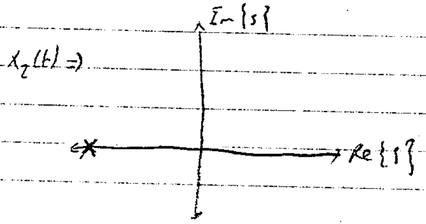


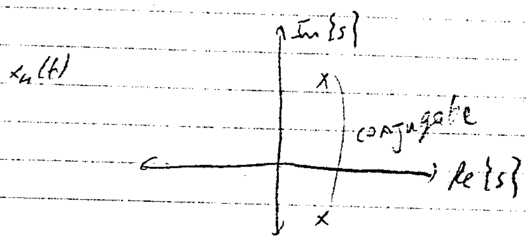


⟹









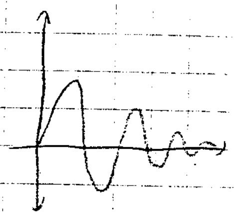
Initial and Final Value Theorems

Initial Value $\Rightarrow$ $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$

Final Value $\Rightarrow$ $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

Ex Step response

$i^{zs}(t) = \frac{1}{4} e^{-3t} \sin 4t \xleftrightarrow[\mathcal{L}]{} I^{zs}(s) = \frac{1}{s^2+6s+25}$



$i^{zs}(0^+) = 0 \xrightarrow[\text{theorem}]{\text{initial value}} \lim_{s \to \infty} s I^{zs}(s) = 0$

$i^{zs}(\infty) = 0 \xrightarrow[\text{theorem}]{\text{final value}} \lim_{s \to 0} s I^{zs}(s) = 0$

Some Important Details

To use Initial Value Theorem $s = \infty$ should be in R.O.C

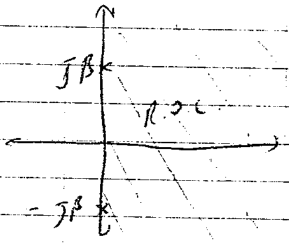
To use Final Value Theorem $s = 0$ should be in R.O.C.

Eg. $\mathcal{L}\{\cos(\beta t)\} = \frac{s}{s^2+\beta^2} \xrightarrow[\text{value theorem}]{\text{apply final}} \lim_{s \to 0} s \cdot \frac{s}{s^2+\beta^2} = 0$!! but,

$\lim_{t \to \infty} \cos(\beta t)$ does not exists

do not forget R.O.C when $I(s)$ is discussed.

$\mathcal{L}\{\cos(\beta t)\} = \frac{s}{s^2+\beta^2}$ has R.O.C



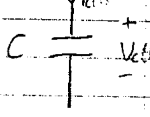
$I(s) = \int_0^\infty \cos(\beta t) e^{-st} dt$

26/04/2010
Pazartesi

Circuit Components in S-Domain

Capacitor:

Time Domain



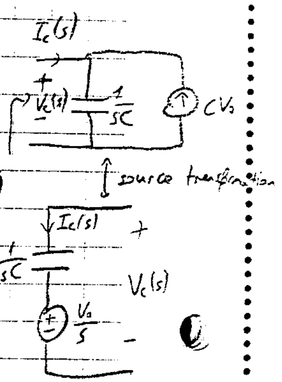
$i_c(t) = C \frac{dV_c(t)}{dt} \xrightarrow{} \mathcal{L}\{\}$

$V_c(0^-) = V_0$

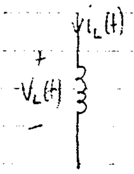
S-Domain

$I_c(s) = Cs V_c(s) - C\overset{V_0}{V_c(0^-)}$

$V_c(s) = \frac{I_c(s)}{sC} + \frac{V_0}{s} \longrightarrow$



Inductor!



Time Domain

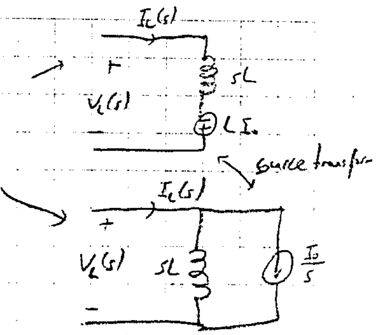
$V_L(t) = L\frac{di_L(t)}{dt}$

$i_L(0^-) = I_0 \xrightarrow{} \mathcal{L}\{\}$

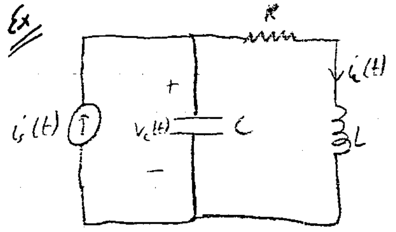
S-Domain

$V_L(s) = Ls\,I_L(s) - LI_0$

$I_L(s) = \frac{V_L(s)}{sL} + \frac{I_0}{s}$

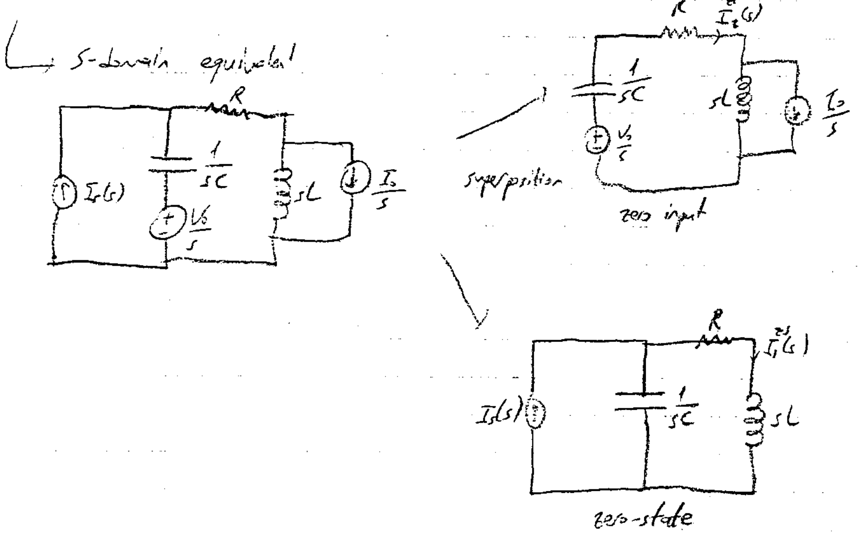


Ex



$V_C(0^-) = V_0$

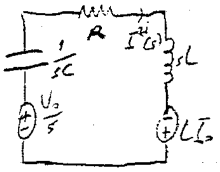
$I_L(0^-) = I_0$



Zero state $\Rightarrow$

$$I_1^{zs}(s) = I_s(s)\,\frac{1/sC}{\frac{1}{sC}+(sL+R)} = \frac{1}{LCs^2+sRC+1}\,I_s(s) = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}\,I_s(s)$$

Zero input $\Rightarrow$



$$I_1^{zi}(s) = \frac{V_0/s + LI_0}{\frac{1}{sC}+R+sL} = \frac{\frac{V_0}{L}+sI_0}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

$$I_1^{zs}(s) = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}\,I_s(s) \Rightarrow \left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)I_1(s) = \frac{I_s(s)}{LC}$$

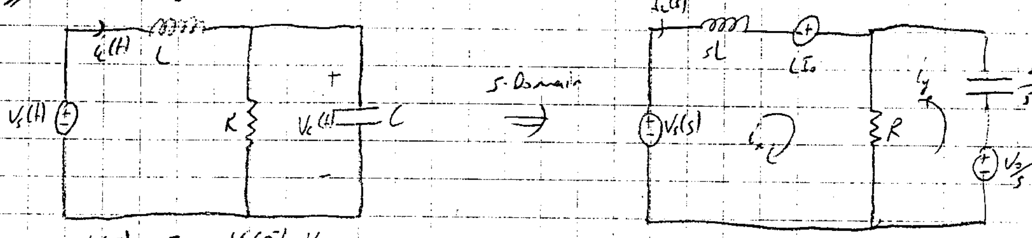
$\downarrow \mathcal{L}^{-1}\{\}$

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)i_1^{zs}(t) = \frac{i_s(t)}{LC}$$

natural freq: (roots of char. poly)

poles of $I_1^{zs}(s)$ solution $I_1^{zs}(s) = \dfrac{\frac{1}{LC} I_s(s)}{\underbrace{s^2+\frac{R}{L}s+\frac{1}{LC}}_{\text{roots of this}}}$

Ex// Mesh analysis in S-Domain



$i_L(0^-) = I_0 \quad V_C(0^-) = V_0$

multiply 2nd row with sC^{-1}

$$\begin{bmatrix} sL+R & R \\ sCR & 1+sCR \end{bmatrix} \begin{bmatrix} i_x(s) \\ i_y(s) \end{bmatrix} = \begin{bmatrix} V_s(s)+LI_0 \\ CV_0 \end{bmatrix} \overset{\text{Cramer's}}{\underset{\text{Rule}}{\Rightarrow}} I_x(s) = \frac{\begin{vmatrix} V_s(s)+LI_0 & R \\ V_0 C & 1+sRC \end{vmatrix}}{\begin{vmatrix} sL+R & R \\ sRC & 1+sRC \end{vmatrix}}$$

$$\Rightarrow I_x(s) = \frac{(V_s(s)+LI_0)(1+sRC) - V_0CR}{(sL+R)(1+sRC)-sR^2C} = \frac{V_s(s)[1+sRC] + (LI_0 - RCV_0) + sRCLI_0}{s^2RLC + s(L+R^2C-R^2C)+R},$$

$$\Rightarrow I_x(s) = \frac{\frac{1}{RCL}+\frac{s}{L}}{s^2 - \frac{s}{RC} + \frac{1}{LC}} V_s(s) + \frac{\frac{I_0}{RC} - \frac{V_0}{L} + sI_0}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Let $R = 8\Omega$, $L = 6H$ $C = \frac{1}{48}F$

$$I_x(s) = \underbrace{\frac{\left(1+\frac{s}{6}\right)}{s^2+6s+8} V_s(s)}_{\text{zero state}} + \underbrace{\frac{6I_0 - \frac{V_0}{6} + sI_0}{s^2+6s+8}}_{\text{zero input}}$$

zero input $\Rightarrow I_x^{zi}(s) = \dfrac{-\frac{1}{2}\left(2I_0 - \frac{V_0}{6}\right)}{s+4} + \dfrac{\frac{1}{2}\left(4I_0 - \frac{V_0}{6}\right)}{s+2}$ zero state $= I_x^{zs}(s) = \dfrac{\left(1+\frac{s}{6}\right)}{s^2+6s+8} V_s(s)$

Assume: We have zero-input case, problem is find initial cond. to excite mode with $\lambda=-2$

$$i_x^{zi}(t) = \left[\left(\frac{V_0}{12} - I_0\right)e^{-4t} + \left(2I_0 - \frac{V_0}{12}\right)e^{-2t}\right]u(t)$$

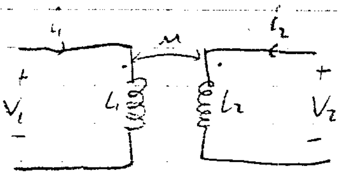
then $V_0 = 12I_0$ $\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \alpha \begin{bmatrix} 12 \\ 1 \end{bmatrix} \rightarrow$ excites $\lambda=-2$ mode only

Assume: $V_s(t) = 16\cos 2t\, u(t)$ find $i_x^{zs}(t)$.

$$I_x^{zs}(s) = \frac{1+\frac{s}{6}}{s^2+6s+8} \cdot \frac{16\, s}{s^2+4} = \frac{-\frac{4}{3}}{s+2} + \frac{8/15}{s+4} + \frac{\frac{4}{5}s + \frac{32}{15}}{s^2+4}$$

$$i_x^{zs}(t) = -\frac{4}{3}e^{-2t} + \frac{8}{15}e^{-4t} + \frac{4}{5}\cos(2t) + \frac{16}{15}\sin(2t)$$

Coupled Inductor

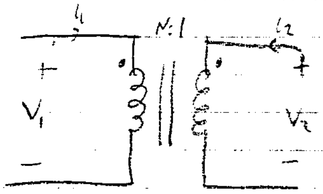


$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} D\, i_1(t) \\ D\, i_2(t) \end{bmatrix} \qquad \begin{matrix} i_1(0^-) = I_1 \\ i_2(0^-) = I_2 \end{matrix}$$

$\hookrightarrow$ Laplace

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} sI_1(s) - I_1 \\ sI_2(s) - I_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} sI_1(s) \\ sI_2(s) \end{bmatrix} - \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Transformer



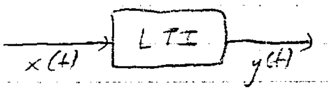
$$\frac{i_1(t)}{i_2(t)} = \frac{-1}{N} \qquad \frac{V_1(t)}{V_2(t)} = \frac{N}{1}$$

$\hookrightarrow$ Laplace

$$\frac{I_1(s)}{I_2(s)} = -\frac{1}{N} \; ; \quad \frac{V_1(s)}{V_2(s)} = N$$

29/04/2010
Persembe

NETWORK FUNCTIONS



output at zero state

$$H(s) = \frac{\mathcal{L}\{y^{zs}(t)\}}{\mathcal{L}\{x(t)\}}$$

Network function

input

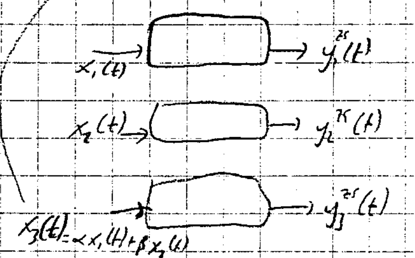
$$\frac{I_L^{zs}(s)}{V_s(s)} = \frac{s}{s+1} \longrightarrow (s+1)\, I_L^{zs}(s) = V_s(s)\, s$$

$\mathcal{L}^{-1}\{\}$

$$(D+1)\, i_L^{zs}(t) = \frac{d}{dt} v_s(t)$$

output (unknown)

input



$$\frac{\mathcal{L}\{y_1^{zs}(t)\}}{\mathcal{L}\{x_3(t)\}} = H(s)$$

$x_3(t) = \alpha x_1(t) + \beta x_2(t)$

$$\mathcal{L}\{y_3^{zs}(t)\} = H(s)\,\mathcal{L}\{x_3(t)\}$$

$$\mathcal{L}\{y_3^{zs}(t)\} = H(s)\,\mathcal{L}\{\alpha x_1(t)\} + H(s)\,\mathcal{L}\{\beta x_2(t)\}$$

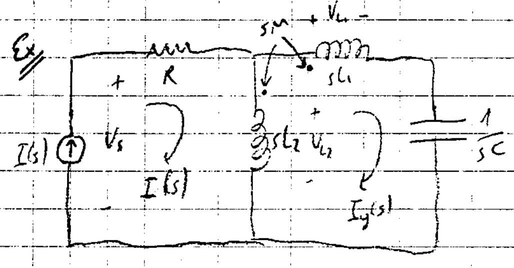
$\mathcal{L}^{-1}\{\}$ $\mathcal{L}^{-1}\{\}$

$$y_3^{zs}(t) = \alpha y_1^{zs}(t) + \beta y_2^{zs}(t)$$

The ZS solutions obey the superposition principle so ZS solution can be found by superposition method

If initial conditions are non-zero, they should be treated seperately by ZI solution

Ex//



← Note no initial conditions since network func. are always expressed in ZS.

$$Z(s) = \frac{V(s)}{I(s)}$$

$$V(s) = I_s(s)R + sL_2(I(s) - I_y(s)) + sMI_y(s)$$

$$KVL: -V_{L_2} + V_{L_1} + I_y(s)\frac{1}{sC} = 0$$

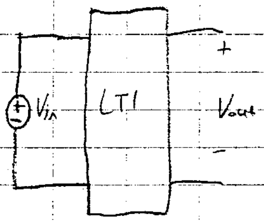
$$-(sL_2(I(s) - I_y(s)) + sMI_y(s)) + (sL_1I_y(s) + sM(I(s) - I_y(s)) + I_y(s)\frac{1}{sC} = 0$$

Find I_y in terms of input $I(s)$

↓

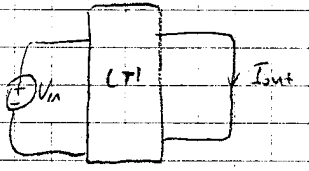
Replace $I_y(s)$ in the output relation and express output in terms of $I(s)$ (input)

$$Z(s) = \frac{V(s)}{I(s)} = \frac{R + s^2RC(L_1 + L_2 - 2M) - s^3 \ldots}{s^2C(L_1 + L_2 - 2M) + 1}$$



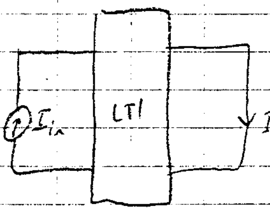
$$H_v(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

↑ voltage transfer function

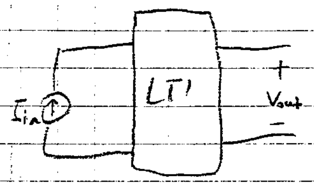


Transfer admittance

$$\frac{I_{out}(s)}{V_{in}(s)}$$

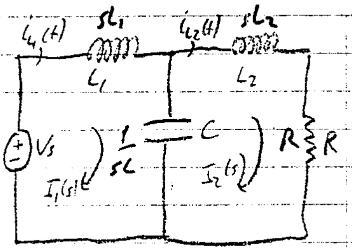


→ current transfer function



Transfer Impedance

Ex//



KVL: $-V_s(s) + sL_1 I_1(s) + \frac{1}{sC}(I_1(s) - I_2(s)) = 0$

$\frac{1}{sC}(I_2(s) - I_1(s)) + sL_2 I_2(s) + R I_2(s) = 0$

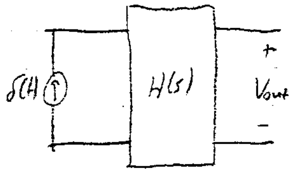
$$\Bigg) \begin{bmatrix} sL_1 + \frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & \frac{1}{sC} + sL_2 + R \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \underset{\uparrow}{\frac{1}{\Delta(s)}} \begin{bmatrix} \frac{1}{sC} + sL_2 + R & \frac{1}{sC} \\ \frac{1}{sC} & sL_1 + \frac{1}{sC} \end{bmatrix} \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix}$$

$$\Delta(s) = (sL_1 + \frac{1}{sC})(sL_2 + \frac{1}{sC} + R) - \frac{1}{s^2C^2}$$

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{sC} + sL_2 + R}{\Delta(s)} \cdot V_s(s) \\ \frac{1/sC}{\Delta(s)} \cdot V_s(s) \end{bmatrix} \qquad H(s) = \frac{I_1(s)}{V_s(s)} = \frac{\frac{1}{sC} + sL_2 + R}{\Delta(s)}$$

Network Functions and Impulse Response



$V_{out}(s) = H(s) \cdot \mathcal{L}\{\delta(t)\}$

$V_{out}(s) = H(s) \cdot 1$

$$H(s) = \mathcal{L}\{\underbrace{V_{out}(t) \text{ for } \delta(t) \text{ input}}_{\text{impulse response (defined for) } t \geq 0}\}$$

↙ impulse response in "s domain"

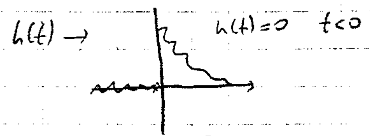
$V_{out}(s) = H(s) I_{in}(s)$

$\mathcal{L}^{-1}\{\}\downarrow$

$$V_{out}(t) = \int_{-\infty}^{\infty} h(\tau)\, i_{in}(t-\tau)\, d\tau \quad \longleftarrow \text{convolution}$$

$$\boxed{V_{out}(t) = \underset{\text{convolution}}{h(t) * i_{in}(t)}}$$

About the integral



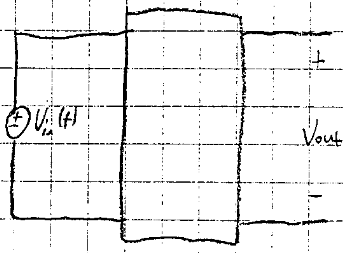
$i_{in}(t)$ given for $t > 0$

$i_{in}(t) = 0$ for $t < 0$

$$\Rightarrow V_{out}(t) = \int_{-\infty}^{\infty} h(\tau)\, I_{in}(t-\tau)\, d\tau = \underset{\substack{\text{since } h(\tau) \\ = 0 \text{ in the} \\ \text{integration} \\ \text{range}}}{\cancel{\int_{-\infty}^{0}}} + \int_{0}^{t} + \underset{\substack{\text{since } i_{in}(t-\tau) = 0 \\ \text{for integration} \\ \text{range}}}{\cancel{\int_{t}^{\infty}}}$$

$$\boxed{= \int_{0}^{t} h(\tau)\, i_{in}(t-\tau)\, d\tau}$$

Step Response



$$V_{out}(s) = H(s)V_{in}(s) \leftarrow \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$V_{out}^{step}(s) = H(s)\frac{1}{s}$$

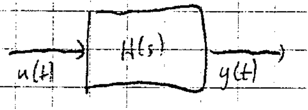
$\downarrow \mathcal{L}^{-1}\{\ \}$

$$V_{out}^{step}(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \int_{0^-}^{t} h(\tau)d\tau$$

OR By Convolution;

$$V_{out}^{step}(t) = \int_{-\infty}^{\infty} h(\tau)V_{in}(t-\tau)d\tau = \int_{0^-}^{t} h(\tau)\underbrace{\overset{\overset{u(t-\tau)}{\downarrow}}{V_{in}(t-\tau)}}_{1}d\tau = \int_{0^-}^{t} h(\tau)d\tau$$

Ex//



$$H(s) = \frac{1}{(s+2)(s+3)} = \frac{V_{out}^{zs}(s)}{V_{in}(s)}$$

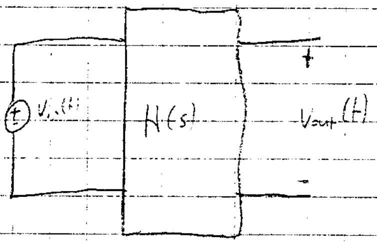
$$V_{out}^{step}(s) = H(s)\frac{1}{s}$$

$$= \frac{1}{(s+2)(s+3)}\frac{1}{s} = \underbrace{\frac{\frac{1}{-2}}{s+2} + \frac{\frac{1}{3}}{s+3}}_{\text{response related natural frequencies}} + \underbrace{\frac{\frac{1}{6}}{s}}_{\text{response related to step}}$$

$$\Rightarrow V_{out}(t) = -\frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{6}$$

03/05/2010
Pazartesi

Sinousoidal Steady State Response



$$H(s) = \frac{V_{out}(s)}{V_{IN}(s)}$$

$$\mathcal{L}\{V_{in}(t)\} = \mathcal{L}\{\underbrace{A\cos(wt+\phi)}_{\cos(wt)\cos\phi}\} = A\cos\phi\mathcal{L}\{\cos wt\} - A\sin\phi\mathcal{L}\{\sin wt\}$$

$-\sin(wt)\sin\phi$

$V_{in}(t) = A\cos(wt+\phi)$

$\downarrow$

$V_{out}(t) = ?$

$$\mathcal{L}\{V_{in}(t)\} = \frac{A\cos\phi\, s}{s^2+w^2} - \frac{A\sin\phi\, w}{s^2+w^2} = A\frac{(\cos\phi)s - (\sin\phi)w}{s^2+w^2}$$

$$V_{out}(s) = H(s)V_{IN}(s) = H(s)\left[A\frac{(\cos\phi)s - (\sin\phi)s}{s^2+w^2}\right] = \frac{K}{s-jw} + \frac{K^*}{s+jw} + \frac{A_1}{s-\lambda_1} + \frac{A_2}{s-\lambda_2} + \dots + \frac{A_N}{s-\lambda_N}$$

let's focus on the response due to sinousoidal input;

$K = ?$

$$K = H(s)\,A\left(\frac{\cos\phi\, s - \sin\phi\,\omega}{(s+j\omega)(s-j\omega)}\right)(s-j\omega)\Bigg|_{s=j\omega}$$

$$K = H(j\omega)\,A\,\frac{\cos\phi\, j\omega - \sin\phi\,\omega}{2j\omega} = H(j\omega)\,\frac{A}{2}\,[\cos\phi + j\sin\phi] = \underbrace{H(j\omega)\frac{A}{2}\,e^{j\phi}}_{K}$$

$$\mathcal{L}^{-1}\left\{\frac{K}{s-j\omega} + \frac{K^*}{s+j\omega}\right\} = Ke^{j\omega t} + K^*e^{-j\omega t} = 2\,\mathrm{Re}\{Ke^{j\omega t}\}$$

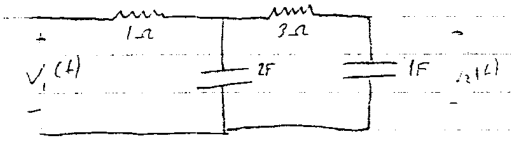
$$= 2\,\mathrm{Re}\{|K|\,e^{j(\omega t + \measuredangle K)}\} = 2|K|\cos(\omega t + \measuredangle K)$$

$$K = \frac{A}{2}|H(j\omega)|\,e^{j(\measuredangle(H(j\omega)) + \phi)}$$

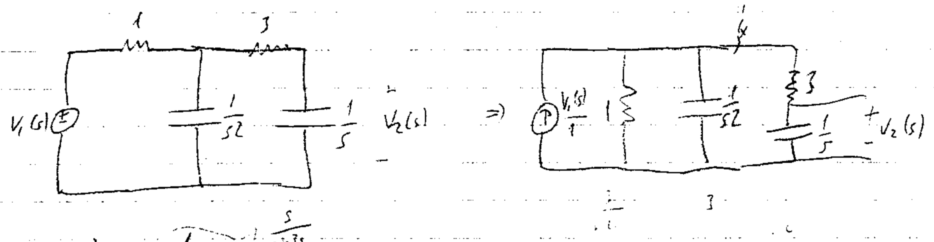
$$\underbrace{V_{out}(t) = A\,|H(j\omega)|\cos(\omega t + \phi + \measuredangle H(j\omega))}$$

The output at steady state
(free of natural response terms)

Ex//



a) $V_2^{ss}(t) = ?$ for $V_1(t) = 4\cos(\frac{1}{2}t + 30°)$



$$i_x(s) = \frac{\frac{1}{\frac{1}{s}+3}}{1 + s2 + \frac{1}{\frac{1}{s}+3}} \cdot \frac{V_1(s)}{1}$$

$\frac{1}{\frac{1}{s}+3} \rightarrow \frac{s}{1+3s}$

$$V_2(s) = i_x(s)\frac{1}{s} = \frac{s}{(1+2s)(1+3s)+s} \cdot \frac{1}{s}\,V_1(s) = \underbrace{\boxed{\frac{\frac{1}{6}}{s^2+s+\frac{1}{6}}}}_{H(s)}\,V_1(s)$$

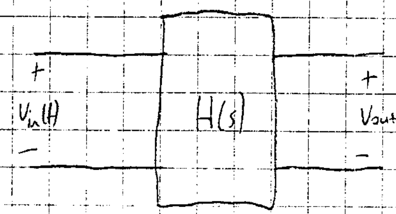
a) $V_2(t) = 4\,|H(j\tfrac{1}{2})|\cos(\tfrac{1}{2}t + 30° + \measuredangle H(j\tfrac{1}{2}))$

$$H(j\tfrac{1}{2}) = \frac{\frac{1}{6}}{-\frac{1}{4}+\frac{j}{2}+\frac{1}{6}} = \frac{1}{-0.5+j3} = \frac{1}{\sqrt{9+\frac{1}{4}}}\,e^{-j(\pi - \tan^{-1}3/0.5)}$$

$\sqrt{9+\frac{1}{4}} \Rightarrow \sim 3$

If $4\cos(\frac{1}{2}t + 30°) \longrightarrow \boxed{H(s)} \longrightarrow \frac{4}{3}\cos\left(\frac{1}{2}t + 30° + (\tan^{-1}\frac{3}{0.5} - \pi)\right)$

FREQUENCY RESPONSE



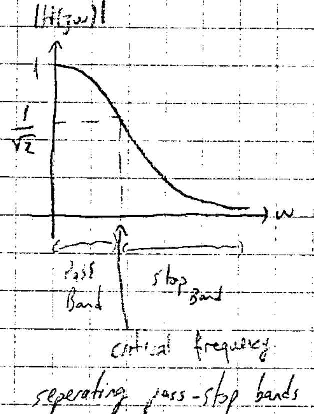
$V_{in}(t) = A_{in}\cos(\omega t+\phi_{in})$

$V_{out}(t) = A_{in}|H(j\omega)|\cos(\omega t+\phi_{in}+\angle H(j\omega))$

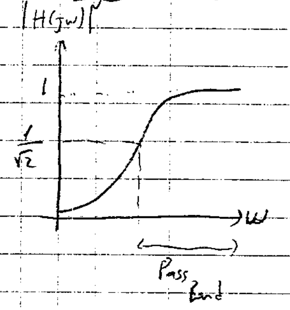
Gain function - $|H(j\omega)|$ (Magnitude Response)

Phase function: $\angle H(j\omega)$ (Phase Response)

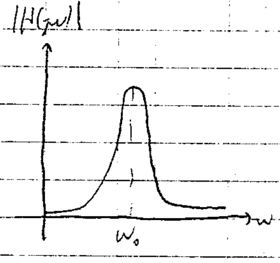
Low Pass



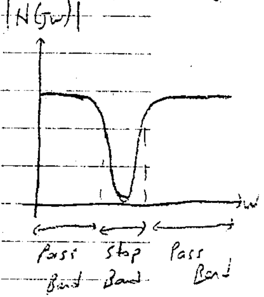
High Pass



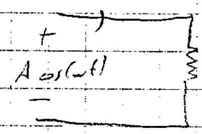
Band-Pass



Band Stop

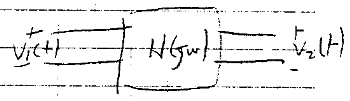


Decibel



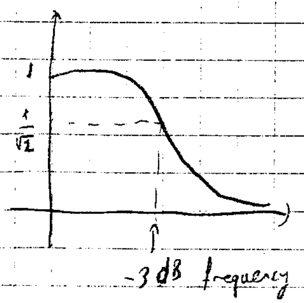
$$P_{1\Omega}^{avg} = \frac{A^2}{2}R \leftarrow R=1\Omega = \frac{A^2}{2}$$

↑ square of RMS of $A\cos(\omega t)$



⇒ The avg. power at the input is $\frac{A^2}{2}$ = AVG. power dissipated over 1Ω resistor.

Decibel $\Rightarrow 10\log_{10}\left(\frac{\text{Power at the output}}{\text{Power at the input}}\right) = 10\log_{10}\left(\frac{\frac{|AH(j\omega)|^2}{2}R}{\frac{A^2}{2}R}\right) = 10\log_{10}(|H(j\omega)|^2)\, dB$



$\lvert H(j\omega)\rvert^2$	dB $(10\log_{10}(\lvert H(j\omega)\rvert^2)$
1	0
2	3
3	4.77
4	6
5	7
6	7.78
7	8.45
8	9
9	9.54
10	10

$10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$

Frequency Response of 1st Order Circuits

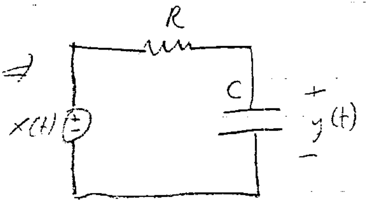
a) Low-Pass

$H(s) = \frac{K}{s+\alpha}$ $\xrightarrow{X(s)} \boxed{H(s)} \rightarrow Y(s)$

$$\frac{Y(s)}{X(s)} = \frac{K}{s+\alpha} \Rightarrow Y(s)s + \alpha Y(s) = K X(s)$$

$$\boxed{\frac{d}{dt}y(t) + \alpha y(t) = K x(t)}$$

⇒



$$-x(t) + (C\dot{y}(t))R + y(t) = 0$$

$$\dot{y}(t) + \underbrace{\frac{1}{RC}}_{\alpha} y(t) = \underbrace{\frac{1}{RC}}_{K} x(t)$$

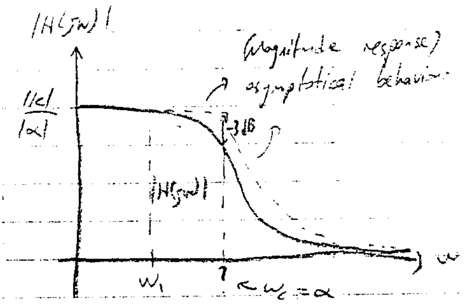
$x(t) = A\cos(\omega t + \phi_{in})$; $y(t) = A|H(j\omega)|\cos(\omega t + \phi_{in} + \measuredangle H(j\omega))$

$|H(j\omega)|$: Magnitude response

$$|H(j\omega)| = \left|\frac{K}{\alpha + j\omega}\right| = \frac{|K|}{\sqrt{\omega^2+\alpha^2}}$$

$$\measuredangle H(j\omega) = \measuredangle\left(\frac{K}{\alpha + j\omega}\right) = \measuredangle K - \tan^{-1}\frac{\omega}{\alpha}$$

06/05/2010
Perşembe



$$|H(j\omega)| = \frac{|K|}{\omega\sqrt{1+\left(\frac{\alpha}{\omega}\right)^2}} \Rightarrow |H(j\omega)| \cong \begin{cases} \frac{|K|}{|\alpha|} & \omega \ll |\alpha| \\ \frac{|K|}{\omega} & \omega \gg |\alpha| \end{cases}$$

$$= \frac{|K|}{\alpha\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}}$$

-3 dB freq: A special freq. at which $|H(j\omega)|$ is $\frac{1}{\sqrt{2}}$ times $(\max|H(j\omega)|)$
$\Downarrow$
Half power

$\max|H(j\omega)| = \frac{|K|}{|\alpha|}$ (at $\omega=0$) $\Rightarrow \frac{\max|H(j\omega)|}{\sqrt{2}} = \frac{|K|}{|\alpha|\sqrt{1+(\frac{\omega}{\alpha})^2}} \Rightarrow 1+(\frac{\omega}{\alpha})^2 = 2$

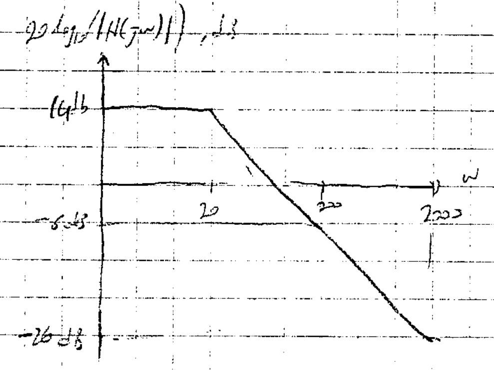
$(\frac{\omega}{\alpha})^2 = 1 \Rightarrow \boxed{\omega_c = \alpha}$

Interestingly $\Rightarrow$ The point when two asymptotes meet is the critical freq.

low freq asym $\Rightarrow \frac{|K|}{|\alpha|}$ High freq asym $\Rightarrow \frac{|K|}{\omega}$

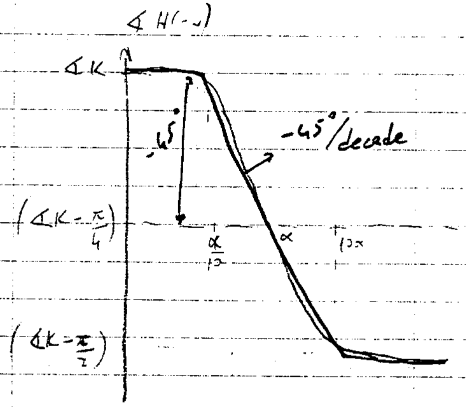
In magnitude response plots, we use dB scale and that changes the plot with linear scale as follows. (Matlab notation loglog scale)

Let $H(s) = \frac{100}{s+20}$ $K=100$
$\alpha = 20 \leftarrow \omega_c = 20$



Phase response

$\measuredangle H(j\omega) = \measuredangle K - \tan^{-1}\left(\frac{\omega}{\alpha}\right)$



10/05/2010
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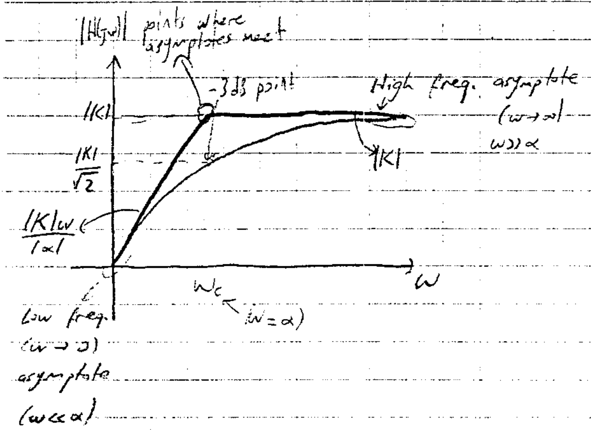
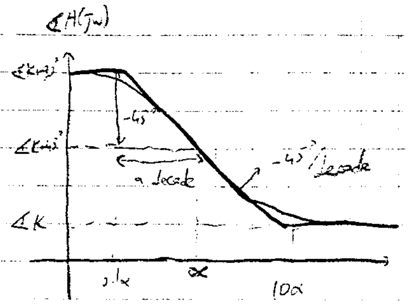
First Order High Pass Response

$H(s) = K\frac{s}{s+\alpha}$ $\longrightarrow \boxed{H(s)} \longrightarrow$
LTI

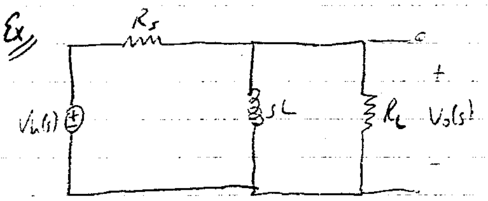
Magnitude Response: $|H(j\omega)| = |K|\frac{|\omega|}{|j\omega+\alpha|} = |K|\frac{\omega}{\sqrt{\omega^2+\alpha^2}}$

Phase Response: $\measuredangle H(j\omega) = \measuredangle K + \measuredangle j\omega - \measuredangle(j\omega+\alpha) = \measuredangle K + 90° - \tan^{-1}\left(\frac{\omega}{\alpha}\right)$



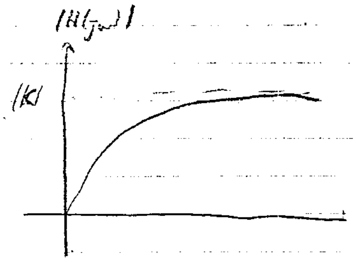


Ex



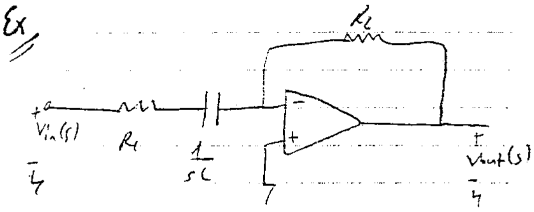
$$H(s) = \frac{sLR_L}{sL(R_s+R_L)+R_sR_L} = \frac{K\,s}{s+\alpha}$$

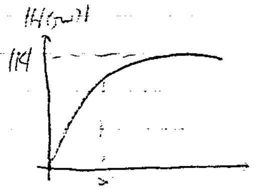
$$= \frac{s}{s+\frac{R_sR_L}{L(R_s+R_L)}} \; \frac{\cancel{L}R_L}{\cancel{L}(R_s+R_L)} \rightarrow \text{Note } K<1!$$



Since $|K|<1$ I have less power at the output than input power

Ex



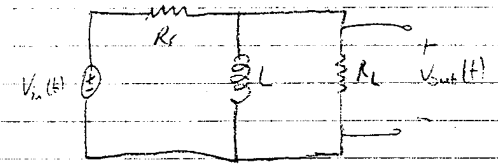


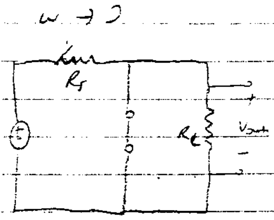
$$H(s) = -\frac{R_2}{R_1}\;\frac{s}{s+\frac{1}{RC}}$$

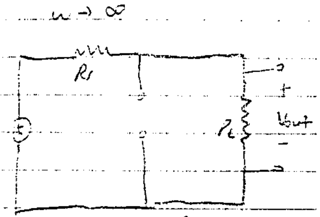
$K<0$

→ depending on R_2, output power will be greater than input power at pass band freq.

Understanding the frequency behaviour of simple circuit by inspection;





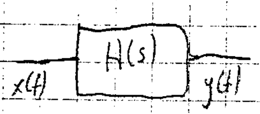


$V_{out}(t) = 0$ for $\omega \to 0$

$V_{out}(t) = \frac{R_L}{R_s+R_L}$ for $\omega \to \infty$

Bandpass - Bandstop Filters using 1st Order Circuits

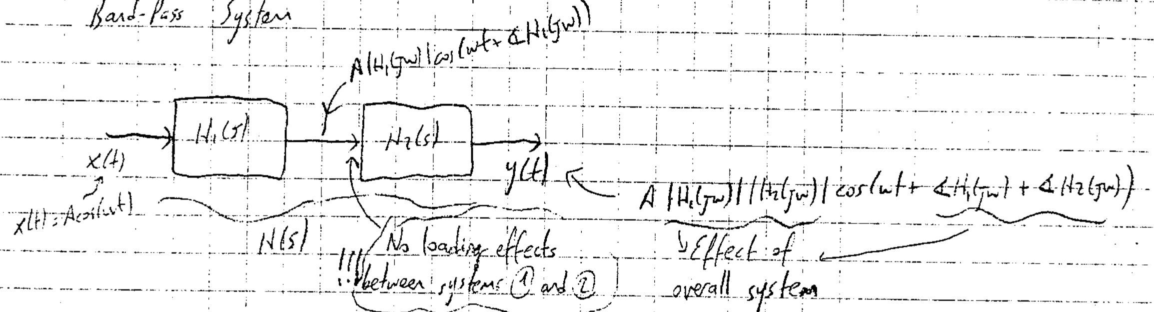
13/05/2010
Perşembe



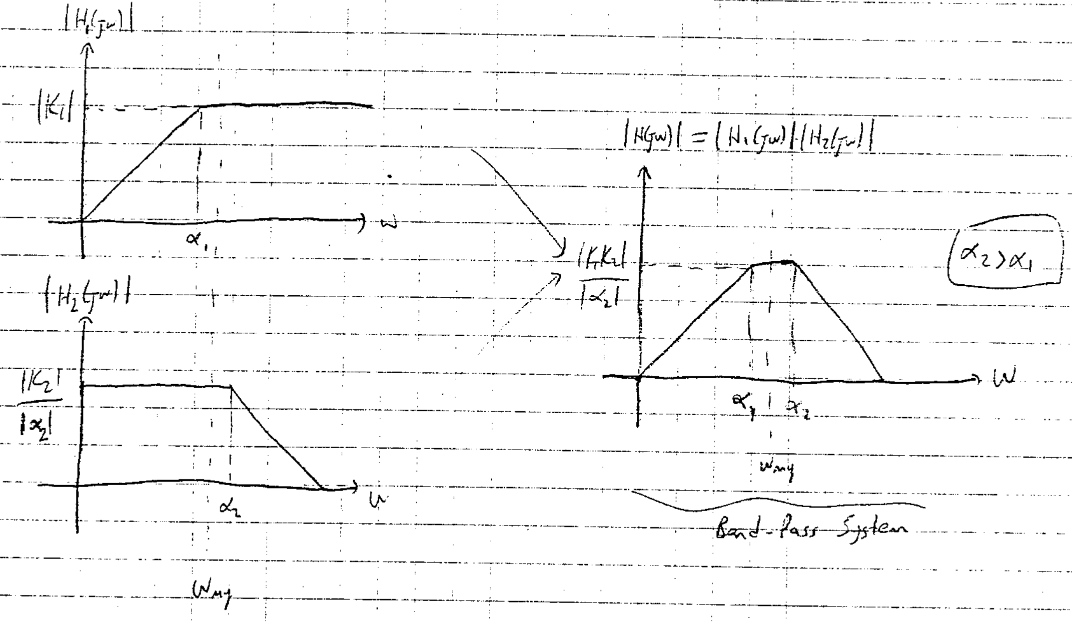
$H_1(s) = K_1 \frac{s}{s+\alpha_1}$ ← High-pass system

$H_2(s) = K_2 \frac{1}{s+\alpha_2}$ ← low pass system

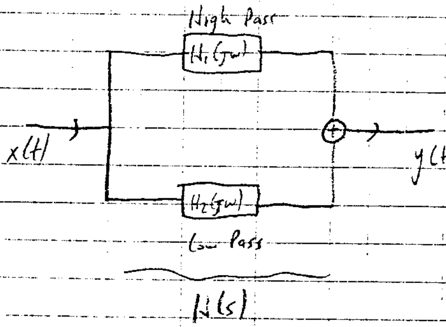
Band-Pass System

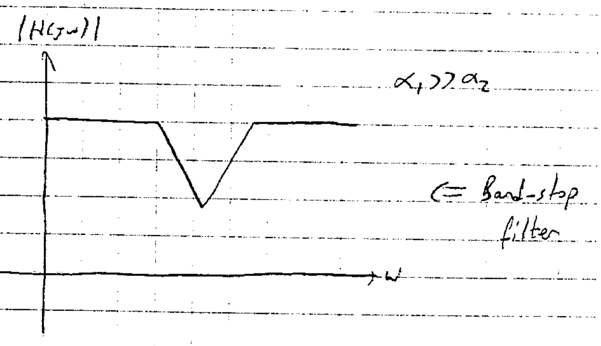


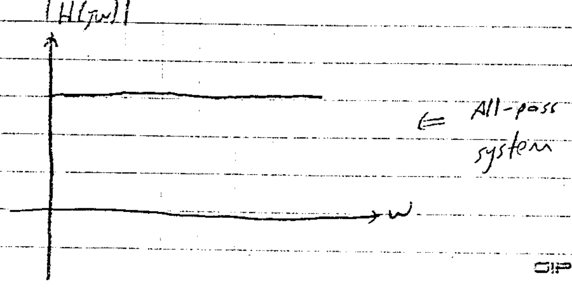
$H(s) = H_1(s)\,H_2(s)$ (We have not shown this result in detail; it follows from "convolution" result)



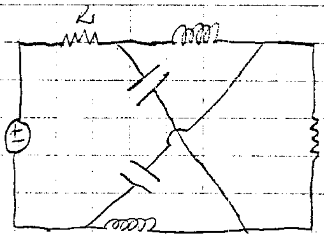
Parallel Combination







Ex//



All pass system

$\frac{L}{C} = R_L^2$

$$H(j\omega) = \frac{R_L}{R_L+R_s} \frac{1-j\omega C R_L}{1+j\omega C R_L}$$

$$|H(j\omega)| = \frac{R_L}{R_L+R_s} \quad ; \quad \measuredangle H(j\omega) = 2\tan^{-1}(-\omega C R_L)$$

SECOND ORDER CIRCUITS

Ⓐ Band-Pass System

$$H(s) = K \frac{s}{s^2+2\gamma\omega_0 s+\omega_0^2}$$

$s^2+2\gamma\omega_0 s+\omega_0^2$: char. poly

$$H(s) = K \frac{s}{s-\lambda_1} \frac{1}{s-\lambda_2}$$

- distinct and real roots $\Rightarrow \lambda_1 \neq \lambda_2$ -
- identical roots $\Rightarrow \lambda_1 = \lambda_2 \rightarrow H(s) = K \frac{s}{(s-\lambda_1)^2}$
- complex conjugate roots $\Rightarrow \lambda_1 = \lambda_2^*$

2nd order system with complex poles. Requires some special interest

$$H(s)\Big|_{s=j\omega} = K \frac{j\omega}{-\omega^2+2\gamma\omega_0 j\omega+\omega_0^2}$$

$$= K \frac{1}{\frac{-\omega^2}{j\omega}+\frac{2\gamma\omega_0 j\omega}{j\omega}+\frac{\omega_0^2}{j\omega}} = K \frac{1}{2\gamma\omega_0+j\left[\omega-\frac{\omega_0^2}{\omega}\right]}$$

$$= K \frac{1}{\omega_0\left(2\gamma+j\left[\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right]\right)}$$

$$\Rightarrow |H(j\omega)| = \frac{|K|}{\omega_0} \frac{1}{\sqrt{(2\gamma)^2+\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}} \qquad \measuredangle H(j\omega) = \measuredangle K - \tan^{-1}\left[\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)/2\gamma\right]$$

$$s^2+2\gamma\omega_0 s+\omega_0^2 = 0$$

$$s_{1,2} = \{-\gamma\omega_0 \mp \omega_0\sqrt{\gamma^2-1}\}$$

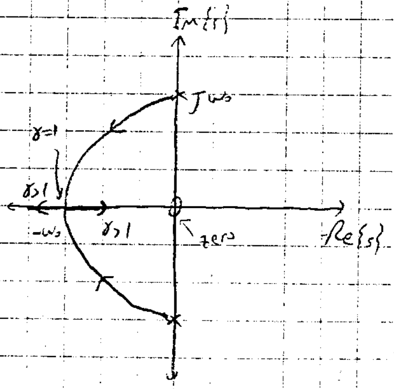
① $\gamma = 1 \longrightarrow s_{1,2} = \{-\omega_0\}$ double roots (critically damped) $\lambda_{1,2} = -\omega_0$

② $\gamma > 1 \longrightarrow$ distinct roots (over damped)

③ $\gamma = 0 \longrightarrow s_{1,2} = \{-j\omega_0\}$

④ $|\gamma| < 1 \longrightarrow$ Complex roots.

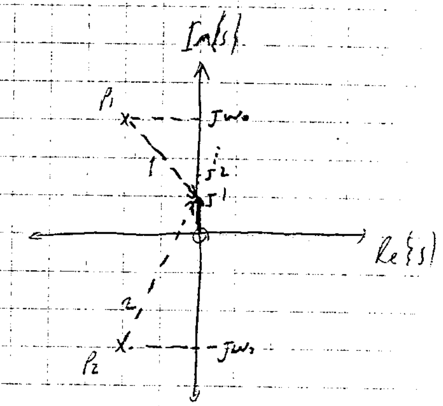
Pole-Zero Plot for $H(s)$



$$H(s) = K \frac{s}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

γ : damping coefficient

ω_0 : resonant frequency (rad/sec



$$H(s) = K \frac{s}{(s-p_1)(s-p_2)} \qquad |H(j\omega)| = |H(s)|\Big|_{s=j\omega}$$

$$|H(j\omega)| = \frac{|K| \cdot |j\omega|}{|j\omega - p_1| \cdot |j\omega - p_2|} = |K| \cdot \frac{\text{Length of blue solid vector}}{\text{Length of blue shaded vector 1} \cdot \text{Length of blue shaded vector 2}}$$

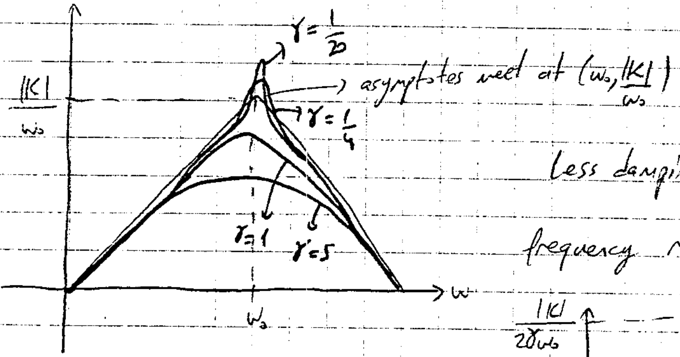
Returning to Algebraic Discussion

$$H(j\omega) = \frac{\frac{K}{\omega_0}}{2\gamma + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \qquad \underset{\omega}{\arg\max}\left(|H(j\omega)|\right) = \omega_0$$

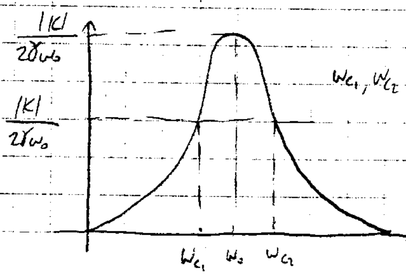
$$|H(j\omega_0)| = \max_{\omega} |H(j\omega)| = \frac{|K|}{2\gamma\omega_0}$$

① $\omega \ll \omega_0 \longrightarrow H(j\omega) = \dfrac{K/\omega_0}{2\gamma + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \cong \dfrac{K/\omega_0}{2\gamma + j\left(-\frac{\omega_0}{\omega}\right)} \cong \dfrac{K/\omega_0}{j\frac{\omega_0}{\omega}} \cong j\dfrac{K\omega}{\omega_0^2}$

② $\omega \gg \omega_0 \longrightarrow H(j\omega) \cong \dfrac{\frac{K}{\omega_0}}{2\gamma + j\frac{\omega}{\omega_0}} \cong -j\dfrac{K}{\omega}$



Less damping (γ smaller) results in peaky frequency response



$\omega_{c_1}, \omega_{c_2}$: are the cut-off freq

$$|H(jw)| = \left| \frac{K/w_0}{2\gamma + j\underbrace{\left(\frac{w_0}{w} - \frac{w}{w_0}\right)}_{\mp 2\gamma}} \right|_{w=w_{c1}, w_{c2}} = \frac{|K|/w_0}{\sqrt{2}\, 2\gamma}$$

Then cut off frequencies satisfy the following quadratic.

$$\left(\frac{w}{w_0} - \frac{w_0}{w}\right) = 2\gamma$$

$w^2 - 2\gamma w_0 w - w_0^2 = 0$ ← solution gives me the cut off frequencies

roots $= \{ w_0 \left(\gamma \mp \sqrt{1+\gamma^2}\right) \}$ the meaningful root is the positive one

$$w_{c2} = w_0 \left(\gamma + \sqrt{1+\gamma^2}\right)$$

For w_{c1}; we repeat same calculation for $\left(\frac{w}{w_0} - \frac{w_0}{w}\right) = -2\gamma$ and get

$$w_{c1} = w_0 \left(-\gamma + \sqrt{1+\gamma^2}\right)$$

Then Bandwith of the system $= w_{c2} - w_{c1} = 2\gamma w_0$

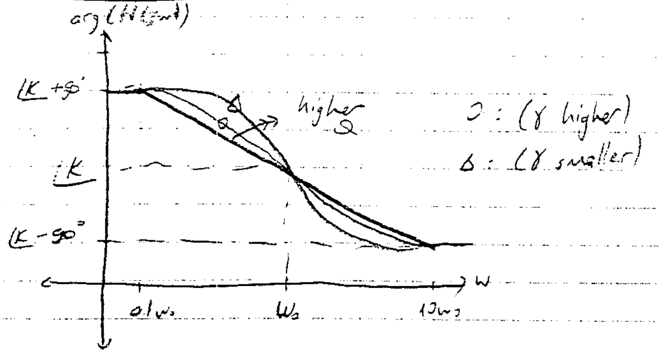
Then Q: Quality factor $= \frac{w_0}{BW} = \frac{1}{2\gamma}$

17/05/2010
Pazartesi

① γ: small ⇒ peaky response → Q: high → High Q filters → Narrow band filters

② γ: large ⇒ broad response → Q: low → low Q filter → Wide band filter

Phase Response (2^{nd} order Band Pass Systems)

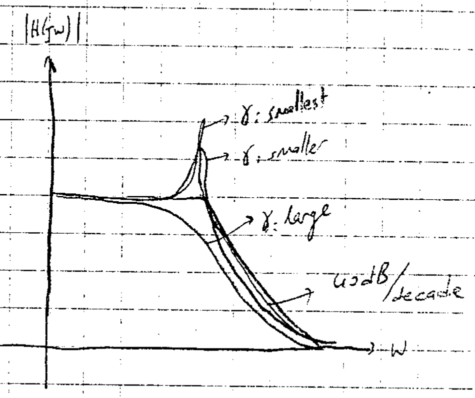


Second Order Low-Pass System

$$H(s) = \frac{K}{s^2 + 2\gamma\omega_0 s + \omega_0^2} \; ; \quad H(j\omega) = \frac{K}{\omega_0^2 - \omega^2 + j\,2\gamma\omega_0\omega}$$

$$\nearrow \; |H(j\omega)| = \frac{|K|}{\sqrt{(\omega_0 - \omega)^2 + 4\gamma^2\omega_0^2\omega^2}}$$

$$\searrow \; \measuredangle H(j\omega) = \measuredangle K - \tan^{-1}\left(\frac{2\gamma\omega_0\omega}{\omega_0^2 - \omega^2}\right)$$



① $\omega \ll \omega_0 \rightarrow |H(j\omega)| = \frac{|K|}{\omega_0^2}$

② $\omega \gg \omega_0 \rightarrow |H(j\omega)| = \frac{|K|}{\omega^2}$

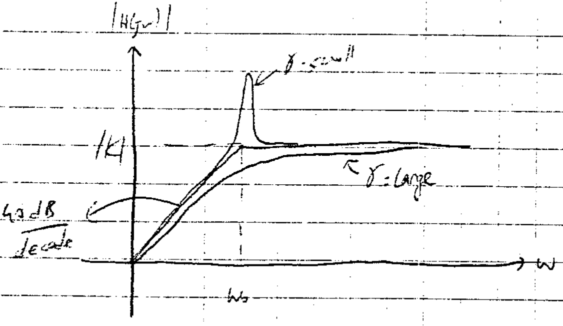
③ $|H(j\omega)| = \frac{|K|}{2\gamma\omega_0^2}$

Second Order High-Pass System

$$H(s) = \frac{Ks^2}{s^2 + 2\gamma\omega_0 s + \omega_0^2}$$

$$H(j\omega) = \frac{K(-\omega^2)}{\omega_0^2 - \omega^2 + j2\gamma\omega_0\omega} = \frac{-K\omega^2}{\omega_0^2\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{2\gamma\omega}{\omega_0}\right)}$$

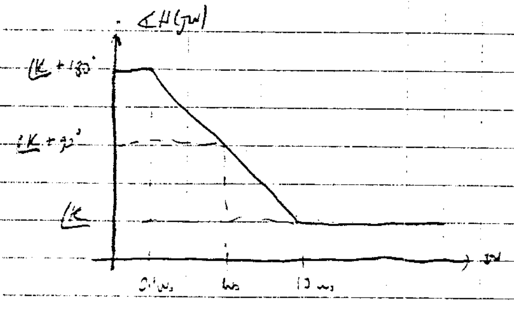
$$\arg(H(j\omega)) = 180^\circ + \angle K - \tan^{-1}\left(\frac{2\gamma\omega/\omega_0}{1 - \left(\frac{\omega}{\omega_0}\right)^2}\right)$$



① $\omega \ll \omega_0 \rightarrow H(j\omega) = \frac{-K\omega^2}{\omega_0^2} \; ; \; |H(j\omega)| = \frac{|K|\omega^2}{\omega_0^2}$

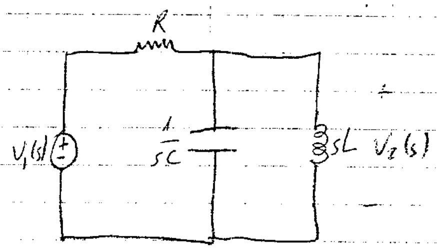
② $\omega \gg \omega_0 \Rightarrow H(j\omega) = \frac{-K\omega^2}{-\omega^2} = K \; ; \; |H(j\omega)| = |K|$

③ $\omega = \omega_0 \Rightarrow |H(j\omega)| = \frac{|K|}{2\gamma}$



20/05/2010
Persembe

Ex



Find the type of filter and $H(s)$

L.P → $w=0$ → $V_2(s)=0$ ⇓ not L.P

HP → $w=\infty$ → $V_2(s)=0$ ⇓ not HP

$$H(s)=\frac{V_2(s)}{V_1(s)}=\frac{\left(\frac{1}{sC}\,//\,sL\right)}{R+\left(\frac{1}{sC}\,//\,sL\right)}=\frac{s/RC}{s^2+\frac{s}{RC}+\frac{1}{LC}}$$

← Compare this form with previously studied LP, HP, Band Pass 2nd Order Systems

→ Band pass

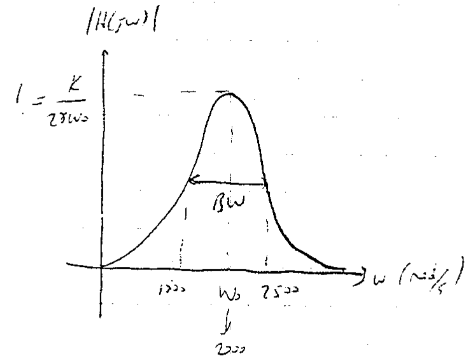
$$\Rightarrow H(s)=\underbrace{K}_{\frac{1}{RC}}\frac{s}{s^2+\underbrace{2\delta w_0}_{\frac{1}{RC}}s+\underbrace{w_0^2}_{\frac{1}{LC}}}$$

Let $L=0.25\,H$ $R=1k\Omega$ $C=1\mu F$

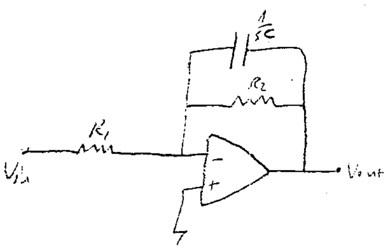
$$w_0=\frac{1}{\sqrt{LC}}=2\,krad/sec\rightarrow\left(f_0=\frac{w_0}{2\pi}=318.3\,Hz\right)$$

$$BW=2w_0\delta=\frac{1}{RC}=1\,krad/sec\rightarrow(BW=159.15\,Hz)$$

$$Q=\frac{w_0}{BW}=2$$



Ex



$$H(s)=\frac{-R_2}{R_1}\frac{1}{1+sR_2C}$$

GIPTA

24/05/2010
Pazartesi

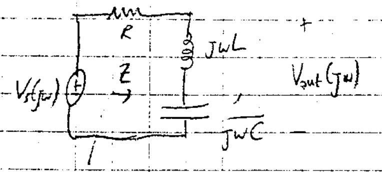
Scaling:

Used to modify a given filter to

i) a realizable filter (with components on the shelf)

ii) Changing (scaling) gain-phase response

1- Magnitude Scaling



$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

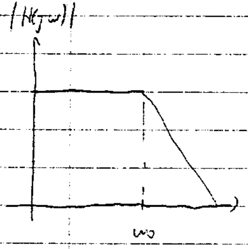
$$I_s(j\omega) = \frac{V_s(j\omega)}{R + j\omega L + \frac{1}{j\omega C}} \Rightarrow \frac{V_{out}(j\omega)}{V_s(j\omega)} = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} \quad (*)$$

Replace $R \rightarrow k_m R$
$L \rightarrow k_m L$
$C \rightarrow \frac{C}{k_m}$

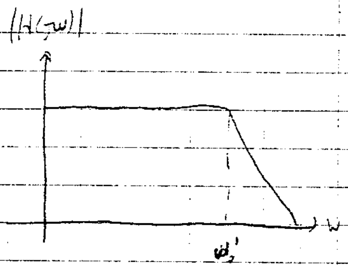
After replacement (*)

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{(j\omega L + \frac{1}{j\omega C})k_m}{(R + j\omega L + \frac{1}{j\omega C})k_m}$$ ↳ no change in the transfer function after replacement.

2- Frequency Scaling



Before scaling



After scaling.

$$H(j\omega) = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

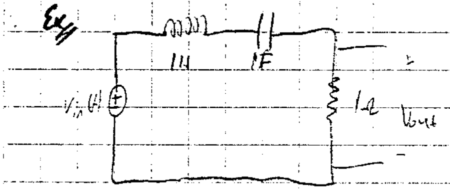
Replace $R \rightarrow R$
$L \rightarrow \frac{L}{k_f}$
$C \rightarrow \frac{C}{k_f}$

After scaling

$$H(j\omega) = \frac{\frac{j\omega L}{k_f} + \frac{1}{\frac{j\omega C}{k_f}}}{R + \frac{j\omega L}{k_f} + \frac{1}{\frac{j\omega C}{k_f}}}$$

$$\overset{after}{H}(j\omega) = \overset{before}{H}\left(\frac{j\omega}{k_f}\right)$$ select $k_f = \frac{\omega_0'}{\omega_0}$

Ex/



$$W_0 = \frac{1}{\sqrt{LC}} = 1 \ rad/sec$$

$$BW = 2\gamma w_0 = \frac{R}{L} = 1 \ rad/sec \qquad Q = \frac{W_0}{BW} = 1$$

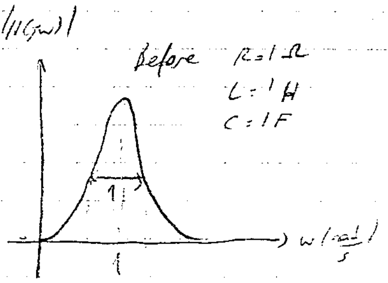
Scale the circuit s.t. the resonant freq. is at 500 Hz. and use a 2μF cap.

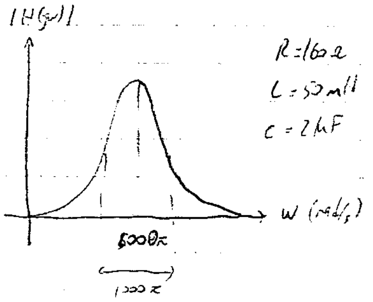
$$w_0' = 2\pi f = 1000\pi \qquad kf = \frac{w_0'}{w_0} = 1000\pi$$

$$R \xrightarrow{kf} R \xrightarrow{km} Rkm \rightarrow 160\Omega$$

$$L \xrightarrow{kf} \frac{L}{kf} \xrightarrow{km} L\frac{km}{kf} \rightarrow 50mH$$

$$C \xrightarrow{kf} \frac{C}{kf} \xrightarrow{km} \frac{C}{kf\,km} = 2\mu F \Rightarrow \frac{1}{1000\pi\, km} = \frac{2}{10^6} \Rightarrow km = \frac{500}{\pi} \sim 160$$





BODE PLOTS

Ex/ $H(s) = 12500 \frac{(s+10)}{(s+50)(s+500)}$

1) Bring H(s) into standard form

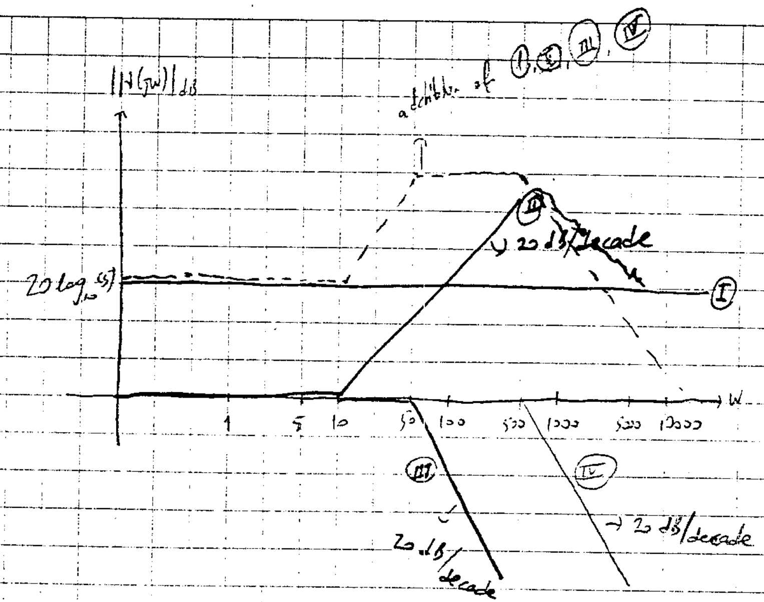
$$H^{standard}(s) = \frac{K(1+s/\alpha_1)}{\left(1+\frac{s}{\alpha_2}\right)\left(1+\frac{s}{\alpha_3}\right)} \Rightarrow H(s) = \overset{5}{\cancel{12500}} \frac{\cancel{10}\left(1+\frac{s}{10}\right)}{\cancel{50}\left(1+\frac{s}{50}\right)\cancel{500}\left(1+\frac{s}{500}\right)} = 5\frac{\left(1+\frac{s}{10}\right)}{\left(1+\frac{s}{50}\right)\left(1+\frac{s}{500}\right)}$$

2) Find critical freq.

$$w_{critical} = \{50, 500\}$$

3) $|H(jw)| = \frac{5\,|1+jw/10|}{\left|1+\frac{jw}{50}\right|\left|1+\frac{jw}{500}\right|}$

$$|H(jw)|_{dB} = 20\log_{10}|H(jw)| = 20\log_{10}(5) + 20\log_{10}|1+jw/10| - 20\log_{10}\left|1-\frac{jw}{50}\right| - 20\log_{10}\left|1+\frac{jw}{500}\right|$$



$$|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)|$$

$$= 20\log_{10}(5) \quad (I)$$

$$+ 20\log_{10}\left(1+\frac{j\omega}{10}\right) \quad (II)$$

$$- 20\log_{10}\left(1+\frac{j\omega}{50}\right) \quad (III)$$

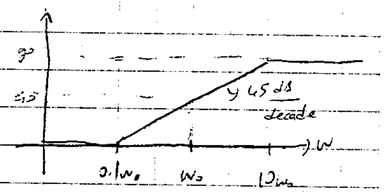
$$- 20\log_{10}\left(1+\frac{j\omega}{500}\right) \quad (IV)$$

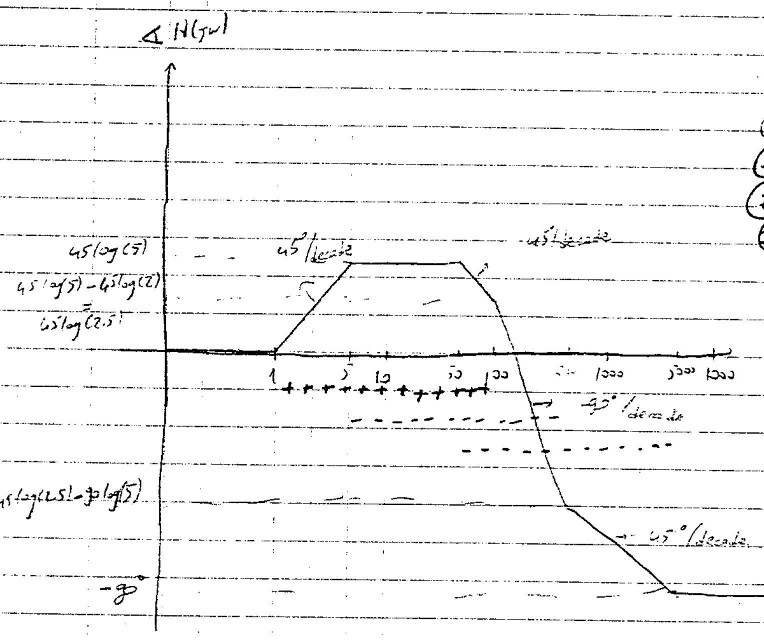
$$\omega_{critical} = \{10, 50, 500\}$$

Phase Plot

(I) $\cdot \measuredangle 5$ (II) $\measuredangle\left(1+\frac{j\omega}{10}\right)$ (III) $-\measuredangle\left(1+j\omega/50\right)$ (IV) $-\measuredangle\left(1+\frac{j\omega}{500}\right)$

$$\measuredangle H(j\omega) = (I) + (II) + (III) + (IV)$$





(I) ⟹ 0

(II) ⟹ 45 dB/decade between [1, 100]

(III) ⟹ −45 dB/decade between [5, 500]

(IV) ⟹ −45 dB/decade between [50, 5000]

27/05/2010

Perşembe

Bode Plots with 2nd Order Systems

Let's examine the following form

$$A(s) = s^2 + 2\gamma\omega_0 s + \omega_0^2$$

Note: $A(s)$ can be factorized as $(s+\alpha_1)(s+\alpha_2)$ (when $\gamma > 1$) (α_1, α_2: real) and then reduces $A(s)$ to the multiplication of 1st order systems.

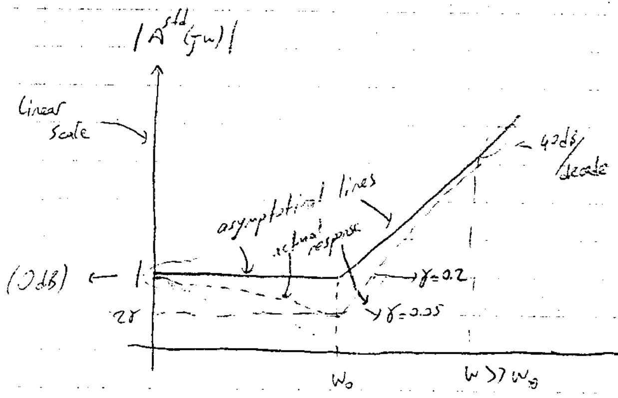
Then assume ($\gamma < 1$); (roots are imaginary)

To start the analysis write A(s) in the standard form.

$$A(s) = \omega_0^2\left(1+\frac{2\gamma s}{\omega_0}+\frac{s^2}{\omega_0^2}\right)$$

$$A^{std}(s) = 1+\frac{2\gamma}{\omega_0}s+\frac{s^2}{\omega_0^2}\Big|_{s=j\omega} \xrightarrow{|\cdot|} \sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\left(\frac{2\gamma\omega}{\omega_0}\right)^2} \longrightarrow 20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\left(\frac{2\gamma\omega_0}{\omega}\right)^2}$$

$$\xrightarrow{\measuredangle} \tan^{-1}\left(\frac{2\gamma\omega/\omega_0}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$



Case 1 : $\omega << \omega_0$

Case 2 : $|A^{std}(j\omega)|_{dB} = 20\log_{10}\left|\sqrt{\left(\frac{\omega}{\omega_0}\right)^4+(\ldots)\left(\frac{\omega}{\omega_0}\right)^2+(\ldots)\left(\frac{\omega}{\omega_0}\right)^0}\right| = 20\log_{10}\left(\frac{\omega}{\omega_0}\right)^2$

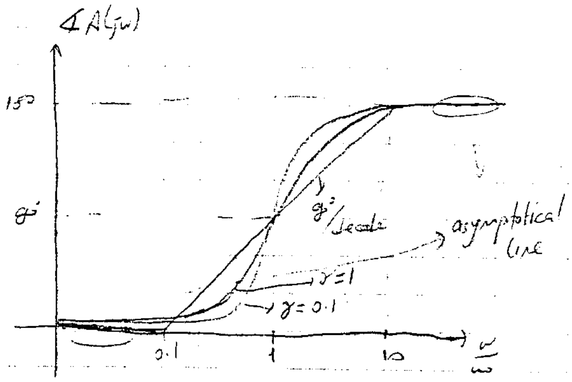
$\approx |A^{std}(j\omega)|$

Case 3 . $\omega=\omega_0 \to |A^{std}(j\omega_0)| = 2\gamma$

$\gamma<1$ for imaginary roots or underdamped system

$Q=\frac{1}{2\gamma}$: High Q , small γ

Note . $\gamma=\frac{1}{2}$; $\xrightarrow[2\gamma=1]{}$ actual curve passes through the intersection of asymptotes

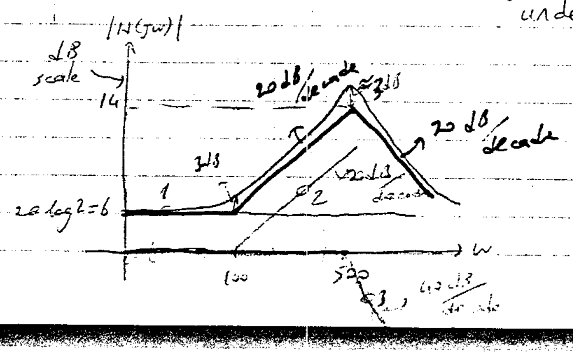


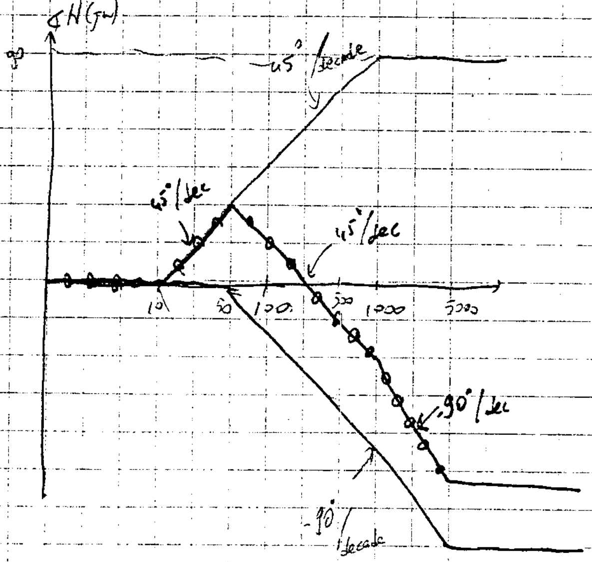
$\measuredangle A(j\omega) = \tan^{-1}\left(\frac{2\gamma\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$

Ex $H(s) = \frac{5000(s+100)}{s^2+400s+(500)^2} = \frac{5000\cdot100\left(1+\frac{s}{100}\right)}{500^2\left(1+\frac{400}{500^2}s+\left(\frac{s}{500}\right)^2\right)} = \frac{\overset{1}{2}\,\overset{2}{\left(1+\frac{s}{100}\right)}}{\underset{3}{\left(1+\frac{s}{625}+\left(\frac{s}{500}\right)^2\right)}}$

$s^2+\underbrace{400s}_{2\gamma\omega}+\underbrace{500^2}_{\omega_0^2} \Rightarrow \omega_0=500 \;\; \gamma=0.4$

underdamped (2nd order system model should be used!!)





STATE EQUATIONS WITH NON LINEAR ELEMENTS

If an element is non-linear and we would like to express the circuit in the state equation form, we can do the following;

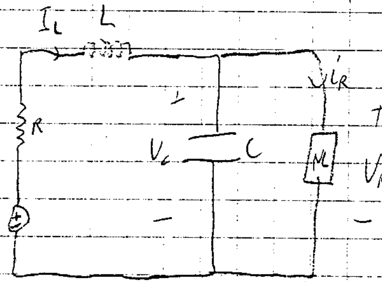
① Check whether component current/voltage controlled

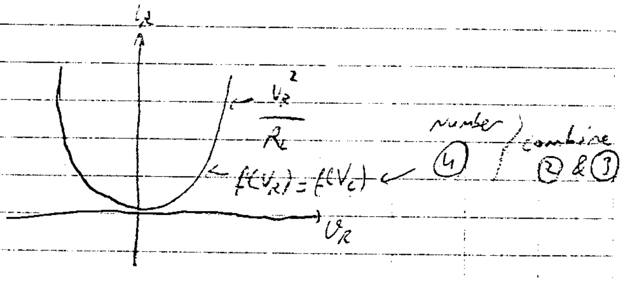
② Try to include the current of the non-linear component if it's current controlled (or its voltage otherwise)

③ Include variables of dynamic components as state variables

④ Try to combine ② and ③

Ex//





$\underline{KCL}$: $C\dot{V}_c(t) = -\overset{f(V_c)}{i_R} + i_L$: ✓ RHS contains only state variables

$\underline{KVL}$. $-V_s(t) + RI_L + L\dot{I}_L + V_c = 0$

$$L\dot{I}_L(t) = -V_c - RI_L + V_s(t) \ : ✓$$

$$\Rightarrow \quad \dot{V}_c(t) = -\frac{V_c^2(t)}{R_L C} + \frac{i_L}{C}$$

$$\dot{I}_L(t) = -\frac{V_c(t)}{L} - \frac{R}{L} I_L(t) + \frac{V_s(t)}{L}$$

Zero Input

$V_c(0^-) = V_0$ $\quad I_L(0^-) = I_0$

$$\dot{V}_c(t) = -\frac{V_c^2(t)}{R_L C} + \frac{i_L(t)}{C}$$

$$\dot{I}_L(t) = -\frac{V_c(t)}{L} - \frac{R}{L} I_L(t)$$

$V_c(t) \triangleq V_c \qquad I_L(t) \triangleq I_L$

$\dot{V}_c(t) = 0 \Rightarrow -\frac{V_c^2}{R_L C} + \frac{I_L}{C} = 0 \Rightarrow V_c^2 = I_L R_L \Rightarrow V_c = \pm\sqrt{I_L R_L}$

$\dot{I}_L(t) = 0 \Rightarrow -\frac{V_c}{L} - \frac{R}{L} I_L = 0 \Rightarrow V_c = -R I_L$