

A Matlab Example on Circuit Characterization by Node, Modified Node, Mesh and State Equations

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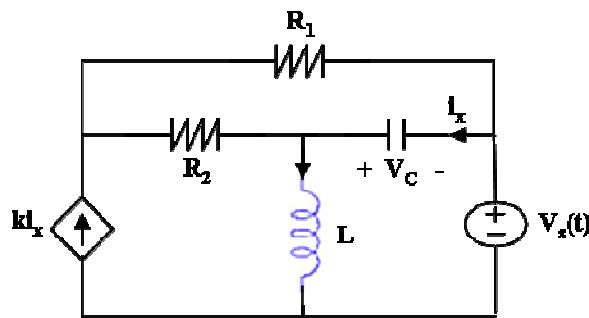
(Prepared for Circuit Theory-II Course (EE 202) on March 09, 2011)

The following circuit is from ZPS-1 (Problem 1c). In these notes, we show how to find the natural frequencies of the circuit from different circuit characterizations (as in Problem 1c) and show how to use Matlab to do the manual calculations. (We also examine the stability of the system when some of its parameters are left as free variables.)

Our goal is to illustrate different characterizations of circuit, to show the calculation of natural frequencies, to discuss the stability concept and emphasize the ease of Matlab usage in similar problems.

Students are invited to closely examine the Matlab codes and do similar experiments. For students who have not used Matlab before, some tutorials are available at:

<http://www.eee.metu.edu.tr/~ee301/material.html>



Part 1:

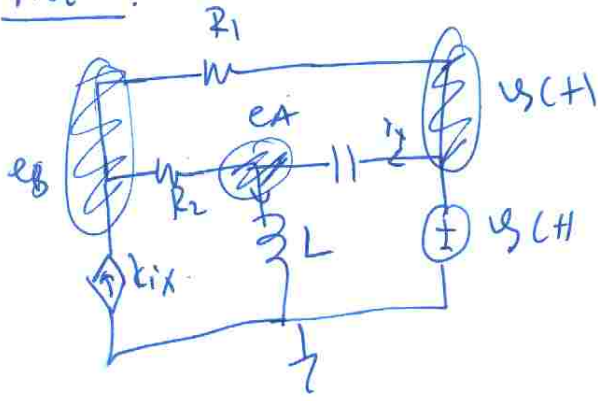
- Find Node, Mesh, Modified Node and State Equation characterizations of the circuit.
- Set $R_1 = \frac{1}{3}$; $R_2 = \frac{1}{2}$; $C = \frac{1}{3}$; $L = \frac{3}{4}$ and $k = 5$ in the characterizations found in part 1a. Find the numeric values of the natural frequencies. Show that all characterizations lead to the same set of natural frequencies.

Part 2:

In this part we set $R_1 = \frac{1}{3}$; $R_2 = \frac{1}{2}$; $C = \frac{1}{3}$; $L = \frac{3}{4}$ as before, but leave k as a free variable.

- Using the state equations, find the characteristic polynomial of the circuit. The coefficients of the polynomial should include k as a parameter.
- Examine the stability of the circuit for various values of k . Sketch the locations of the natural frequencies on the complex plane. Determine for which values of k the system is unstable.

Nock:



KCL @ e_A →

$$\frac{e_A - e_B}{R_2} + i_L(t) + \frac{1}{L} \int e_A dt + C(e_A - e_C) = 0$$

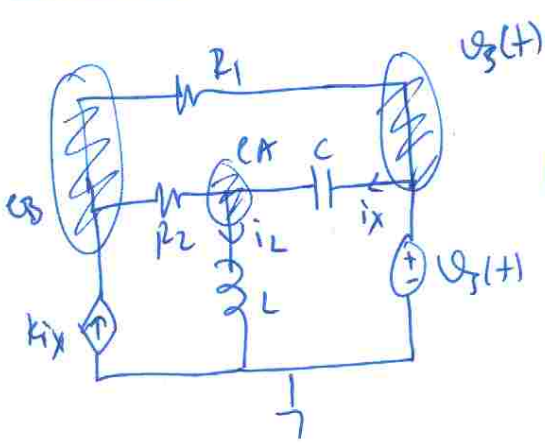
KCL @ e_B →

$$\frac{e_B - e_C}{R_1} + \frac{e_B - e_A}{R_2} - k i_x = 0$$

$$\begin{bmatrix} \frac{1}{R_2} + \frac{1}{L} D + CD & -\frac{1}{R_2} \\ -\frac{1}{R_2} + kCD & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} -i_L(t) \\ 0 \end{bmatrix} + \begin{bmatrix} CD \\ - \\ \frac{1}{R_1} + kCD \end{bmatrix} u_s(t)$$

$$\begin{bmatrix} CD^2 + \frac{1}{R_2} D + \frac{1}{L} & -\frac{1}{R_2} D \\ -\frac{1}{R_2} + kCD & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} CD^2 \\ \frac{1}{R_1} + kCD \end{bmatrix} u_s(t)$$

MNA:



KCL @ eA:

$$(1): \frac{e_A - e_B}{R_2} + i_L + C(\dot{e}_A - \dot{v}_3(t)) = 0$$

KCL @ eB:

$$(2): \frac{e_B - e_A}{R_2} + \frac{e_B - v_3}{R_1} - kix = 0$$

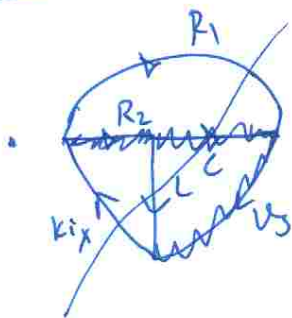
\swarrow
 $C(\dot{v}_3 - \dot{e}_A)$

Inductor L:

$$(3): e_A - L D i_L = 0$$

$$\begin{bmatrix} \frac{1}{R_2} + CD & -\frac{1}{R_2} & 1 \\ -\frac{1}{R_2} + kCD & \frac{1}{R_2} + \frac{1}{R_1} & 0 \\ 1 & 0 & -LD \end{bmatrix}
 \begin{bmatrix} e_A \\ e_B \\ I_L \end{bmatrix}
 =
 \begin{bmatrix} CD \\ \frac{1}{R_1} + kCD \\ 0 \end{bmatrix}
 \quad v_3$$

Stack:



$$C \dot{v}_c = -i_{R1} - i_L + k i_x$$

$$i_{R1} = \frac{v_{R1}}{R1} = \frac{v_{R2} + v_c}{R1} = \frac{i_{R2} R2 + v_c}{R1}$$

$$= (k i_x - i_{R1}) \frac{R2}{R1} + \frac{v_c}{R1}$$

$$i_{R1} = \frac{1}{R1 + R2} [-R2 k C \dot{v}_c + v_c]$$

$$C \dot{v}_c = \frac{1}{R1 + R2} [+R2 k C \dot{v}_c] - \frac{1}{R1 + R2} v_c - i_L - k C \dot{v}_c$$

$$\boxed{C \left(1 + k - \frac{R2 k}{R1 + R2} \right) \dot{v}_c = -\frac{1}{R1 + R2} v_c - i_L} \quad (1)$$

$$\boxed{L \dot{I}_L = v_c + v_s} \quad (2)$$

$$\begin{bmatrix} \dot{v}_c \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sigma(R1 + R2)} & -\frac{1}{\sigma} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_c \\ I_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \mathbf{g}$$

$$\sigma = C \left(1 + k - \frac{R2 k}{R1 + R2} \right)$$

$$= C \left(\frac{R1 k + 1}{R1 + R2} \right)$$

EE202 : ZPS-1 Problem 1c (Part 1)

Finding different characterizations of the circuit and the natural frequencies

Contents

- [State Equation](#)
- [Node Analysis](#)
- [MNA \(Modified Node Analysis\)](#)
- [Mesh Analysis](#)

```
syms R1 R2 k L C %introduce the symbolic variables
syms D %D is for d/dt
```

State Equation

```
gamma = C*(1+k-R2*k/(R1+R2));
A=[ -1/(gamma*(R1+R2)) -1/gamma; ...
    1/L 0];
B=[0; 1/L];

%xdot(t) = A x(t) + Bvs(t)
Anum = subs(A, {R1,R2,k,L,C}, {1/3 1/2 5 3/4 1/3}),
Bnum = subs(B, {R1,R2,k,L,C}, {1/3 1/2 5 3/4 1/3});
```

Anum =

```
-1.2000    -1.0000
 1.3333         0
```

Characteristic Polynomial (calculated from state equation characterization)

```
det(D*eye(2)-Anum)
```

ans =

```
D^2+6/5*D+4/3
```

The natural frequencies are

```
eig(Anum)
```

ans =

```
-0.6000 + 0.9866i
-0.6000 - 0.9866i
```

Node Analysis

```
G = [ C*D^2+1/R2*D+1/L -1/R2*D ; ...
```

```

-1/R2+k*C*D  1/R1+1/R2];
Gnum = subs(G, {R1,R2,k,L,C}, {1/3 1/2 5 3/4 1/3}),

```

Gnum =

```

[ 1/3*D^2+2*D+4/3,      -2*D]
[      -2+5/3*D,        5]

```

Characteristic Polynomial (calculated from node equation characterization)

det (Gnum)

ans =

```
5*D^2+6*D+20/3
```

The natural frequencies are

roots([5 6 20/3])

ans =

```

-0.6000 + 0.9866i
-0.6000 - 0.9866i

```

MNA (Modified Node Analysis)

```

M = [1/R2+C*D -1/R2 1; ...
      -1/R2+k*C*D 1/R2+1/R1 0; ...
      1 0 -L*D];
Mnum = subs(M, {R1,R2,k,L,C}, {1/3 1/2 5 3/4 1/3}),

```

Mnum =

```

[ 2+1/3*D,      -2,      1]
[ -2+5/3*D,      5,      0]
[      1,        0,     -3/4*D]

```

Characteristic Polynomial (calculated by modified node analysis)

det (Mnum)

ans =

```
-15/4*D^2-9/2*D-5
```

The natural frequencies are

```
roots([-15/4 -9/2 -5])
```

```
ans =
```

```
-0.6000 + 0.9866i  
-0.6000 - 0.9866i
```

Mesh Analysis

```
Z = [(R1+R2*(1-k))*D+1/C -k*R2*D+1/C; ...  
      1/C+L*k*D^2 1/C+L*(k+1)*D^2];
```

```
Znum = subs(Z, {R1,R2,k,L,C}, {1/3 1/2 5 3/4 1/3}),
```

```
Znum =
```

```
[ -5/3*D+3, -5/2*D+3]  
[ 3+15/4*D^2, 3+9/2*D^2]
```

Characteristic Polynomial (calculated by mesh equation characterization)

```
det(Znum)
```

```
ans =
```

```
5/2*D+15/8*D^3+9/4*D^2
```

The natural frequencies are

```
roots([15/8 9/4 5/2 0])
```

```
ans =
```

```
0  
-0.6000 + 0.9866i  
-0.6000 - 0.9866i
```

The additional root at zero is due to differentiations to get rid off the integral terms. Please see the derivation of mesh equations.

EE202 : ZPS-1 Problem 1c (Part 2)

On the stability of the circuit for various "k" values

Contents

- [State Equation](#)
- [Characteristic Polynomial](#)
- [Natural frequencies](#)
- [Case 1: k > -2.5](#)
- [Case 2: k < -2.5](#)

```
syms R1 R2 k L C %introduce the symbolic variables
syms D %D is for d/dt
```

State Equation

```
gamma = C*(1+k-R2*k/(R1+R2));
A=[ -1/(gamma*(R1+R2)) -1/gamma; ...
    1/L 0];
B=[0; 1/L];

%xdot(t) = A x(t) + Bvs(t)
Anumk = subs(A, {R1,R2,L,C}, {1/3 1/2 3/4 1/3});
Bnumk = subs(B, {R1,R2,L,C}, {1/3 1/2 3/4 1/3});
```

Anumk =

```
[ -18/5/(1+2/5*k),    -3/(1+2/5*k) ]
[          4/3,          0]
```

Characteristic Polynomial

```
collect(det(D*eye(2)-Anumk), 'D'),
```

ans =

```
D^2+18/(5+2*k)*D+20/(5+2*k)
```

$$\text{Char. Poly}(\lambda) = \lambda^2 + \frac{18}{2k+5}\lambda + \frac{20}{2k+5}$$

Natural frequencies

```
natfreq = eig(Anumk), %natural freq. as a function of "k"
```

natfreq =

```
1/(5+2*k)*(-9+(-19-40*k)^(1/2))
1/(5+2*k)*(-9-(-19-40*k)^(1/2))
```

$$\lambda = \left\{ \frac{1}{2k+5}(-9 - \sqrt{-40k-19}), \frac{1}{2k+5}(-9 + \sqrt{-40k-19}) \right\}$$

Case 1: k > -2.5

This is the case that the system is stable.

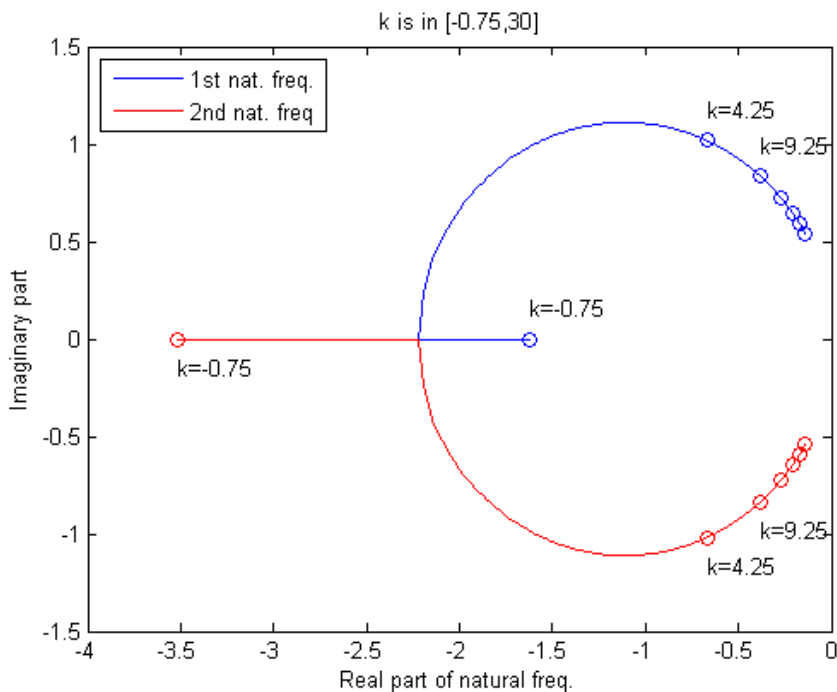
Now, we sketch the natural frequencies on the complex plane.

```
kvec=-0.75:0.025:30; %this "k" values will be examined
ind=0;
natfreq_num = zeros(2,length(kvec));
for kval=kvec
    ind=ind+1;
    natfreq_num(:,ind) = subs(natfreq,'k',kval);
end;
inc = round(5/(kvec(2)-kvec(1))); %increment

plot(real(natfreq_num(1,:)),imag(natfreq_num(1,:)), 'b'); hold on;
plot(real(natfreq_num(2,:)),imag(natfreq_num(2,:)), 'r');
plot(real(natfreq_num(1,1:inc:end)),imag(natfreq_num(1,1:inc:end)), 'bo');
plot(real(natfreq_num(2,1:inc:end)),imag(natfreq_num(2,1:inc:end)), 'ro');
xlabel('Real part of natural freq. '); ylabel('Imaginary part')
hold off;
legend('1st nat. freq.', '2nd nat. freq.', 'Location', 'NorthWest')
text(real(natfreq_num(2,1)), imag(natfreq_num(2,1))-0.15, ['k=' num2str(kvec(1))]);
text(real(natfreq_num(2,1+inc)), imag(natfreq_num(2,1+inc))-0.15, ['k=' num2str(kvec(1+inc))]);
text(real(natfreq_num(2,1+2*inc)), imag(natfreq_num(2,1+2*inc))-0.15, ['k=' num2str(kvec(1+2*inc))]);

text(real(natfreq_num(1,1)), imag(natfreq_num(1,1))+0.15, ['k=' num2str(kvec(1))]);
text(real(natfreq_num(1,1+inc)), imag(natfreq_num(1,1+inc))+0.15, ['k=' num2str(kvec(1+inc))]);
text(real(natfreq_num(1,1+2*inc)), imag(natfreq_num(1,1+2*inc))+0.15, ['k=' num2str(kvec(1+2*inc))]);

title('k is in [-0.75,30]');
```



Note that

- $k=-0.75$, two natural frequencies are real and negative valued
- as k increases natural frequencies are approaching together
- the approach continues until $k=-19/40=-0.475$ (Why?)
- when $k>-0.475$, the natural frequencies are imaginary and have negative real parts and the system is stable.

The comments given above can also be checked from

$$\lambda = \left\{ \frac{1}{2k+5}(-9 - \sqrt{-40k-19}), \frac{1}{2k+5}(-9 + \sqrt{-40k-19}) \right\}$$

Case 2: $k < -2.5$

This is the case that the system is unstable!

```

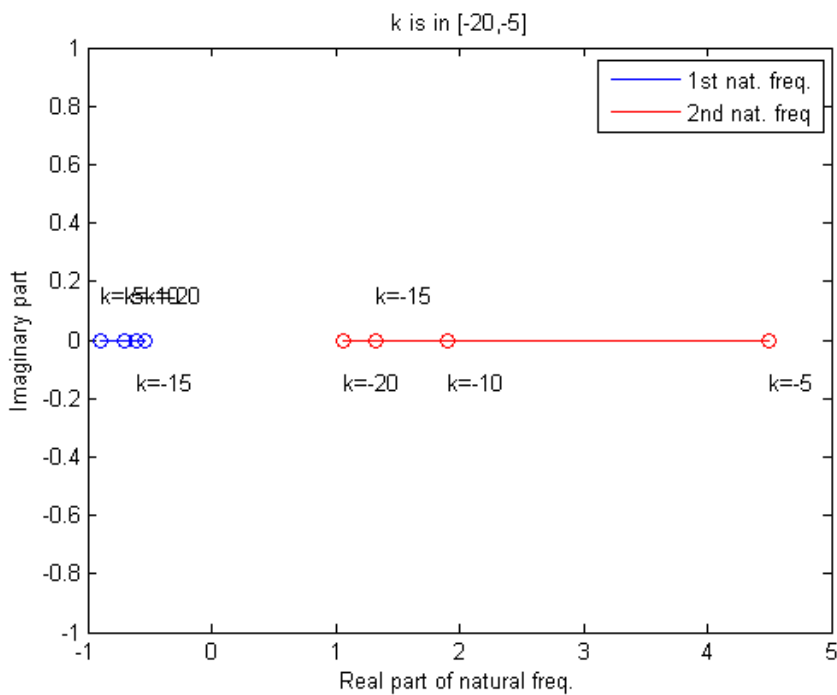
kvec=linspace(-20,-5,1024); %this "k" values will be examined
ind=0;
natfreq_num = zeros(2,length(kvec));
for kval=kvec
    ind=ind+1;
    natfreq_num(:,ind) = subs(natfreq,'k',kval);
end;
inc = round(5/(kvec(2)-kvec(1))); %increment

plot(real(natfreq_num(1,:)),imag(natfreq_num(1,:)),'b'); hold on;
plot(real(natfreq_num(2,:)),imag(natfreq_num(2,:)),'r');
plot(real(natfreq_num(1,1:inc:end)),imag(natfreq_num(1,1:inc:end)),'bo');
plot(real(natfreq_num(2,1:inc:end)),imag(natfreq_num(2,1:inc:end)),'ro');
xlabel('Real part of natural freq. '); ylabel('Imaginary part')
hold off;
legend('1st nat. freq.','2nd nat. freq')
text(real(natfreq_num(2,1)),imag(natfreq_num(2,1))-0.15,['k=' num2str(kvec(1))]);
text(real(natfreq_num(2,1+inc)),imag(natfreq_num(2,1+inc))+0.15,['k=' num2str(kvec(1+inc))]);
text(real(natfreq_num(2,1+2*inc)),imag(natfreq_num(2,1+2*inc))-0.15,['k=' num2str(kvec(1+2*inc))]);
text(real(natfreq_num(2,1+3*inc)),imag(natfreq_num(2,1+3*inc))-0.15,['k=' num2str(kvec(1+3*inc))]);

text(real(natfreq_num(1,1)),imag(natfreq_num(1,1))+0.15,['k=' num2str(kvec(1))]);
text(real(natfreq_num(1,1+inc)),imag(natfreq_num(1,1+inc))-0.15,['k=' num2str(kvec(1+inc))]);
text(real(natfreq_num(1,1+2*inc)),imag(natfreq_num(1,1+2*inc))+0.15,['k=' num2str(kvec(1+2*inc))]);
text(real(natfreq_num(1,1+3*inc)),imag(natfreq_num(1,1+3*inc))+0.15,['k=' num2str(kvec(1+3*inc))]);

title('k is in [-20,-5]');

```



Note that

- $k=-5$, the natural freq. shown with red color is real and its value is around 4.
- as k goes to $-\infty$, the natural frequencies are getting "closer" to each other
- but one of them is always positive and the other is negative valued for this case (Why?) hence the system is unstable.

The comments given above can also be checked from

$$\lambda = \left\{ \frac{1}{2k+5}(-9 - \sqrt{-40k-19}), \frac{1}{2k+5}(-9 + \sqrt{-40k-19}) \right\}$$