36. Let $S(t)$ denote the price of a security at time t. A popular model for the process $\{S(t), t \geqslant 0\}$ supposes that the price remains unchanged until a "shock" occurs, at which time the price is multiplied by a random factor. If we let $N(t)$ denote the number of shocks by time t, and let X_i denote the i^{th} multiplicative factor, then this model supposes that

$$S(t) = S(0) \prod_{i=1}^{N(t)} X_i$$

where $\prod_{i=1}^{N(t)} X_i$ is equal to 1 when $N(t) = 0$. Suppose that the X_i are independent exponential random variables with rate μ; that $\{N(t), t \geqslant 0\}$ is a Poisson process with rate λ; that $\{N(t), t \geqslant 0\}$ is independent of the X_i; and that $S(0) = s$.

(a) Find $E[S(t)]$
(b) Find $E[S^2(t)]$

37. Cars cross a certain point in the highway in accordance with a Poisson process with rate $\lambda = 3$ per minute. If Reb blindly runs across the highway, then what is the probability that she will be uninjured if the amount of time that it takes her to cross the road is s seconds? (Assume that if she is on the highway when a car passes by, then she will be injured.) Do it for $s = 2, 5, 10, 20$.

38. Suppose in Exercise 37 that Reb is agile enough to escape from a single car, but if she encounters two or more cars while attempting to cross the road, then she will be injured. What is the probability that she will be unhurt if it takes her s seconds to cross? Do it for $s = 5, 10, 20, 30$.

39. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Assuming this theory, find

(a) the mean lifetime of an individual,
(b) the variance of the lifetime of an individual.

Also approximate

(c) the probability that an individual dies before age 67.2.
(d) the probability that an individual reaches age 90.
(e) the probability that an individual reaches age 100.

***40.** Show that if $\{N_i(t), t \geqslant 0\}$ are independent Poisson processes with rate λ_i, $i = 1, 2$, then $\{N(t), t \geqslant 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$ where $N(t) = N_1(t) + N_2(t)$.

41. In Exercise 40 what is the probability that the first event of the combined process is from the N_1 process?

42. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the nth event. Find

(a) $E[S_4]$
(b) $E[S_4|N(1) = 2]$ ♦
(c) $E[N(4) - N(2)|N(1) = 3]$

43. Customers arrive at a two-server service station according to a Poisson process with rate λ. Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters service first with server 1 and then with server 2. If the service times at the servers are independent exponentials with respective rates μ_1 and μ_2, what proportion of entering customers completes their service with server 2?

44. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.

(a) Find the probability that her waiting time is 0.
(b) Find her expected waiting time.

Hint: Condition on the time of the first car.

45. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ, that is independent of the nonnegative random variable T with mean μ and variance σ^2. Find

(a) $\mathrm{Cov}(T, N(T))$
(b) $\mathrm{Var}(N(T))$

46. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ, that is independent of the sequence $X_1, X_2, \ldots$ of independent and identically distributed random variables with mean μ and variance σ^2. Find

$$\mathrm{Cov}\left(N(t), \sum_{i=1}^{N(t)} X_i\right)$$

47. Consider a two-server parallel queueing system where customers arrive according to a Poisson process with rate λ, and where the service times are exponential with rate μ. Moreover, suppose that arrivals finding both servers busy immediately depart without receiving any service (such a customer is said to be lost), whereas those finding at least one free server immediately enter service and then depart when their service is completed.

(a) If both servers are presently busy, find the expected time until the next customer enters the system.

(a) What is the distribution of $N(n)$?
(b) What is the distribution of T_r?
(c) What is the distribution of I_r?
(d) Given that $N(n) = r$, show that the unordered set of r days on which events occurred has the same distribution as a random selection (without replacement) of r of the values $1, 2, \ldots, n$.

57. Events occur according to a Poisson process with rate $\lambda = 2$ per hour.

(a) What is the probability that no event occurs between 8 P.M. and 9 P.M.?
(b) Starting at noon, what is the expected time at which the fourth event occurs?
(c) What is the probability that two or more events occur between 6 P.M. and 8 P.M.?

58. Pulses arrive at a Geiger counter in accordance with a Poisson process at a rate of three arrivals per minute. Each particle arriving at the counter has a probability $\frac{2}{3}$ of being recorded. Let $X(t)$ denote the number of pulses recorded by time t minutes.

(a) $P\{X(t) = 0\} = ?$
(b) $E[X(t)] = ?$

59. There are two types of claims that are made to an insurance company. Let $N_i(t)$ denote the number of type i claims made by time t, and suppose that $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent Poisson processes with rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The amounts of successive type 1 claims are independent exponential random variables with mean \$1000 whereas the amounts from type 2 claims are independent exponential random variables with mean \$5000. A claim for \$4000 has just been received; what is the probability it is a type 1 claim?

***60.** Customers arrive at a bank at a Poisson rate λ. Suppose two customers arrived during the first hour. What is the probability that

(a) both arrived during the first 20 minutes?
(b) at least one arrived during the first 20 minutes?

61. A system has a random number of flaws that we will suppose is Poisson distributed with mean c. Each of these flaws will, independently, cause the system to fail at a random time having distribution G. When a system failure occurs, suppose that the flaw causing the failure is immediately located and fixed.

(a) What is the distribution of the number of failures by time t?
(b) What is the distribution of the number of flaws that remain in the system at time t?
(c) Are the random variables in parts (a) and (b) dependent or independent?

62. Suppose that the number of typographical errors in a new text is Poisson distributed with mean λ. Two proofreaders independently read the text. Suppose that each error is independently found by proofreader i with probability p_i, $i = 1, 2$. Let X_1 denote the number of errors that are found by proofreader 1 but not by proofreader 2. Let X_2 denote the number of errors that are found by proofreader 2 but not by proofreader 1. Let X_3 denote the number of errors that are found by both proofreaders. Finally, let X_4 denote the number of errors found by neither proofreader.

(a) Describe the joint probability distribution of X_1, X_2, X_3, X_4.
(b) Show that

$$\frac{E[X_1]}{E[X_3]} = \frac{1-p_2}{p_2} \quad \text{and} \quad \frac{E[X_2]}{E[X_3]} = \frac{1-p_1}{p_1}$$

Suppose now that λ, p_1, and p_2 are all unknown.

(c) By using X_i as an estimator of $E[X_i]$, $i = 1, 2, 3$, present estimators of p_1, p_2, and λ.
(d) Give an estimator of X_4, the number of errors not found by either proofreader.

63. Consider an infinite server queueing system in which customers arrive in accordance with a Poisson process and where the service distribution is exponential with rate μ. Let $X(t)$ denote the number of customers in the system at time t. Find

(a) $E[X(t+s)|X(s) = n]$
(b) $\mathrm{Var}[X(t+s)|X(s) = n]$

Hint: Divide the customers in the system at time $t+s$ into two groups, one consisting of "old" customers and the other of "new" customers.

***64.** Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ. The bus departs at time t. Let X denote the total amount of waiting time of all those who get on the bus at time t. We want to determine $\mathrm{Var}(X)$. Let $N(t)$ denote the number of arrivals by time t.

(a) What is $E[X|N(t)]$?
(b) Argue that $\mathrm{Var}[X|N(t)] = N(t)t^2/12$.
(c) What is $\mathrm{Var}(X)$?

65. An average of 500 people pass the California bar exam each year. A California lawyer practices law, on average, for 30 years. Assuming these numbers remain steady, how many lawyers would you expect California to have in 2050?

66. Policyholders of a certain insurance company have accidents at times distributed according to a Poisson process with rate λ. The amount of time from when the accident occurs until a claim is made has distribution G.

Hint: Make use of the identity

$$m(t+h) - m(t) = m'(t)h + o(h)$$

***84.** Let $X_1, X_2, \ldots$ be independent and identically distributed nonnegative continuous random variables having density function $f(x)$. We say that a record occurs at time n if X_n is larger than each of the previous values $X_1, \ldots, X_{n-1}$. (A record automatically occurs at time 1.) If a record occurs at time n, then X_n is called a *record value*. In other words, a record occurs whenever a new high is reached, and that new high is called the record value. Let $N(t)$ denote the number of record values that are less than or equal to t. Characterize the process $\{N(t), t \geqslant 0\}$ when

(a) f is an arbitrary continuous density function.
(b) $f(x) = \lambda e^{-\lambda x}$.

Hint: Finish the following sentence: There will be a record whose value is between t and $t + dt$ if the first X_i that is greater than t lies between. . . .

85. An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 5$ per week. If the amount of money paid on each policy is exponentially distributed with mean \$2000, what is the mean and variance of the amount of money paid by the insurance company in a four-week span?

86. In good years, storms occur according to a Poisson process with rate 3 per unit time, while in other years they occur according to a Poisson process with rate 5 per unit time. Suppose next year will be a good year with probability 0.3. Let $N(t)$ denote the number of storms during the first t time units of next year.

(a) Find $P\{N(t) = n\}$.
(b) Is $\{N(t)\}$ a Poisson process?
(c) Does $\{N(t)\}$ have stationary increments? Why or why not?
(d) Does it have independent increments? Why or why not?
(e) If next year starts off with three storms by time $t = 1$, what is the conditional probability it is a good year?

87. Determine

$$\mathrm{Cov}[X(t), X(t+s)]$$

for 15 hours daily. Approximate the probability that the total daily withdrawal is less than \$6000.

89. Some components of a two-component system fail after receiving a shock. Shocks of three types arrive independently and in accordance with Poisson processes. Shocks of the first type arrive at a Poisson rate λ_1 and cause the first component to fail. Those of the second type arrive at a Poisson rate λ_2 and cause the second component to fail. The third type of shock arrives at a Poisson rate λ_3 and causes both components to fail. Let X_1 and X_2 denote the survival times for the two components. Show that the joint distribution of X_1 and X_2 is given by

$$P\{X_1 > s, X_1 > t\} = \exp\{-\lambda_1 s - \lambda_2 t - \lambda_3 \max(s, t)\}$$

This distribution is known as the *bivariate exponential distribution.*

90. In Exercise 89 show that X_1 and X_2 both have exponential distributions.

***91.** Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables. Show that the probability that the largest of them is greater than the sum of the others is $n/2^{n-1}$. That is, if

$$M = \max_j X_j$$

then show

$$P\left\{M > \sum_{i=1}^{n} X_i - M\right\} = \frac{n}{2^{n-1}}$$

Hint: What is $P\{X_1 > \sum_{i=2}^{n} X_i\}$?

92. Prove Equation (5.22).

93. Prove that

(a) $\max(X_1, X_2) = X_1 + X_2 - \min(X_1, X_2)$ and, in general,

(b) $$\max(X_1, \ldots, X_n) = \sum_1^n X_i - \sum\sum_{i<j} \min(X_i, X_j) + \sum\sum\sum_{i<j<k} \min(X_i, X_j, X_k) + \cdots + (-1)^{n-1} \min(X_i, X_j, \ldots, X_n)$$

Show by defining appropriate random variables $X_i, i = 1, \ldots, n$, and by taking expectations in part (b) how to obtain the well-known formula