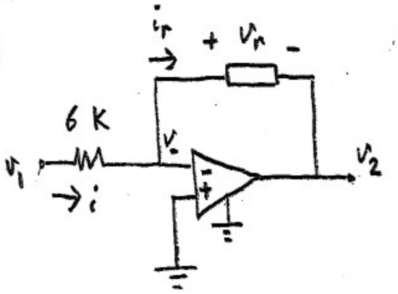
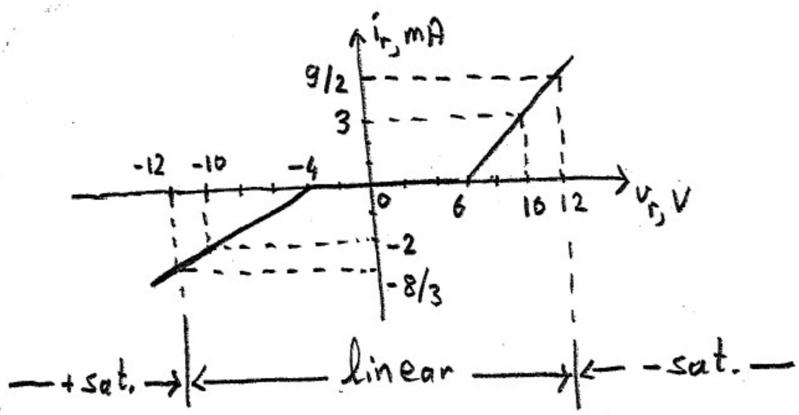
On the Third Midterm Examination (Fall 2012)

1)



$E_{sat} = 12$ V



The voltages are in volts; the currents are in milliamperes.

$v_+ = 0$, $i = i_r$, $i_r = f(v_r)$; $v_1 = 6i + v_-$, $v_r = v_- - v_2$

$$f(v_r):\quad i_r = \begin{cases} \frac{1}{3}(v_r+4), & v_r < -4 \\ 0, & -4 \le v_r \le 6 \\ \frac{3}{4}(v_r - 6), & v_r > 6 \end{cases}$$

linear region: $v_- = 0$, $|v_2| \le 12$ V

$v_1 = 6i$, $v_r = -v_2 \Rightarrow \boxed{i = v_1/6}$, $|v_r| \le 12$ V ; $\boxed{\frac{v_1}{6} = f(-v_2)}$

$v_r = 12$ V $\Rightarrow i_r = 9/2$ mA, $v_1 = 27$ V; $v_r = -12$ V $\Rightarrow i_r = -\frac{8}{3}$ mA, $v_1 = -16$ V

$-\frac{8}{3} \le i \le \frac{9}{2}$, $\boxed{-16 \le v_1 \le 27}$

+ saturation region: $v_- < 0$, $v_2 = 12$ V

$v_1 < 6i$, $v_r < -12$ V $\Rightarrow i_r < -\frac{8}{3}$ mA, $\boxed{v_1 < -16 \text{ V}}$

$$\boxed{\begin{array}{l} v_1 = 6i + v_r + 12 \\ i = f(v_r) \end{array}}$$

- saturation region: $v_- > 0$, $v_2 = -12$ V

$v_1 > 6i$, $v_r > 12$ V $\Rightarrow i_r > \frac{9}{2}$ mA, $\boxed{v_1 > 27 \text{ V}}$

$$\boxed{\begin{array}{l} v_1 = 6i + v_r - 12 \\ i = f(v_r) \end{array}}$$

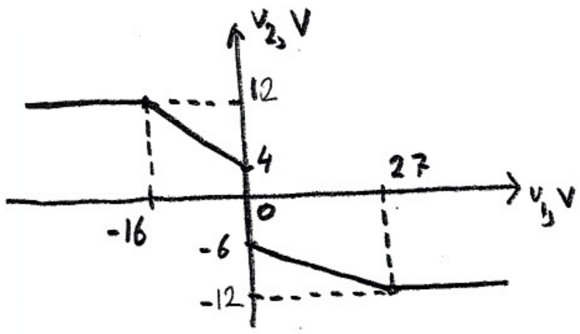
Transfer characteristic:

linear region:

$-12 \le v_r < -4 \; (4 < v_2 \le 12): \; \frac{v_1}{6} = \frac{1}{3}(-v_2+4) \Rightarrow v_2 = -\frac{1}{2}v_1 + 4 \quad (-16 \le v_1 < 0)$

$-4 \le v_r \le 6 \quad (-6 \le v_2 \le 4): \; \frac{v_1}{6} = 0 \Rightarrow v_1 = 0$

$6 < v_r \le 12 \quad (-12 \le v_2 < -6): \; \frac{v_1}{6} = \frac{3}{4}(-v_2-6) \Rightarrow v_2 = -\frac{2}{9}v_1 - 6 \quad (0 < v_1 \le 27)$



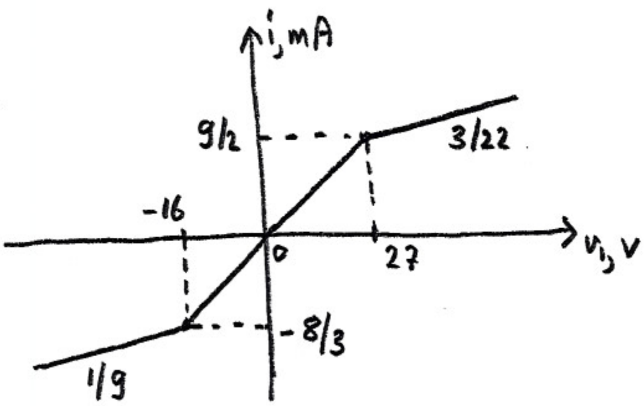
Input characteristic:

+ saturation region:

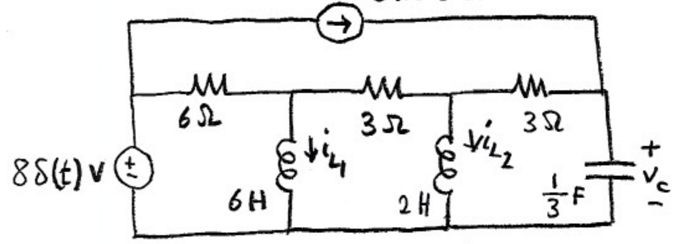
$$v_1 = 6i + (3i-4) + 12 = 9i + 8 \Rightarrow i = \frac{1}{9}v_1 - \frac{8}{9}$$

- saturation region:

$$v_1 = 6i + \left(\frac{4}{3}i + 6\right) - 12 = \frac{22}{3}i - 6 \Rightarrow i = \frac{3}{22}v_1 + \frac{9}{11}$$



2)



$v_c(0^-) = -4$ V

$i_{L_1}(0^-) = 1$ A

$i_{L_2}(0^-) = 3$ A

For a LTI capacitor: $v_c(t_1^+) = v_c(t_1^-) + \frac{1}{C}\int_{t_1^-}^{t_1^+} i(t)\,dt$

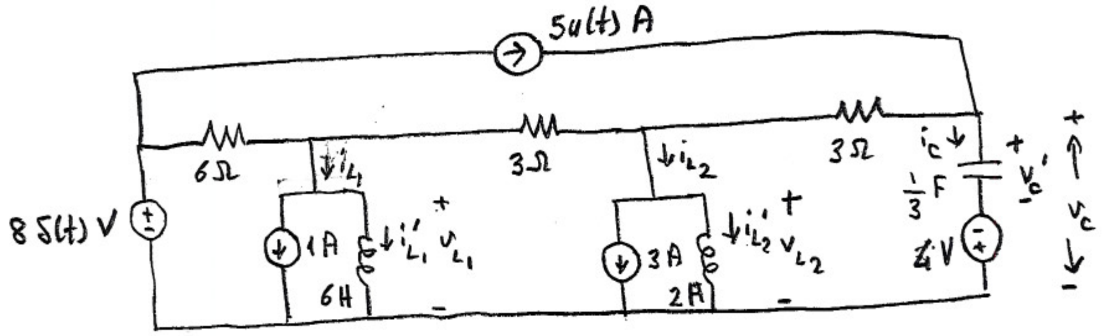
If $i(t)$ is bounded at $t=t_1$, i.e., either $i(t)$ is continuous at t_1 or has a jump discontinuity at t_1, then $\int_{t_1^-}^{t_1^+} i(t)\,dt = 0$ and $v_c(t_1^+) = v_c(t_1^-)$, i.e., $v_c(t)$ is continuous at t_1.

If $i(t)$ is impulsive at $t=t_1$: $i(t) = K\delta(t-t_1) + i'(t)$, where $i'(t)$ is bounded at t_1

$\int_{t_1^-}^{t_1^+} (K\delta(t-t_1) + i'(t))\,dt = K \Rightarrow v_c(t_1^+) = v_c(t_1^-) + \frac{K}{C}$, i.e., $v_c(t)$ has a jump discontinuity at t_1, the jump amount is K/C.

If the capacitor is full, then q_c is constant and $i = \frac{dq_c}{dt} = 0$, i.e., the capacitor behaves like an open circuit.

For a LTI inductor: The dual case.



$v_c'(0^-) = 0$

$i'_{L_1}(0^-) = 0$

$i'_{L_2}(0^-) = 0$

Let $x(t)$ denote any current or voltage. Write

$x(t) = x_1(t) + x_2(t) + x_3(t)$

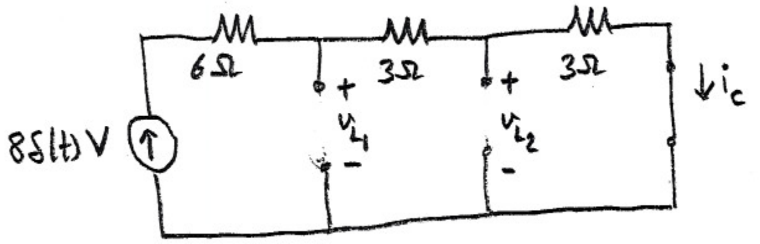
- $x_3(t)$: due to the impulse source
- $x_2(t)$: due to the step source

} the zero-state solution

- $x_1(t)$: due to the constant sources (initial conditions) } the zero-input solution

$x_1(t)$ and $x_2(t)$ are bounded.

For $x_3(t)$, set the initial conditions to zero (kill the constant sources) and kill the step source. Replace the inductors with open circuits (zero initial currents) and the capacitor with a short circuit (zero initial voltage).



$$i_c = \frac{8\delta(t)}{6+3+3} = \frac{2}{3}\delta(t) \text{ A}$$

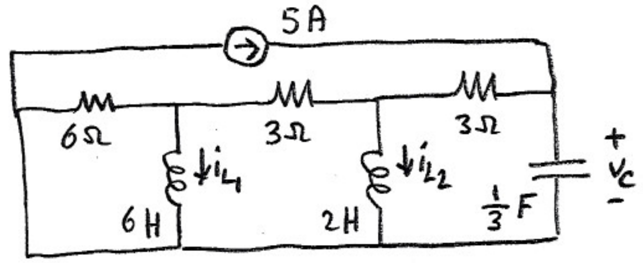
$$v_{L_2} = 3 i_c = 2\delta(t) \text{ V}$$

$$v_{L_1} = (3+3) i_c = 4\delta(t) \text{ V}$$

Hence $v_{c_3}(0^+) = \frac{2/3}{1/3} = 2\text{ V}$, $i_{L_{1_3}}(0^+) = \frac{4}{6} = \frac{2}{3}\text{ A}$, $i_{L_{2_3}}(0^+) = \frac{2}{2} = 1\text{ A}$.

Then $v_c(0^+) = -4+2 = -2\text{V}$, $i_{L_1}(0^+) = 1+\frac{2}{3} = \frac{5}{3}\text{ A}$, $i_{L_2}(0^+) = 3+1 = 4\text{ A}$

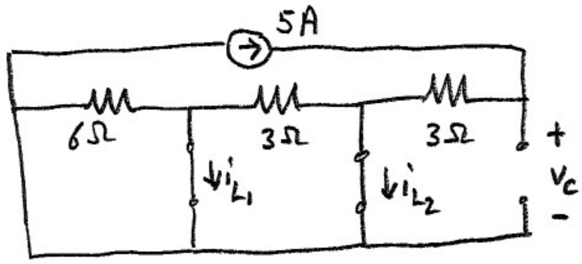
For $t>0$:



The circuit is passive. The input is constant.

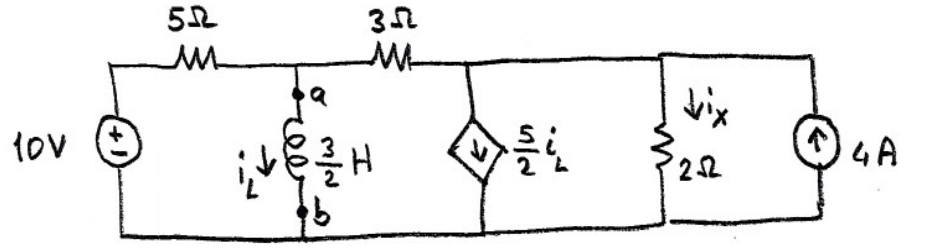
At $t=\infty$, the capacitor and the inductors are full. Then

$t=\infty$:



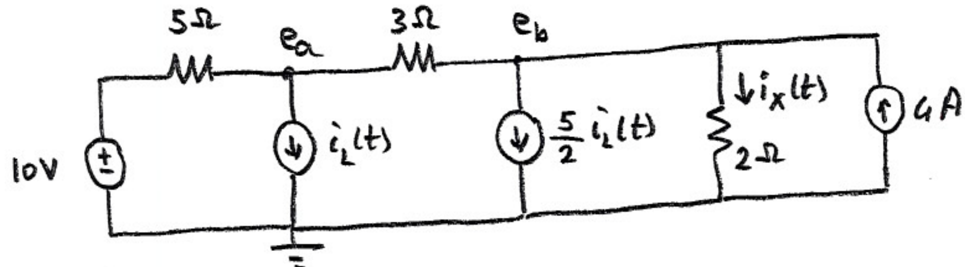
$i_{L_2} = 5\text{ A}$, $i_{L_1} = 0$, $v_c = 3 \cdot i_{L_2} = 15\text{ V}$

3)



$i_L(0)=3$ A

Suppose that $i_L(t)$ is known. Replacing the inductor with an independent current source of current $i_L(t)$,



Solving this circuit, $i_x(t)$ is determined.

Method 1 : Node analysis

$$\left.\begin{aligned} &\frac{e_a-10}{5}+i_L+\frac{e_a-e_b}{3}=0 \\ &\frac{e_b-e_a}{3}+\frac{5}{2}i_L+\frac{e_b}{2}-4=0 \end{aligned}\right\} \begin{bmatrix} 8/15 & -1/3 \\ -1/3 & 5/6 \end{bmatrix}\begin{bmatrix} e_a \\ e_b \end{bmatrix}=\begin{bmatrix} 2-i_L \\ 4-\frac{5}{2}i_L \end{bmatrix}$$

$$\begin{bmatrix} e_a \\ e_b \end{bmatrix}=\frac{1}{1/3}\begin{bmatrix} 5/6 & 1/3 \\ 1/3 & 8/15 \end{bmatrix}\begin{bmatrix} 2-i_L \\ 4-\frac{5}{2}i_L \end{bmatrix} \Rightarrow e_b=2-i_L+\frac{8}{5}\left(4-\frac{5}{2}i_L\right)=\frac{42}{5}-5i_L$$

$$i_x=e_b/2=\frac{21}{5}-\frac{5}{2}i_L$$

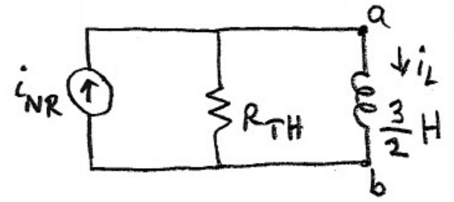
Method 2 : Superposition

$$i_x=\frac{10}{5+3+2}-\frac{5}{5+3+2}i_L+\frac{5+3}{5+3+2}\left(4-\frac{5}{2}i_L\right)$$

$$=1-\frac{1}{2}i_L+\frac{4}{5}\left(4-\frac{5}{2}i_L\right)=\frac{21}{5}-\frac{5}{2}i_L$$

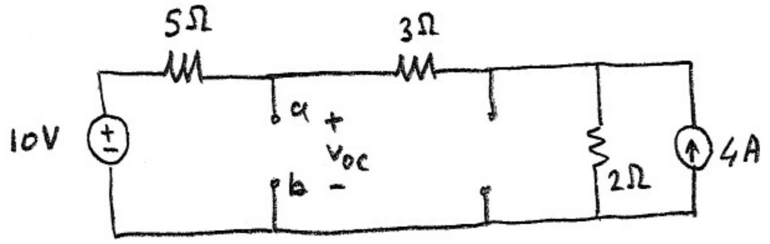
Determination of $i_L(t)$:

Obtain the Norton (or the Thevenin) equivalent of the one-port as seen by the inductor.



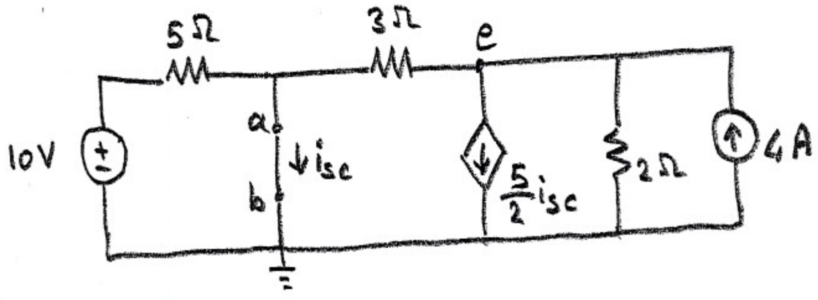
$i_L(0) = 3$ A

v_{TH} (or v_{oc}):



$$v_{oc} = (3+2)\,\frac{10}{5+3+2} + 5\cdot\frac{2}{5+3+2}\,4 = 5+4 = 9\ V$$

i_{NR} (or i_{sc}):



$$i_{sc} = \frac{10}{5} + \frac{e}{3} = \frac{1}{3}e + 2$$

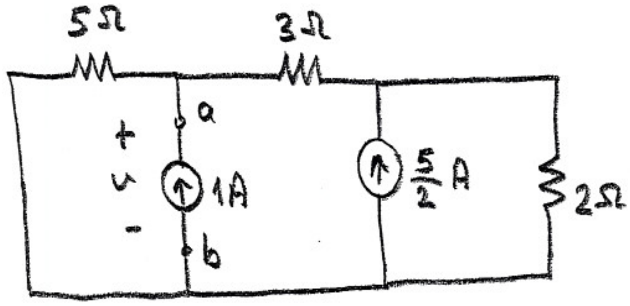
$$\frac{e}{3} + \frac{5}{2}i_{sc} + \frac{e}{2} - 4 = 0 \Rightarrow \left(\frac{1}{3} + \frac{1}{2} + \frac{5}{2}\,\frac{1}{3}\right)e = 4 - \frac{5}{2}\,2$$

$$\frac{5}{3}e = -1 \Rightarrow e = -\frac{3}{5}\ V$$

$$i_{sc} = -\frac{1}{5} + 2 = \frac{9}{5}\ A$$

$$R_{TH} = \frac{v_{oc}}{i_{sc}} = \frac{9}{9/5} = 5\ \Omega$$

Note



$$R_{TH} = \frac{v}{1}$$

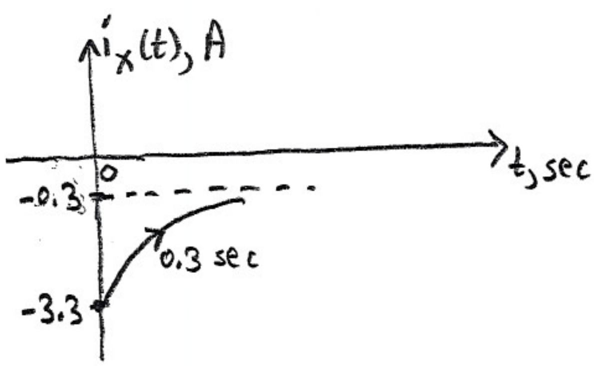
$$v = 5\,\frac{3+2}{3+2+5}\cdot 1 + 5\,\frac{2}{2+3+5}\,\frac{5}{2} = \frac{5}{2} + \frac{5}{2} = 5\ \text{V} \Rightarrow R_{TH} = 5\,\Omega.$$

$$i_L(+\infty) = \frac{9}{5}\ \text{A},\quad \tau = \frac{3/2}{R_{TH}} = \frac{3}{10}\ \text{sec}$$

$$i_L(t) = \frac{9}{5} + \underbrace{\left(3 - \frac{9}{5}\right)}_{6/5} e^{-10t/3}\ \text{A},\ t \geqslant 0$$

$$i_x(t) = \frac{21}{5} - \frac{5}{2}\left(\frac{9}{5} + \frac{6}{5}e^{-10t/3}\right) = \frac{21}{5} - \frac{9}{2} - 3e^{-10t/3}$$

$$= -\frac{3}{10} - 3e^{-10t/3}\ \text{A},\ t \geqslant 0$$



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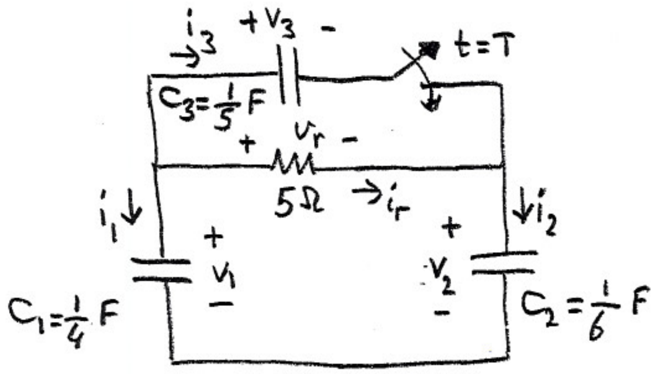


$v_1(0)=-3\text{ V}$

$v_2(0)=7\text{ V}$

$v_3(0)=3.5\text{ V}$

$v_1(T^-)=0$

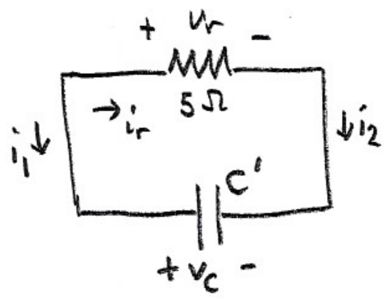
Study the circuit in three parts: $0\le t<T$, $t: T^- \to T^+$, $t>T$.

<u>$0\le t<T$</u>

$i_3=0$ (the switch is open), $v_3=3.5\text{ V}$

C_1 and C_2 are in series; the equivalent capacitance C' is

$$\frac{1}{C'}=\frac{1}{1/4}+\frac{1}{1/6}=4+6=10 \Rightarrow C'=\frac{1}{10}F$$



$v_c=v_1-v_2$

$v_c(0)=-3-7=-10\text{ V}$

$\tau_1=5\times C'=\frac{1}{2}\text{ sec}$

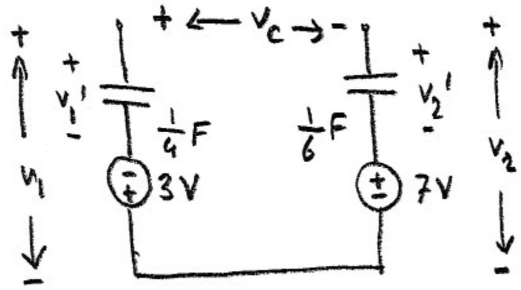
$v_c(t)=-10e^{-2t}\text{ V}$

Method 1: $i_r=v_c/5=-2e^{-2t}\text{A}$, $i_1=-i_r$, $i_2=i_r$

$$v_1(t)=-3+\frac{1}{1/4}\int_0^t(2e^{-2t'})dt'=-3+4(-e^{-2t'})\Big|_0^t=-3+4(1-e^{-2t})=1-4e^{-2t}\text{ V}$$

$$v_2(t)=7+\frac{1}{1/6}\int_0^t(-2e^{-2t'})dt'=7+6(e^{-2t'})\Big|_0^t=7+6(e^{-2t}-1)=1+6e^{-2t}\text{ V}$$

Method 2:



$v_1'(0)=0, \quad v_2'(0)=0$

$$v_1' = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{4}}(v_c+(3+7)) = \frac{2}{5}(v_c+10) = \frac{2}{5}v_c+4$$

$$v_2' = -\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{6}}(v_c+(3+7)) = -\frac{3}{5}(v_c+10) = -\frac{3}{5}v_c-6$$

$$v_1(t) = v_1'(t)-3 = \frac{2}{5}v_c(t)+1 = 1-4e^{-2t}\ \text{V}$$

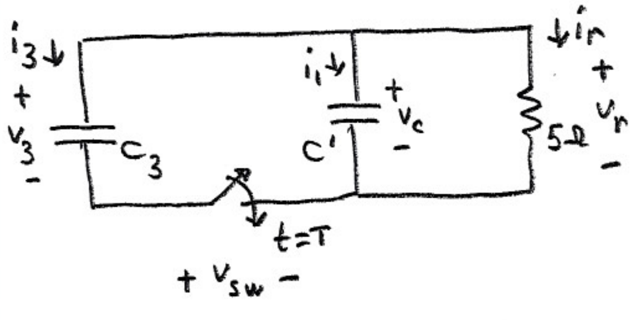
$$v_2(t) = v_2'(t)+7 = -\frac{3}{5}v_c+1 = 1+6e^{-2t}\ \text{V}$$

Note: $v_2 = v_1 - v_c = (1-4e^{-2t})-(-10e^{-2t})$ V

$v_1(T^-)=0: \quad 1-4e^{-2T}=0 \Rightarrow e^{-2T}=\frac{1}{4} \Rightarrow e^{2T}=4 \Rightarrow T=\frac{1}{2}\ln 4 = \ln 2$ sec

$v_2(T^-) = 1+6e^{-2T} = 1+\frac{6}{4} = 2.5$ V

$t: T^- \to T^+$



$v_3(T^-) = 3.5\ \text{V}, \quad v_c(T^-) = 0-2.5 = -2.5\ \text{V}$

$v_3(T^+) = v_c(T^+) = V_0$

$v_3(T^-) \neq v_c(T^-) \Rightarrow i_3 = -i_1$ is impulsive at $t=T$

$$i_3(t) = K\,\delta(t-T) + \underbrace{i_3'(t)}_{\text{bounded at } t=T}$$

$$V_0 = \frac{C_3 v_3(T^-) + C' v_c(T^-)}{C_3 + C'} = \frac{\frac{1}{5}3.5 + \frac{1}{10}(-2.5)}{\frac{1}{5}+\frac{1}{10}} = 1.5\ V$$

$$v_3(T^+) = 3.5 + \frac{1}{1/5}\int_{T^-}^{T^+} i_3(t)\,dt = 3.5 + 5K = V_0 = 1.5 \Rightarrow K = -\frac{2}{5}$$

Note: $v_c(T^+) = -2.5 + \frac{1}{1/10}\int_{T^-}^{T^+}(-i_3(t))\,dt = -2.5 - 10K$

$v_3(T^+) = v_c(T^+) \Rightarrow 3.5 + 5K = -2.5 - 10K \Rightarrow 15K = -6 \Rightarrow K = -2/5$

$v_3(T^+) = v_c(T^+) = 3.5 + 5(-2/5) = 1.5\ V$

$$v_1(T^+) = 0 + \frac{1}{1/4}\int_{T^-}^{T^+}(-i_3(t))\,dt = -4K = 8/5\ V$$

$$v_2(T^+) = 2.5 + \frac{1}{1/6}\int_{T^-}^{T^+} i_3(t)\,dt = 2.5 + 6K = 1/10\ V$$

$$\{ v_2(T^+) = v_1(T^+) - V_0 = 1.6 - 1.5 = 0.1\ V \}$$

$t > T$

C_3 and C' are in parallel; the equivalent capacitance C'' is

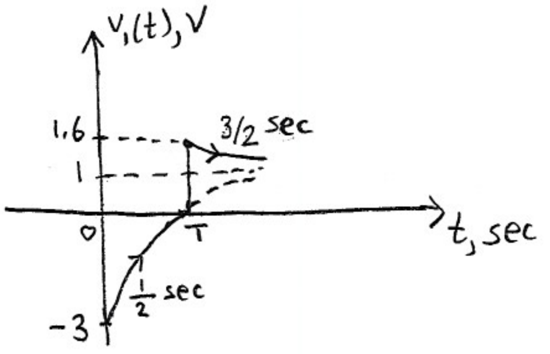
$$C'' = C_3 + C' = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}\ F$$

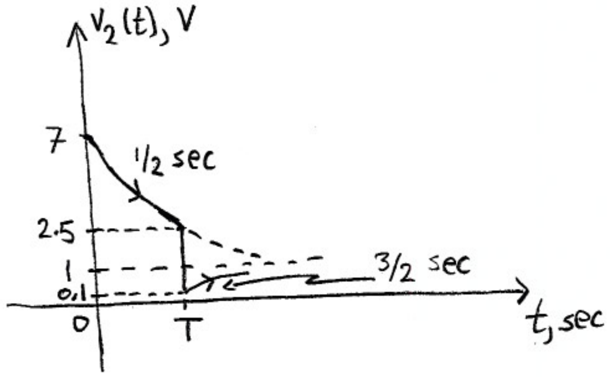
$$\tau_2 = 5 \cdot C'' = 3/2\ \text{sec}$$

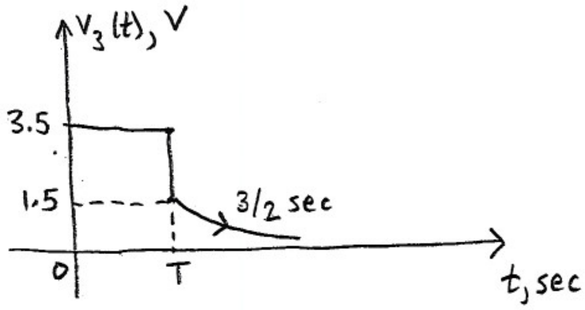
$$v_c(t) = v_3(t) = 1.5\, e^{-\frac{2}{3}(t-T)}\ V$$

$$v_1(t) = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{4}}(v_c - 1.5) + 1.6 = \frac{2}{5}(v_c - 1.5) + 1.6 = \frac{2}{5}v_c + 1 = 1 + 0.6\, e^{-\frac{2}{3}(t-T)}\ V$$

$$v_2(t) = -\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{6}}(v_c - 1.5) + 0.1 = -\frac{3}{5}(v_c - 1.5) + 0.1 = -\frac{3}{5}v_c + 1 = 1 - 0.9\, e^{-\frac{2}{3}(t-T)}\ V$$







Energy Considerations:

The energy delivered to the resistor on $[0, T)$:

$$W_r[0,T) = \int_0^T 5\, i_r^2(t)\, dt = 20 \int_0^T e^{-4t}\, dt = -5\, e^{-4t}\Big|_0^T = 5\,(1 - \underbrace{e^{-4T}}_{(1/4)^2 = \frac{1}{16}}) = \frac{75}{16}\ \text{J}$$

The energy delivered to the resistor on $(T, +\infty)$:

$$W_r(T,+\infty) = \int_T^{+\infty} \frac{1}{5}\, v_c^2(t)\, dt = \frac{9}{20} \int_T^{+\infty} e^{-\frac{4}{3}(t-T)}\, dt = \frac{9}{20} \int_0^{+\infty} e^{-\frac{4}{3}t'}\, dt'$$

$$(t' \triangleq t - T)$$

$$= -\frac{27}{80}\, e^{-\frac{4}{3}t'}\Big|_0^{+\infty} = \frac{27}{80}\ \text{J}$$

The sum of the stored energies in C_1 and C_2

at $t=0$: $e_a(0) = \frac{1}{2}\frac{1}{4}(-3)^2 + \frac{1}{2}\frac{1}{6}(7^2) = \frac{9}{8} + \frac{49}{12} = \frac{125}{24}$ J

at $t=T^-$: $e_a(T^-) = \frac{1}{2}\frac{1}{4}(0)^2 + \frac{1}{2}\frac{1}{6}(2.5)^2 = \frac{25}{48}$ J

The sum of the stored energies in C_1, C_2 and C_3

at $t = T^-$: $e_b(T^-) = \frac{25}{48} + \frac{1}{2}\frac{1}{5}(3.5)^2 = \frac{419}{240}$ J

at $t = T^+$: $e_b(T^+) = \frac{1}{2}\frac{1}{4}(1.6)^2 + \frac{1}{2}\frac{1}{6}(0.1)^2 + \frac{1}{2}\frac{1}{5}(1.5)^2 = \frac{131}{240}$ J

at $t = \infty$: $e_b(\infty) = \frac{1}{2}\frac{1}{4}(1)^2 + \frac{1}{2}\frac{1}{6}(1)^2 + \frac{1}{2}\frac{1}{5}(0)^2 = \frac{5}{24}$ J

$e_a(0) - e_a(T^-) = \frac{125}{24} - \frac{25}{48} = \frac{75}{16}$ J $\leftarrow W_r\,[0, T)$

$e_b(T^+) - e_b(\infty) = \frac{131}{240} - \frac{5}{24} = \frac{27}{80}$ J $\leftarrow W_r\,(T, +\infty)$

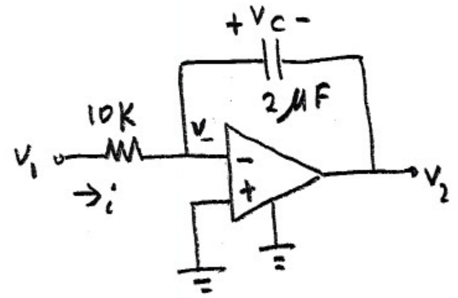
The energy given to the switch:

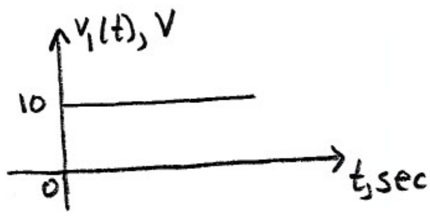
$v_{sw}(T^-) = v_c(T^-) - v_3(T^-) = (0 - 2.5) - 3.5 = -6$ V, $\quad v_{sw}(T^+) = 0$

$$W_{sw} = \int_{T^-}^{T^+} \frac{(-6)+0}{2}\, i_3(t)\,dt = -3K = 1.2 \text{ J}$$

$e_b(T^-) - e_b(T^+) = \frac{419}{240} - \frac{131}{240} = 1.2$ J $\leftarrow W_{sw}$

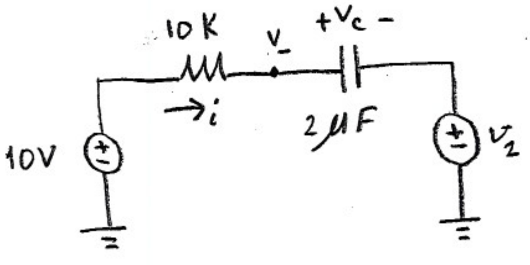
5)





$E_{sat} = 15V$, $v_c(0^-) = -20$ V

t>0:



Preliminaries:

linear region : $v_- = 0$, $|v_2| \leq 15$ V

$i = \frac{10}{10^4} = 10^{-3}$ A, $v_c(t) = v_c(t_0) + \frac{1}{2\times10^{-6}} \int_{t_0}^{t} 10^{-3} dt' = v_c(t_0) + 500(t - t_0)$

+ saturation region : $v_- < 0$, $v_2 = 15$ V

$\tau = 10^4 \times 2\times10^{-6} = 0.02$ sec $\equiv 20$ msec

$v_c(t) = -5 + (v_c(t_1) + 5)e^{-(t-t_1)/\tau}$, $v_-(t) = 10 + (v_c(t_1) + 5)e^{-(t-t_1)/\tau}$

- saturation region : $v_- > 0$, $v_2 = -15$ V

$v_c(t) = 25 + (v_c(t_2) - 25)e^{-(t-t_2)/\tau}$, $v_-(t) = 10 + (v_c(t_2) - 25)e^{-(t-t_1)/\tau}$

$v_c(0^+) = -20$ V $\Rightarrow$ $v_-(0^+) = -20 + v_2$

linear? $v_-(0^+) = -20 + v_2 = 0$ X

+ sat ? $v_-(0^+) = -20 + 15 = -5 < 0$ ✓

- sat ? $v_-(0^+) = -20 - 15 = -35 > 0$ X

At $t = 0^+$, the op-amp is in the + saturation region.

$0 \leq t \leq T_1$ (+ sat)

$v_c(t) = -5 + (-20+5)e^{-t/\tau} = -5 - 15e^{-50t}$ V

$v_-(t) = 10 - 15e^{-50t}$ V; $v_-(T_1) = 0 \Rightarrow 10 - 15e^{-50T_1} \Rightarrow e^{50T_1} = 1.5 \Rightarrow T_1 = \frac{1}{50}\ln(1.5)$ sec

$v_c(T_1) = -15$ V

$T_1 \leq t \leq T_2$ (linear)

$v_c(t) = -15 + 500(t - T_1)$ V

$v_2(t) = -v_c(t)$, $v_c(T_2) = -15 = 15 - 500(T_2 - T_1) \Rightarrow T_2 = T_1 + \frac{3}{50}$ sec

$v_c(T_2) = 15$ V

$t > T_2$ (- sat)

$v_c(t) = 25 + (15-25)e^{-(t-T_2)/\tau} = 25 - 10e^{-50(t-T_2)}$ V

$v_-(t) = 10 - 10e^{-50(t-T_2)}$ V