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EE 201 Circuit Theory I

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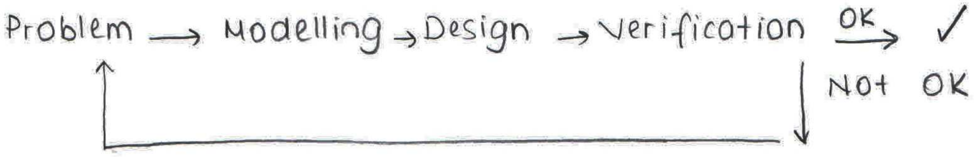


Introduce :

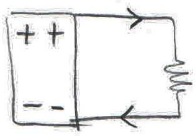
① Models of electric components
② Relations for physical principles
③ Analysis Methods
④ Design principles / examples

charge , current , voltage, power, energy, flux
q : coulombs (c)

① ② charge / sec : Charge transferred per second is current

$$i(t) = \frac{dQ(t)}{dt} \cong \frac{Q(t+\Delta) - Q(t)}{\Delta} = \frac{\text{Coulomb}}{\text{seconds}}$$

Current direction shows the direction of moving (+) charges (sign and direction)



Units : Voltage

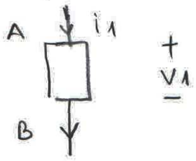
$V_A = V_B + 5$ $V_A = V_B - (-5)$

energy : Joule's
$W = q \cdot \Delta V$

Power : Rate of energy changes (in watts)

$$P(t) = \frac{dW(t)}{dt}$$

Components have mathematical models

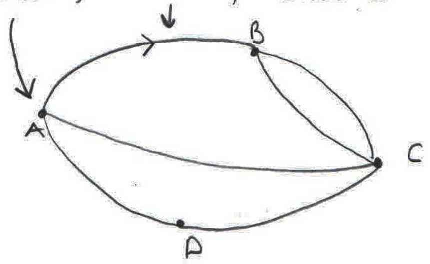


$v_1(t) = R_1 \cdot i_1(t)$ or $i_1(t) = C \dfrac{dv_1(t)}{dt}$ ...

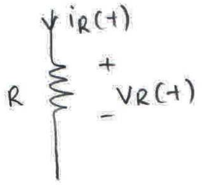
Electrical components

Resistance :

Node , Branch , Network

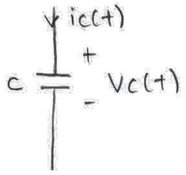


Resistance
R(Ω)



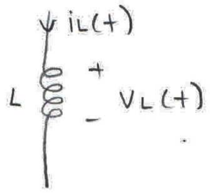
$v_R(t) = R \cdot i_R(t)$

Capacitor
C(F)



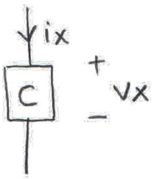
$i_C(t) = C \dfrac{dv_C(t)}{dt}$

Inductor
L(H)

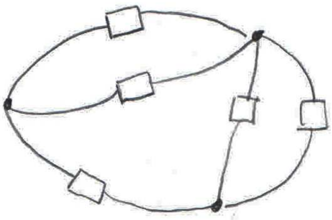


$v_L(t) = L \dfrac{di_L(t)}{dt}$

Passive sign convention :



Lumped components , Lumped circuits



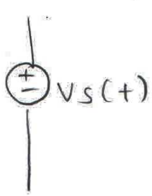
High frequency circuits, wires can act as inductors

$\lambda = c/f$ (Circuit sizes) much smaller (λ)

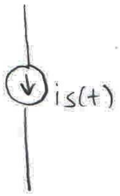
↑

wavelength

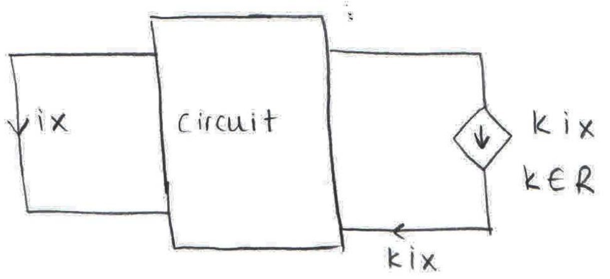
More components ;



Independent voltage source

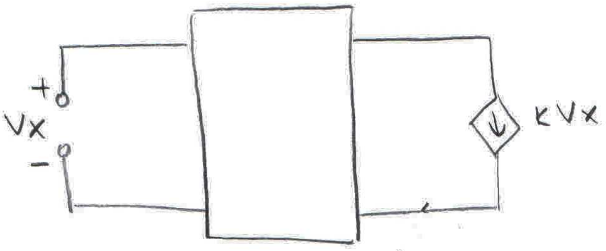


Independent current source

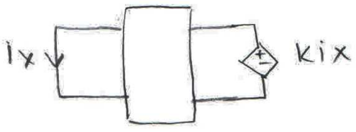


Dependent current source
current controlled current source

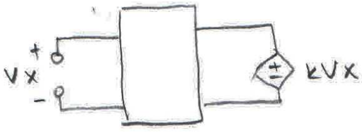
if ix= 1A   kix = k·1= k A



Dependent current source
voltage controlled current source



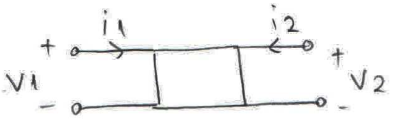
CCVS



VCVS

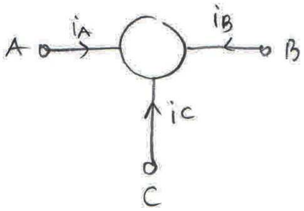
Terminals, Branches

= 1 branch
2 terminals → 1 port circuits



= 2 branches
4 terminals → 2 port circuits

multiport circuits

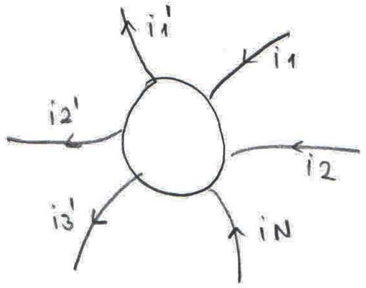


= 3 branches
3 terminals → 3 port circuits

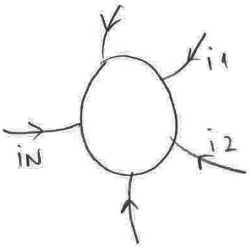
Kirchoff's Laws

1-) Kirchoff's Current Law (KCL)

principle : conservation of charge



$$\underbrace{\sum_{k=1}^{N} i_k}_{\text{Entering charge per sec}} = \underbrace{\sum_{l=1}^{L} i_l'}_{\text{Leaving charge per sec}}$$

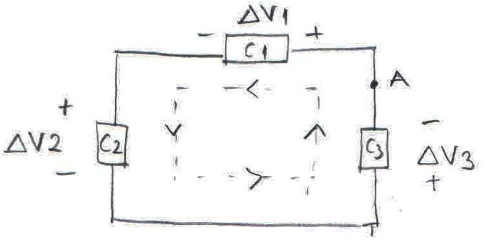


$\sum_{k=1}^{N} i_k = 0$ All entering currents sum to zero

Similarly summation of all leaving currents = 0

2-) Kirchoff's voltage Law (KVL)

principle: conservation of energy

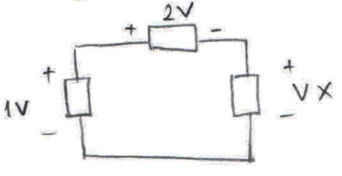


$W_{A \to A} = q\,\Delta V_1 + q\,\Delta V_2 + q\,\Delta V_3 = 0$

$\sum_{k=1}^{3} \Delta V_k = 0$ ↳ number of the branches in the loop

k: branch index in the loop

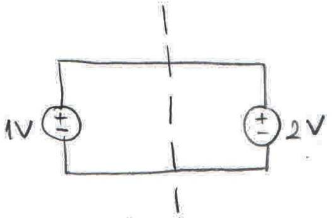
* voltage drops across a loop is equal to zero



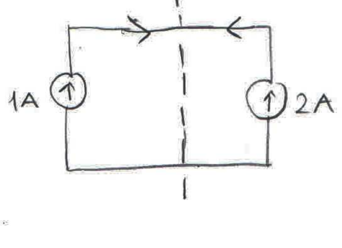
$V_x = ?$

$+2 - 1 + V_x = 0$ then $V_x = -1$ V

*



Idealized circuit Models do not cover this



Idealized models do not apply

Passive and Active components

- Active components (voltage sources, current sources)
  components procedure net energy
- Passive components do not procedure net energy (resistor)

$V(t) = R \cdot i(t)$

$P(t) = V(t)\, i(t)$

$w(t) = \int_{-\infty}^{t} P(\tau)\, d\tau$

$\Big\} P(t) = (i(t))^2 R$ for resistor

So $p(t) > 0 \quad \forall t \quad$ (if $R>0$)

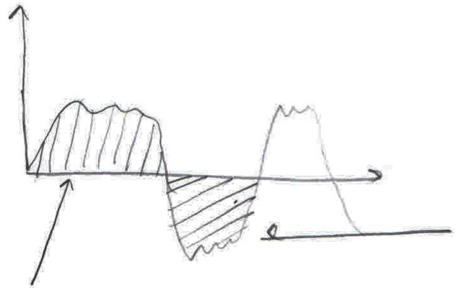
$$w(t) = -\underbrace{\int_{-\infty}^{t} P_R(\tau)\, d\tau}_{\text{positive function}} > 0 \quad (R>0)$$

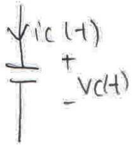
$\longrightarrow$ $W_R(t)$: Energy absorbed by a resistor is always positive

Then $W_R(t) > 0 \quad \forall t \leftarrow$ if component is passive

Capacitors have $P(t)$ which can be positive or negative

$P_c(t) = V_c(t)\, i_c(t)$





$P(t) < 0$ Capacity delivery energy outside the worl

$P(t) > 0$

Capacitor is absorbing power like a resistor

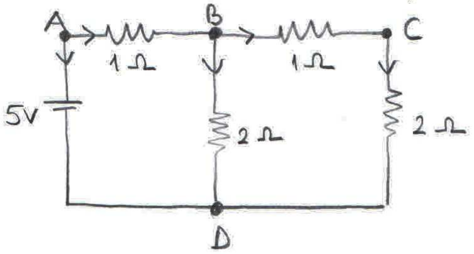
$$W_c(t) = \int_{-\infty}^{t} P_c(\tau)\, d\tau = \int_{-\infty}^{t} C\, \frac{dV_c(\tau)}{d\tau} \cdot Q_c(\tau)\, d\tau = \int_{V=V_c(-\infty)}^{V=V_c(t)} C \cdot u \cdot du$$

$$= C\,\frac{u^2}{2}\Bigg|_{u=V_c(-\infty)}^{u=V_c(t)} = \frac{1}{2}\, C V_c^2(t) > 0$$

$\longrightarrow$ Therefore, capacitor is passive component

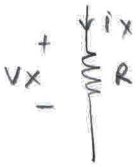
(by taking $V_c(t) = 0$ at infinity)

Graph Theoratical Methods for Circuit Analysis



Conditions;

1- All Ohm's law equations, that is terminal equations for resistor, to be satisfied



$V_x = i_x \cdot R$

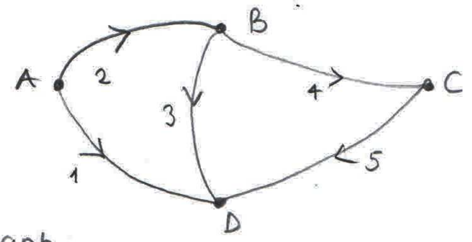
4 equations for resistors

1 equation for voltage source

2- KCL (SO many Of them =nodes)

3- KVL (LOOPS → 3)

Graph ;



Mesh

Loop

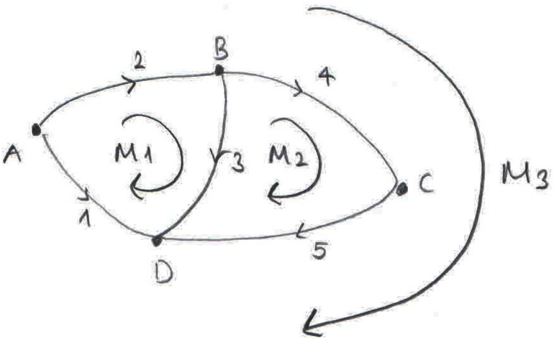
Directed graph

Incidence Matrix:

$$A_a = \begin{array}{c} A\to \\ B\to \\ C\to \\ D\to \\ \text{branches} = \end{array} \begin{array}{c} \left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & -1 \end{array}\right] \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

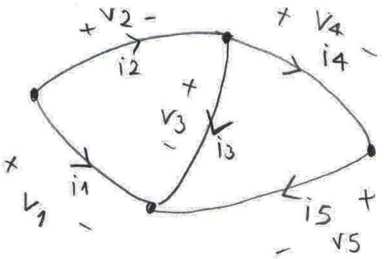
Mesh Matrix

$$M_a = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline M_1 & -1 & 1 & 1 & 0 & 0 \\ M_2 & 0 & 0 & -1 & 1 & 1 \\ M_3 & -1 & 1 & 0 & 1 & 1 \end{array}$$



Mesh Equation in Matrix form

Let $V_1, V_2 \ldots V_5$ be branch voltages



To satisfy KVL;

$$M_a \times \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = 0$$

Then $\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = 0$, $M = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$

Reduced mesh matrix
(Only write equations for inner meshes)

Steps for mesh analysis;

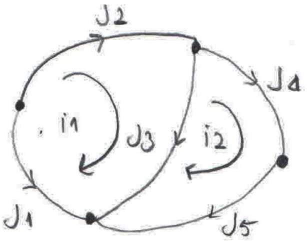
1- Write reduced mesh matrix and form KVL constrants .

$M \cdot V = 0$ $\qquad V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$ $\qquad J = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix}$

branch voltage vector $\qquad$ branch current vector

2- Introduce mesh currents ($i_1, i_2, \ldots$) and express branch currents via mesh currents

$J = M^T \cdot i$



$J_1 = -i_1$

$J_2 = i_1$

$J_3 = i_1 - i_2$

$J_4 = i_2$

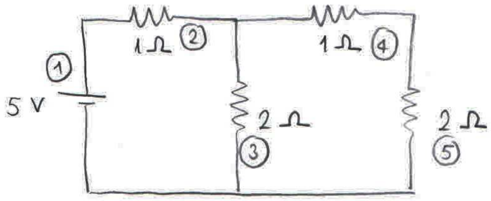
$J_5 = i_2$

$$J = M^T \cdot i$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

3- Write terminal equations

$\underline{V} = R \cdot \underline{J} + \underline{V_c}$



$V_1 = 5$ V

$V_2 = J_2 \cdot 1$

$V_3 = J_3 \cdot 2$

$V_4 = J_4 \cdot 1$

$V_5 = J_5 \cdot 2$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Resistance matrix $\qquad$ voltage source vector

1 - $\underline{M}\,\underline{V} = 0$

2 - $\underline{J} = \underline{M}^T \cdot \underline{i}$

3 - $\underline{V} = \underline{R}\,\underline{J} + \underline{V_s}$

In mesh analysis, mesh currents (i) are unknowns

combine 2 and 3 and get

$$\underline{V} = \underline{\underline{R}}\,\underline{\underline{M}}^T \underline{i} + \underline{V_s}$$

Then multiply from left by M

$$0 = \underline{\underline{M}}\,\underline{V} = (\underline{\underline{M}}\,\underline{\underline{R}}\,\underline{\underline{M}}^T)\,\underline{i} + \underline{\underline{M}}\,\underline{V_s} \quad ; \quad (\underline{\underline{M}}\,\underline{\underline{R}}\,\underline{\underline{M}}^T)\underline{i} = -\underline{\underline{M}}\,\underline{V_s}$$

$$\underline{\underline{M}}\,\underline{\underline{R}}\,\underline{\underline{M}}^T = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & -2 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \underline{V_s} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \qquad (MRM^T)\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = -MVs \longleftrightarrow \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 25/11 \\ 10/11 \end{bmatrix}$$

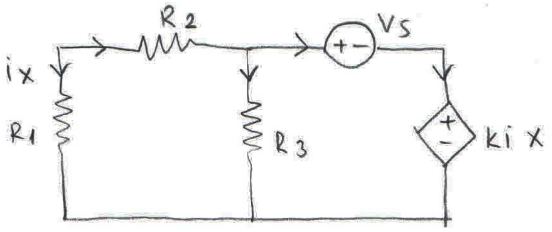
↓

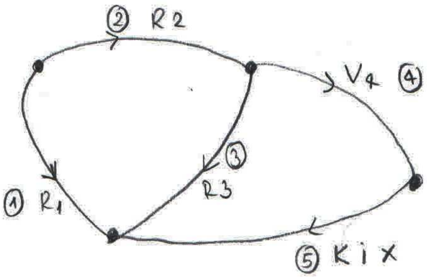
mesh currents (unknowns)

$\underline{J} = \underline{\underline{M}}^T \underline{i}$ branch currents

$\underline{V} = \underline{\underline{R}}\,\underline{J} + \underline{V_s}$ branch voltages

Ex:





$$1-\ M = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

2- $\underline{J} = \underline{\underline{M}}^T \underline{i}$ mesh current unknowns

3- $V_1 = R_1 J_1$ $\quad V_5 = kix = kJ_1$

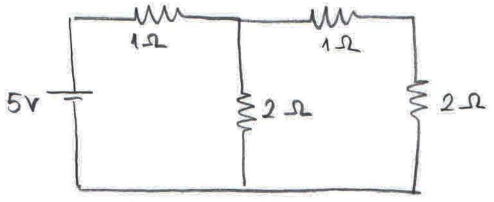
$V_2 = R_2 J_2$

$V_3 = R_3 J_3$

$V_4 = V_s$

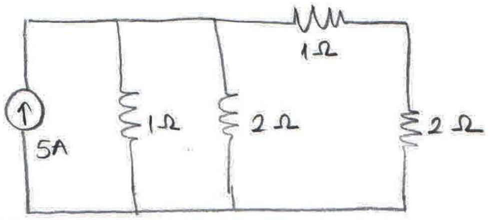
$$V = \underbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{\underline{R}}} \underbrace{\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix}}_{\underline{J}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \end{bmatrix}}_{\underline{V_s}}$$

Node Analysis



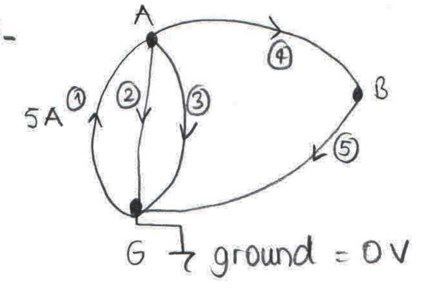
(same example used for mesh analysis)

$\longrightarrow$ source transformation



Solve this using node analysis

1-



2- Write KCL's for every node

$$A_a \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \underline{0} + \underbrace{\begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & -1 \end{bmatrix}}_{\text{incidence matrix}} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{J} = 0 \longrightarrow \begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Reduced incidence matrix

3- Introduce Node voltages (e_k's)

Assume G is at 0V level.

G is ground assign e_A, e_B volts wrt ground to each node

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \underline{V} = \underline{\underline{A}}^T \underline{e} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_A \\ e_B \end{bmatrix} = \begin{bmatrix} -e_A \\ e_A \\ e_A \\ e_A - e_B \\ e_B \end{bmatrix}$$

4- $\underline{J} = \underline{\underline{G}}\,\underline{V} + \underline{J_s}$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/1\Omega & 0 & 0 & 0 \\ 0 & 0 & 1/2\Omega & 0 & 0 \\ 0 & 0 & 0 & 1/1\Omega & 0 \\ 0 & 0 & 0 & 0 & 1/2\Omega \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\underline{A}}\,\underline{J} = 0$ (KCL) $\longrightarrow$ $\underline{V} = \underline{\underline{A}}^T \underline{e}$ $\longrightarrow$ $\underline{J} = \underline{\underline{G}}\,\underline{V} + \underline{J_s}$

↳ e: node voltages

$\underline{\underline{A}}\,\underline{J} = 0 = \underline{\underline{A}}\,\underline{\underline{G}}\,\underline{V} + \underline{\underline{A}}\,\underline{J_S}$ $\qquad \underline{V} = \underline{\underline{A}}^T \underline{e}$

$\boxed{(\underline{\underline{A}}\,\underline{\underline{G}}\,\underline{\underline{A}}^T)\,\underline{e} = -\underline{\underline{A}}\,\underline{J_S}}$ we get node voltages

Duality : (Dual components, dual variables, dual circuits)

Dual variables $(\hat{\ })$

$v \longrightarrow \hat{i}$

$i \longrightarrow \hat{v}$

$q \longrightarrow \hat{\phi} \rightarrow$ flux

$\phi \longrightarrow \hat{q}$

Dual components :

Resistor $(R\ \Omega)$ $\qquad$ $\hat{\text{Resistor}}$ $(R\ \mho\ ,\ \frac{1}{R}\ \Omega)$ ↳ siemens, mho

i.e. $\quad 2\ \Omega \longrightarrow 1/2\ \Omega$

$v = R\,i \longrightarrow \hat{i} = \hat{v}\,\hat{R}$

Capacitor $\longrightarrow$ Inductors

(C Farad) $\qquad$ (C Henry)

Inductor $\qquad$ Capacitor

(L Henry) $\qquad$ (L Farads)

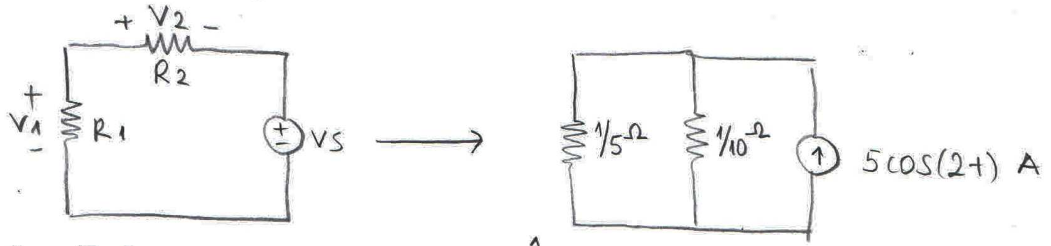
Other Dual Variables

node $\longrightarrow$ mesh.

outer mesh $\longrightarrow$ reference node / ground node

cutset $\longrightarrow$ loop

Dual Circuits :

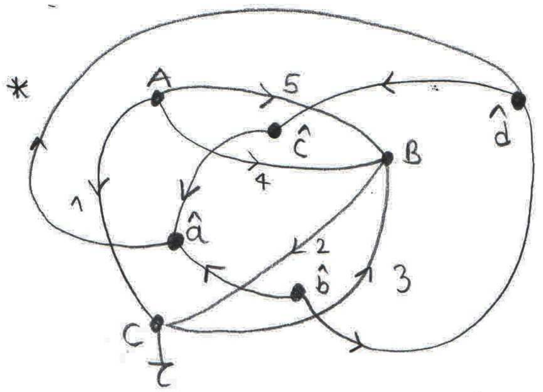


$R_1 = 5\ \Omega$ $\qquad R_1 \longrightarrow \hat{R}_1$

$R_2 = 10\ \Omega$ $\qquad R_2 \longrightarrow \hat{R}_2$

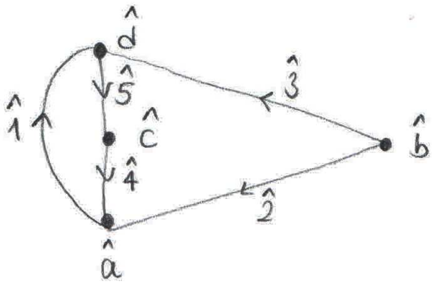
$V_S = 5\cos(2t)\ V$ $\qquad V_S \longrightarrow I_S$

if $V_1 = 3V$ then $J_1 = 3A$



→ outer mesh dots

Dual Graph :



* Draw the circuit

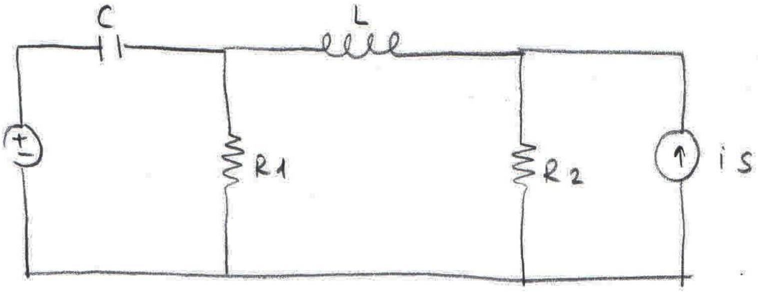
And replace every branch of the dual circuit with a dual component

90° clockwise rotation to find dual graph directions

Every branch is in between 2 dots (two meshes)

find a dual branch for every branch

Ex :



$v_s(t) = 2\cos t$ V

$C = 10\ \mu F$

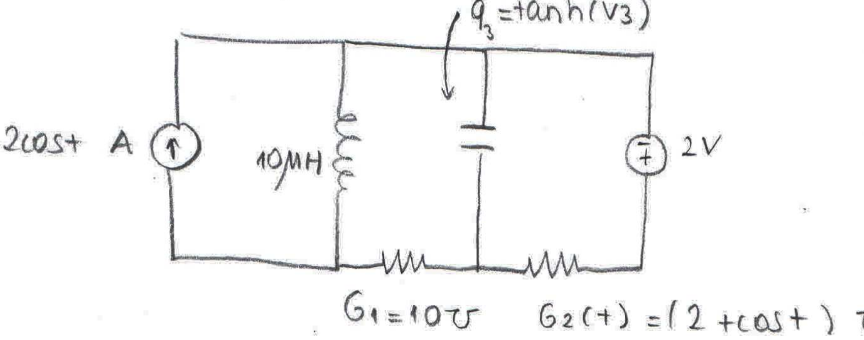
$i_s(t) = 2A$

$\phi_L = \tan h(i_3)$

$R_1 = 10\ \Omega$

$R_2(t) = 2 + \cos t$

Dual Circuit :



→ R resistance Ω

→ $\frac{1}{R} = G$ conductance

℧, siemens, mho

Classification of Circuits

- Resistive (Memoryless) / Dynamics

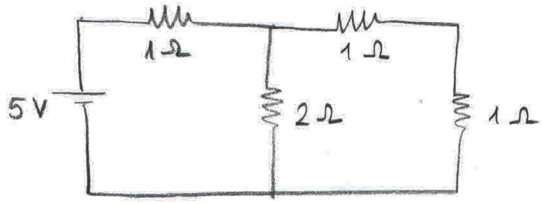
  Linear / Non-linear

  Time invariant / time-varying

  Passive / Active

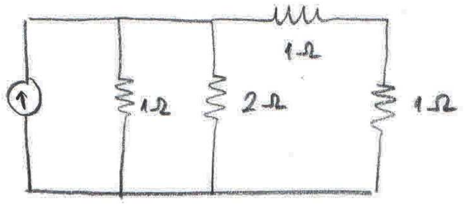
Generalized Branch :

For Graph theoratical Node/Mesh analysis, we may need to form generalized branches



Solve this using mesh analysis

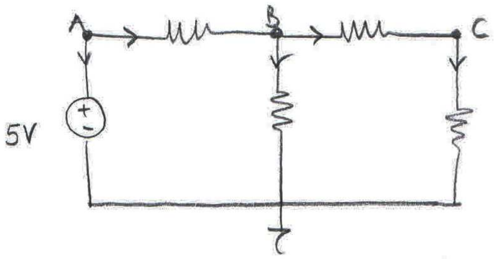
A

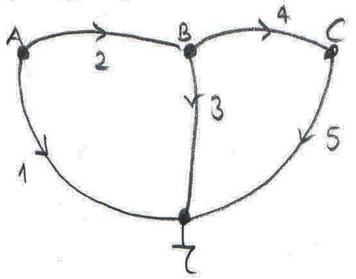


Solve this using node analysis

B

A ≡ B Let's apply node analysis to A





1- $\underline{\underline{A}}\,\underline{J} = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = 0$$

$J_1 = \underline{\underline{G}}\,\underline{V} + J_s$

$J_2 = V_2/1$ $\quad$ J_1 can be any scalar

$J_3 = V_3/2$

$J_4 = V_4/1$

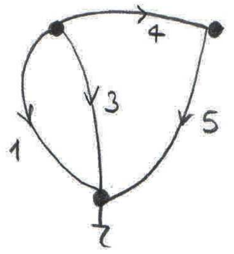
$J_5 = V_5/1$

Generalized branch

Node Analysis

a branch generalized branch

If we use generalized branch, then graph becomes ;

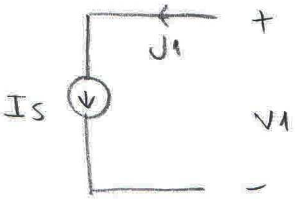


$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

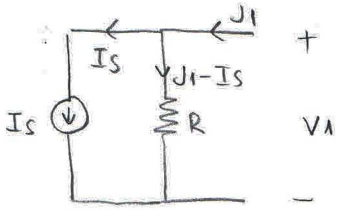
$$\begin{bmatrix} J_1 \\ J_3 \\ J_4 \\ J_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since $J_1 = \frac{V_1 - 5}{1\Omega}$

Mesh Analysis



A relation between J_1 and V_1



$V_1 = R(J_1 - I_s) = RJ_1 - RI_s$

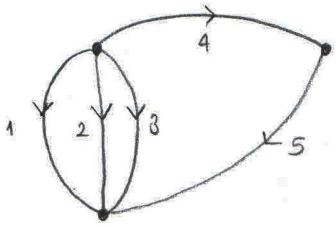
equation expressing V_1 in terms of J_1

Tellegen's Theorem

2 forms

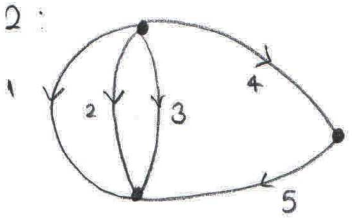
Form 1

Graph

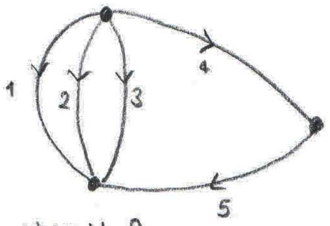


$$\sum_{k=1}^{\#\text{ of branches}} V_k i_k = 0$$

Form 2 :



circuit 1



circuit 2

$\hat{v}_k , \hat{j}_k$ branch variables

$$\sum_{k=1}^{\#branches} v_k \hat{j}_k = \sum_{k=1}^{\#branches} \hat{v}_k j_k = 0$$

Note : Graphs of circuits are identical

Circuit 1	Circuit 2	
$\underline{\underline{A}}\,\underline{j} = 0$	$\underline{\underline{A}}\,\underline{\hat{j}} = 0$	Node analysis
$\underline{v} = \underline{\underline{A}}^T \underline{e}$	$\hat{v} = A^T \hat{e}$	for 2 circuits

$$\sum_{k=1}^{\#branches} v_k \hat{j}_k = \underbrace{\underline{v}^T \underline{\hat{j}}} = (\underline{\underline{A}}^T \underline{e})^T \underline{j} = \underline{e}^T \underbrace{(\underline{\underline{A}}\,\underline{\hat{j}})}_{0} = 0$$

$$\downarrow$$

$$[v_1 \;\; v_2 \;\; v_3 \ldots v_N] \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ \vdots \\ j_N \end{bmatrix} = \sum_{k=1}^{n} v_k j_k$$

1 - $\underline{j}$ (branch currents) are in the nullspace of $\underline{\underline{A}}$

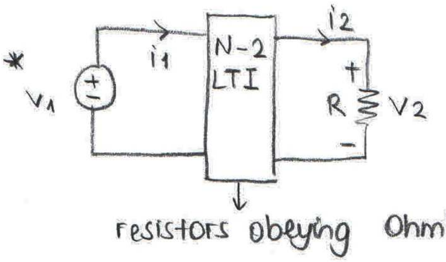
$\underline{\underline{A}}\,\underline{x} = 0$

2 - $\underline{v}$ is the range space of A^T

$A = [c_1 \;\; c_2 \ldots\ldots c_N]$ of column space

3 - So $\underline{j}$ and $\underline{v}$ vectors are in orthogonal spaces of $\underline{\underline{A}}$ matrix

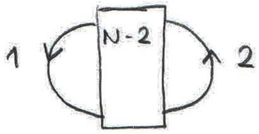
Tellegen's theorem Application (Also related the concept of <u>reciprocity</u>)



Circuit 1	Circuit 2
$R = 1\Omega$	$R = 2\Omega$
$V_1 = 4\ V$	$\hat{V}_1 = 6\ V$
$i_1 = 1A$	$\hat{i}_1 = 1.2\ A$
$V_2 = 1V$	$\hat{V}_2 = ?$
$i_2 = 1A$	

1- $\underline{\underline{V}}^T \underline{J} = 0 \longrightarrow \sum_{k=1}^{N} V_k \hat{J}_k = 0$ 2- $\underline{\underline{\hat{V}}}^T \cdot \underline{J} = 0 \longrightarrow \sum \hat{V}_k J_k = 0$

$\underset{4V}{V_1} \cdot \underset{-1.2A}{\hat{J}_1} + \underset{1V}{V_2} \cdot \underset{\frac{\hat{V}_2}{R_2}}{\hat{J}_2} + \sum_{k=3}^{N} V_k \hat{J}_k = 0$ $\underset{6V}{\hat{V}_1} \underset{-1A}{J_1} + \hat{V}_2 \underset{1A}{J_2} + \sum_{k=3}^{N} \hat{V}_k J_k = 0$

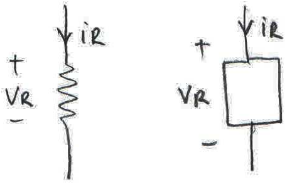


1- $4(-1.2) + \frac{\hat{V}_2}{2} \cdot 1 + \sum R_k J_k \hat{J}_k = 0$

2- $6(-1) + \hat{V}_2 \cdot 1 + \sum R_k \hat{J}_k J_k = 0$

$$\boxed{\hat{V_2} = 2.4 \text{ V}}$$

Linear Time-varying Resistance Circuits



$V_R = f(i_R)$

Some special cases have important consequences

1) Linearity

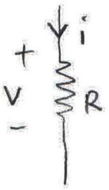
$L\{x_1(t)\} = y_1(t)$
$L\{x_2(t)\} = y_2(t)$

$L\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t)$

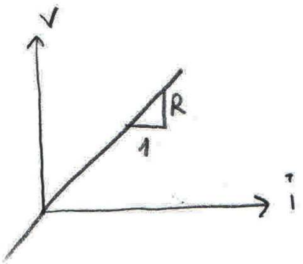
2) $L\{\alpha x_1(t)\} = \alpha y_1(t)$ Linear system

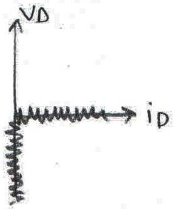
Question; $L\{0\}$ should be 0 (since $L\{x_1(t) - x_1(t)\} = y_1(t) - y_1(t) = 0$

Resistors:



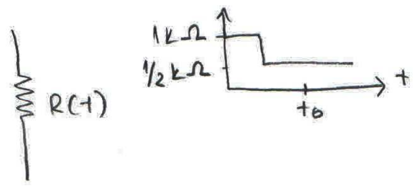
If $V = Ri$ (Ohm's Law)





Diode: nonlinear component

2- Time invariance :



resistance value is not constant

it is time varying

$V_R(t) = R(t)\, i_R(t)$

LTI Resistor $v(t) = R\, i(t)$ $\forall t$ R : constant

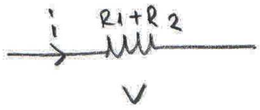
combinational Sources and Resistors (circuit simplification)

Series combination of resistors :

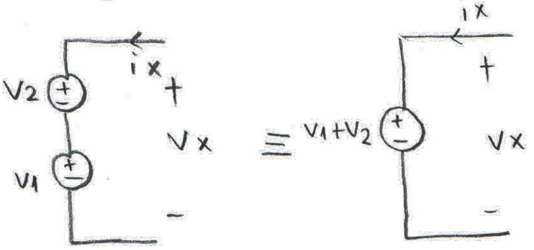
$V = V_1 + V_2$ by KVL

$= iR_1 + iR_2$

$= i(R_1 + R_2)$

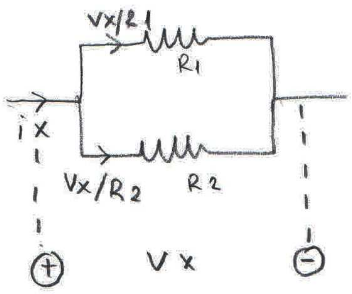


Voltage sources :



$V_x = V_1 + V_2$ $i_x \in R$

Parallel combinations

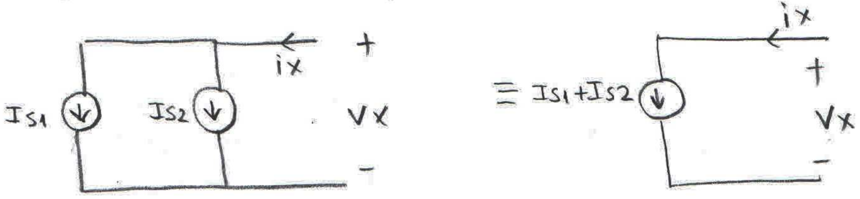


$$i_x = \frac{V_x}{R_1} + \frac{V_x}{R_2}$$

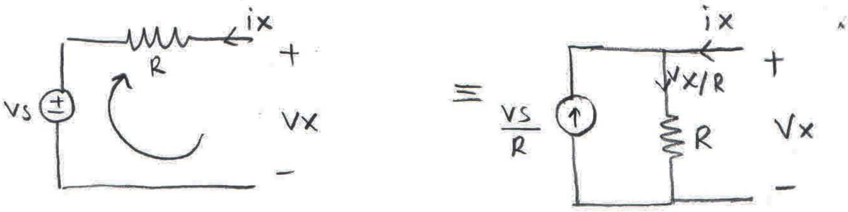
$$i_x = V_x\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

then $V_x = i_x\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$

current sources (parallel)



Source Transformation



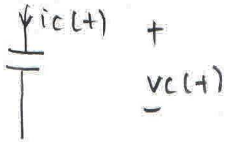
for 1st circuit ; $-V_x + R\,i_x + V_s = 0$ KVL

$$i_x = \frac{V_x - V_s}{R} = \frac{V_x}{R} - \frac{V_s}{R}$$

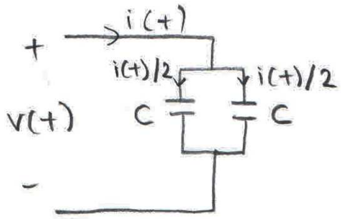
for 2nd circuit ; $i_x = V_x/R - V_s/R$

equality

*

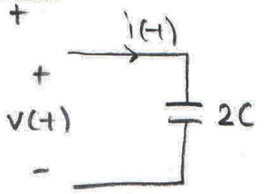


$$i_c(t) = C\frac{dv_c(t)}{dt}$$

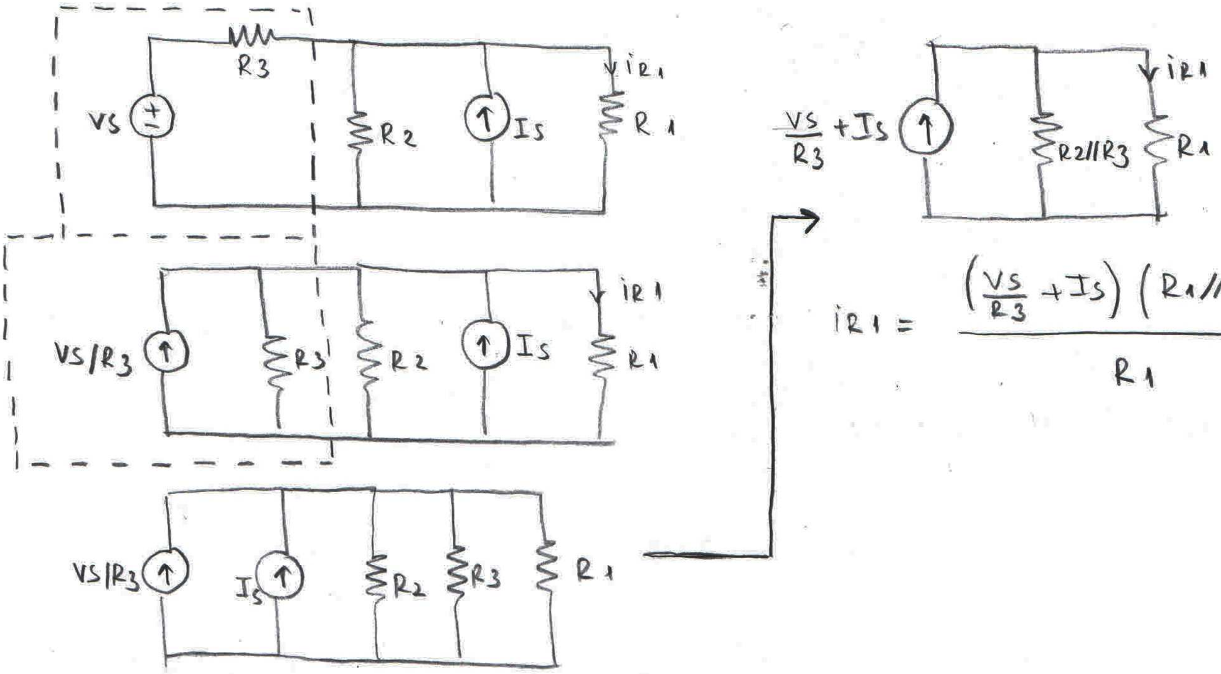


$$i_c(t) = \frac{i(t)}{2} = C\,\frac{dv_c(t)}{dt}$$

$$i(t) = 2C\,\frac{dv_c(t)}{dt}$$



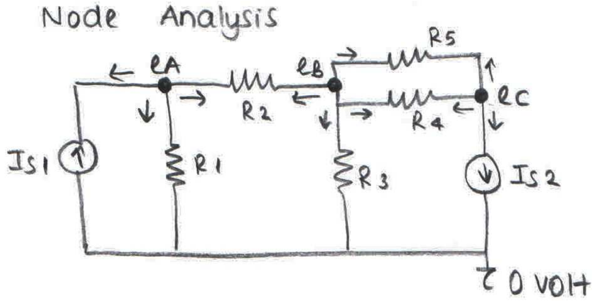
Ex: (source transformation)



$$i_{R1} = \frac{\left(\frac{V_s}{R_3} + I_s\right)\left(R_1//R_2//R_3\right)}{R_1}$$

Node - Mesh Analysis

Node Analysis



Steps : 1. Select a Datum (Ground) node

2. Assign node voltage variables to the remaining nodes

3. Write KCL at every node and solve for node voltages

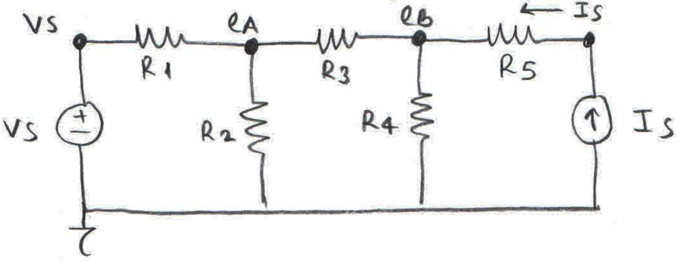
* KCL at eA : $-I_{s1} + \frac{e_A}{R_1} + \frac{e_A - e_B}{R_2} = 0$

KCL at eB : $\frac{e_B - e_A}{R_2} + \frac{e_B}{R_3} + \frac{e_B - e_C}{R_4 // R_5} = 0$

KCL at eC : $I_{s2} + \frac{e_C - e_B}{R_4 // R_5} = 0$

$$\begin{bmatrix} 1/R_1 + 1/R_2 & -1/R_2 & 0 \\ -1/R_2 & 1/R_2 + 1/R_3 + 1/R_4 + 1/R_5 & -1/R_4 - 1/R_5 \\ 0 & -1/R_4 - 1/R_5 & 1/R_4 + 1/R_5 \end{bmatrix} \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix} = \begin{bmatrix} I_{s1} \\ 0 \\ -I_{s2} \end{bmatrix}$$

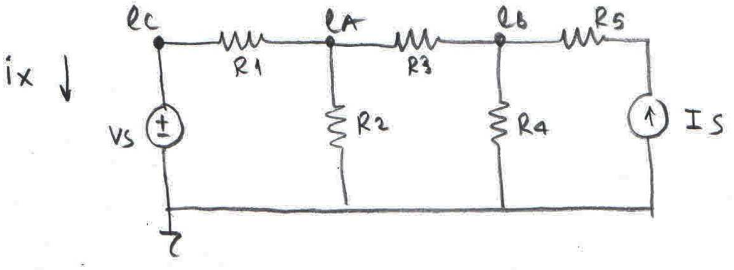
Node Analysis with Voltage Sources



KCL @ eA : $\frac{e_A}{R_2} + \frac{e_A - V_S}{R_1} + \frac{e_A - e_B}{R_3} = 0$

@ eB : $\frac{e_B - e_A}{R_3} + \frac{e_B}{R_4} - I_S = 0$

} can find eA and eB



KCL @ eA : $\frac{e_A - e_C}{R_1} + \frac{e_A}{R_2} + \frac{e_A - e_B}{R_3} = 0$

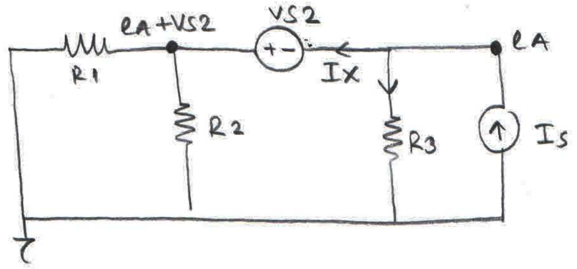
KCL @ eB : $\frac{e_B - e_A}{R_3} + \frac{e_B}{R_4} - I_s = 0$

KCL @ eC : $i_x + \frac{e_C - e_A}{R_1} = 0$

4 equations, 4 unknowns

$\{e_A, e_B, e_C, i_x\}$ $e_C = V_S(t)$

Ex:

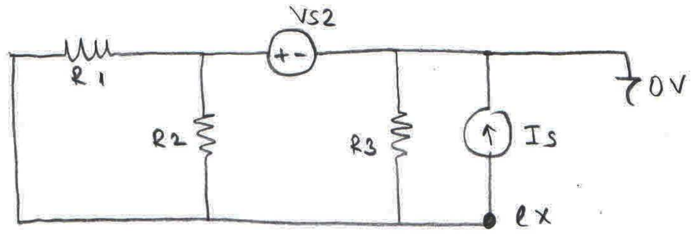


$$\text{KCL @ } e_A: \quad \frac{e_A}{R_3} - I_s + I_x = 0$$

$$\text{@ } e_A + V_{S2}; \quad \frac{e_A + V_{S2}}{R_2} + \frac{e_A + V_{S2}}{R_1} - I_x = 0$$

2 equations
2 unknowns

*

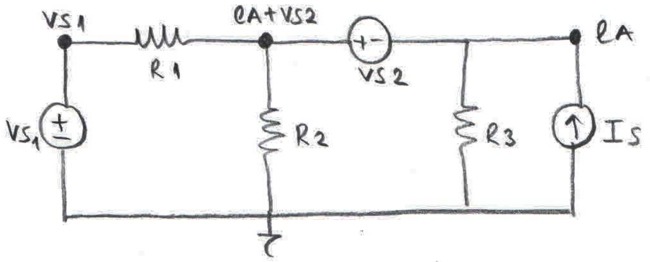


$$\text{KCL @ } e_x; \quad I_s + \frac{e_x}{R_3} + \frac{e_x - V_{S2}}{R_1} + \frac{e_x - V_{S2}}{R_2} = 0$$

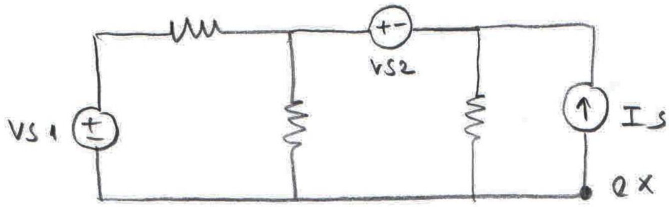
$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) e_x = \left(\frac{V_{S2}}{R_1} + \frac{V_{S2}}{R_2} - I_s\right)$$

$$e_x = \frac{V_{S2}/R_1 + V_{S2}/R_2 - I_s}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Ex:

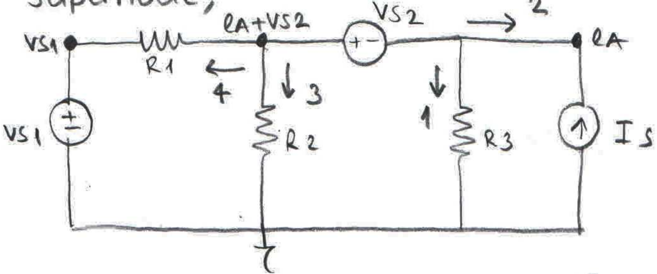


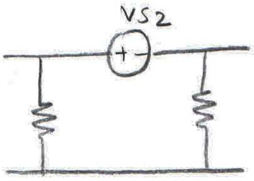
or



2 equations, 2 unknowns

Supernode;



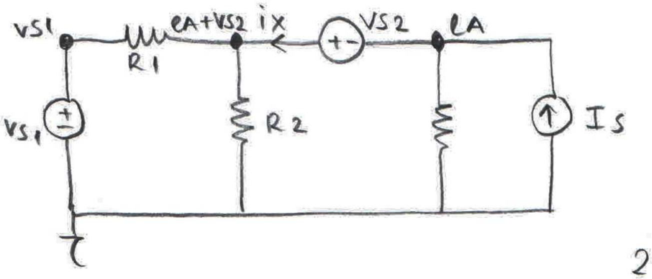


supernode (Always contain a voltage source)

KCL @ Supernode; $\frac{e_A}{R_3} - I_s + \frac{e_A + V_{S2}}{R_2} + \frac{e_A + V_{S2} - V_{S1}}{R_1} = 0$

$$e_A = \frac{I_s - V_{S2}/R_2 + (-V_{S2} + V_{S1})/R_1}{1/R_1 + 1/R_2 + 1/R_3}$$

Without a Super-node



KCL @ e_A = $\frac{e_A}{R_3} - I_s + i_x = 0$

@ $e_A + V_{S2}$ = $\frac{e_A + V_{S2} - V_{S1}}{R_1} - i_x + \frac{e_A + V_{S2}}{R_2} = 0$

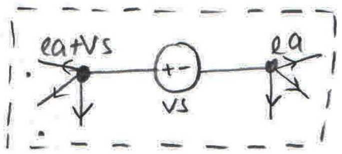
2 equations, 2 unknowns

Node Analysis

Select a ground node

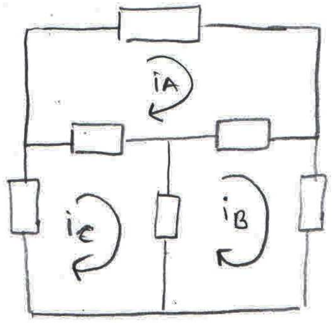
Assign node voltages

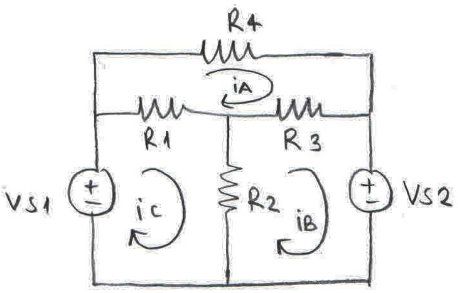
Write KCL equations at nodes (except ground)



Supernode

Mesh Analysis





Assign mesh currents to each mesh.

Express branch currents in terms of mesh currents. Write KVL around each mesh

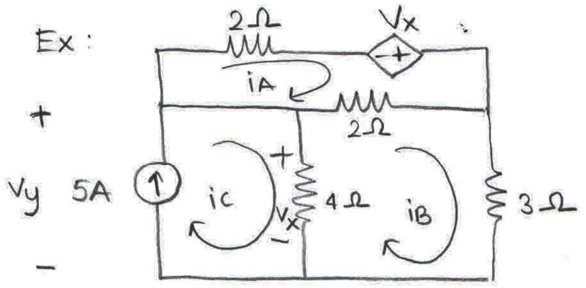
Solve the resulting equation set for mesh currents

Mesh i_A : $R_4 \cdot i_A + R_3(i_A - i_B) + R_1(i_A - i_C) = 0$

Mesh i_B : $R_3(i_B - i_A) + V_{S2} + R_2(i_B - i_C) = 0$

Mesh i_C : $-V_{S1} + R_1(i_C - i_A) + R_2(i_C - i_B) = 0$

Ex :



Mesh analysis is only applicaple to planar circuits .

Node analysis is applicaple to all

Mesh i_A : $2i_A - V_x + 2(i_A - i_B) = 0$

Mesh i_B : $2(i_B - i_A) + 3i_B + 4(i_B - i_C) = 0$

Mesh i_C : $-V_y + 4(i_C - i_B) = 0$ $i_C = 5A$

$V_x = (i_C - i_B)4$

by substituting 5A for i_C, we have

Mesh i_A : $2i_A - \underbrace{4(5 - i_B)}_{V_x} + 2(i_A - i_B) = 0$

Mesh i_B : $4(i_B - 5) + 2(i_B - i_A) + 3i_B = 0$

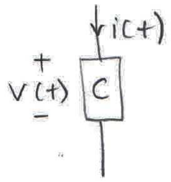
$$\begin{bmatrix} 2+2 & 4-2 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} i_A \\ i_B \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 9 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 9 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \end{bmatrix} \text{ A}$$

$V_x = 8V$ then $P_{5A} = -8 \cdot 5 = -40$ W absorbed , 40 W delivered

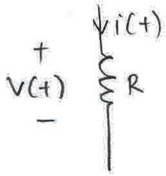
Since it is a source absorbed power is negative

Note on Power calculations ;



$p(t) = v(t)\, i(t)$

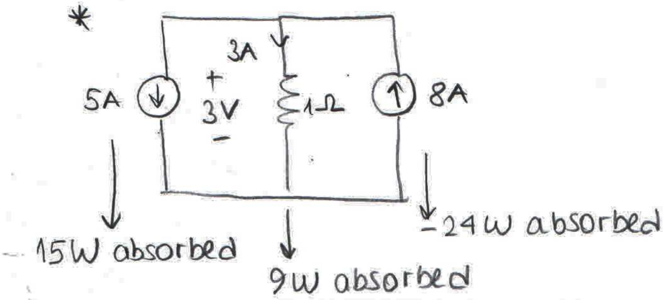
if $p(t) > 0$ instantaneous power is absorbed at time t



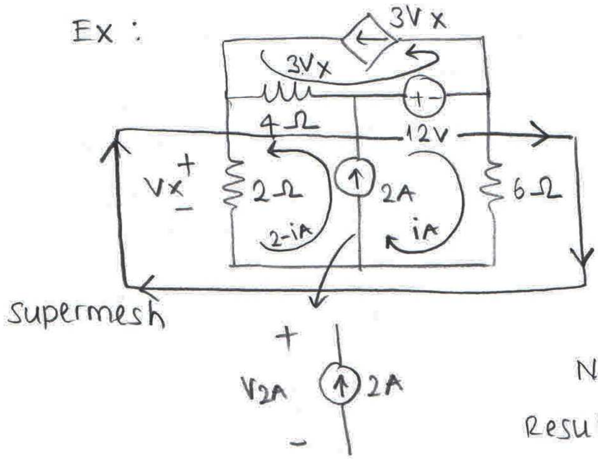
R is always absorbing power

$p(t) < 0$ delivering energy to other components at time t .

*



Ex :



Mesh iA : $12 + 6i_A - V_{2A} = 0$

Mesh $2-i_A$: $-V_{2A} + 4(2-i_A-3V_x) + 2(2-i_A) = 0$

2 equations, 2 unknowns

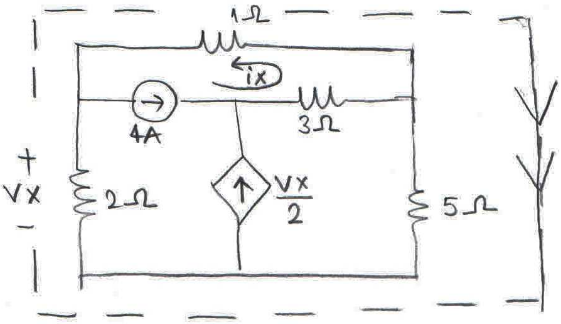
KVL around supermesh

$4(3V_x-(2-i_A)) + 12 + 6i_A - 2(2-i_A) = 0$

Note: (Eqn in Mesh iA) minus (Eqn in (2-iA)) = 0

Results: $i_A = 4A$ $V_x = -4V$

Ex :



KVL around outer mesh ;

$1(-i_x) + 5\left(\frac{V_x}{2} + 4 - i_x\right) + 2(4-i_x) = 0$

$-i_x + 5(i_x - 4 + 4 - i_x) + 2(4-i_x) = 0$

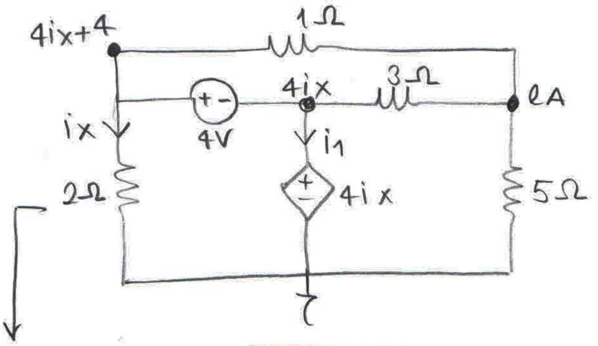
$i_x = 8/3$ A $V_x = -8/3$ V

$V_x = 2(i_x - 4)$

This circuit requires 3 mesh currents, but it has 2 current sources. (one independent, one dependent) then 3-2 = 1 can be sufficient to solve for mesh currents.

For node analysis, same circuit requires 3 nodes and has no voltage sources, so "3-0" = 3 node equations should be solved together

Ex :



KCL @ eA :

$\frac{e_A}{5} + \frac{e_A - 4i_x}{3} + \frac{e_A - (4i_x+4)}{1} = 0$

KCL @ 4ix;

$\frac{4i_x - e_A}{3} + i_1 + (5i_x + 4 - e_A) = 0$

KCL @ 4ix+4;

$-(5i_x + 4 - e_A) + i_x + \frac{4i_x + 4 - e_A}{1} = 0$

$4i_x + 4 = 2i_x$; $\boxed{i_x = -2A}$

$\frac{e_A}{5} + \frac{e_A + 8}{3} + e_A + 4 = 0$

$\frac{23e_A}{15} + \frac{20}{3} = 0$ $\boxed{e_A = -80/23}$

Fundamental loop Method : Similar to graph theoratical mesh analysis (but not equivalent)

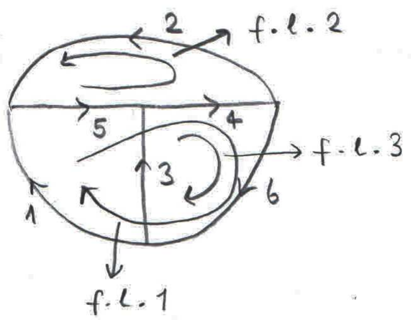
1_ Pick a tree

2_ Write fun-loop eqns ($\underline{B}\ \underline{V}=0$)

3_ Express branch voltages in terms of branch currents ($\underline{V}=\underline{\underline{R}}\,\underline{J}+\underline{V_s}$)

4_ Use $\underline{J}=\underline{\underline{B}}^T\,i$ relation expressing branch current in terms of fundamental loop currents (co-tree currents)

Graph



Tree : Set of connected branches of the graph such that

1_ Branches do not form a loop

2_ Every node is reached by a tree

3_ Tree is connected

Tree branches $\{4,5,6\}$

Co-tree branches $\{1,2,3\}$

* Every co-tree branch and whole-tree makes a loop (fund-loops – Union of a single co-tree branch and tree)

Step 2 :

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 1}_{I} & & & \underbrace{1 \quad 0}_{F} & & 1 \end{array}\right]\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

fundamental loop

* Fundamental loop directions are in the same directions with co-tree

Step 3 :

$\underline{J}=\underline{\underline{B}}^T\,\underline{i}$

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

fundamental loop currents and currents of co-tree branches

Fundamental Cut-set (Similar to graph node analysis (but not equivalent))

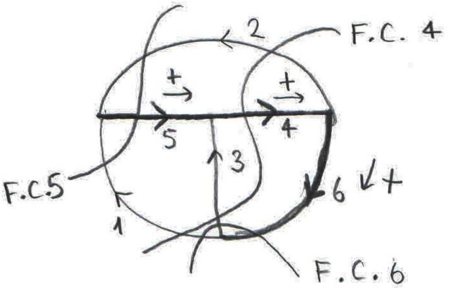
1- Draw graph

2- Select tree

3- Write KCL at fundamental cut-set ($\underline{\underline{Q}}\,\underline{J}=0$)

4- Write branch equations ($\underline{J}=\underline{\underline{G}}\,\underline{V}+\underline{is}$)

5- Use $\underline{V}=\underline{\underline{Q}}^T\underline{e}$ expressing branch voltages in terms of tree branch voltages



Fundamental cut-set

Cut-set : Partitions graph into 2 disjoint sets

1- Should be cut-set

2- Should intersect a single tree branch

Step 2 $\underline{\underline{Q}}\,\underline{J}=\underline{0}$

$$\text{F.C}\begin{bmatrix}4\\5\\6\end{bmatrix}\left[\begin{array}{ccc|ccc} -1&-1&-1&1&0&0\\ -1&-1&0&0&1&0\\ -1&0&-1&0&0&1 \end{array}\right]\begin{bmatrix}J_1\\J_2\\J_3\\J_4\\J_5\\J_6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-F^T$ $\qquad$ I

* Note that +/- sign of cut-set is in the direction of related tree branch. Tree branch always enters to + side

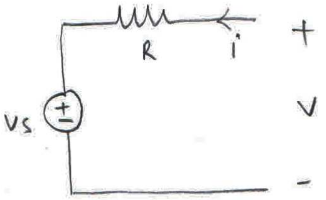
$\underline{\underline{B}} = [\underline{\underline{I}}\,|\,F]$

for the same tree

$Q = [-F^T\,|\,I]$

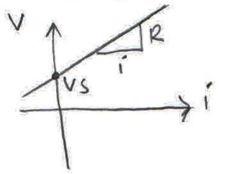
$\underline{\underline{B}}\,Q^T = [I\,|\,F]\begin{bmatrix}-F\\I\end{bmatrix} = F-F=0$ Tellegen's theorem for cut-set thm

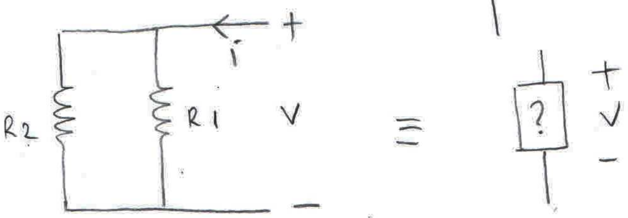
Input i/V calculations



$-V_s+R(-i)+V=0$

$V=V_s+Ri$

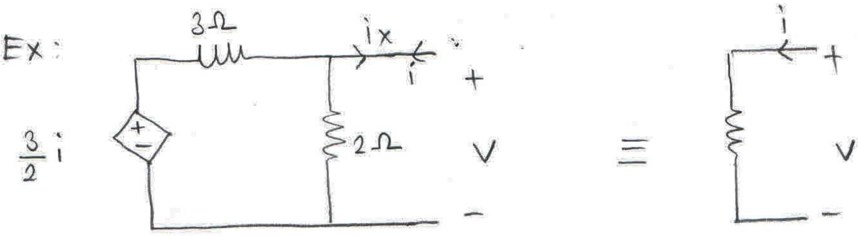




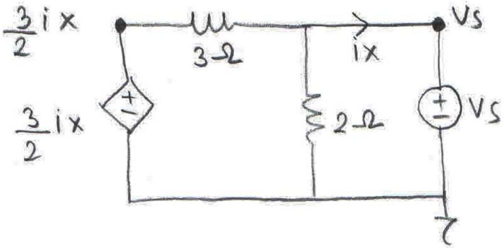
what is the mapping between i and v

$i=\frac{V}{R_1}+\frac{V}{R_2}$

Ex :



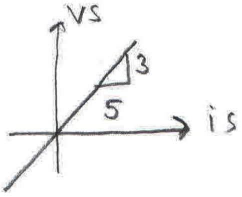
Let's assume that



Find ix in terms of Vs

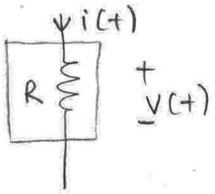
$$\frac{V_S}{2} + i_x = \left(\frac{3}{2} i_x - V_S\right) 1/3$$

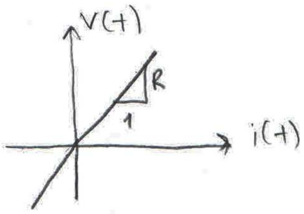
$$i_x = -\frac{5}{3} V_S$$



So I can use both independent voltage source and independent current source for i-V characteristic.

Linearity : Linear components, linear relations → linear systems





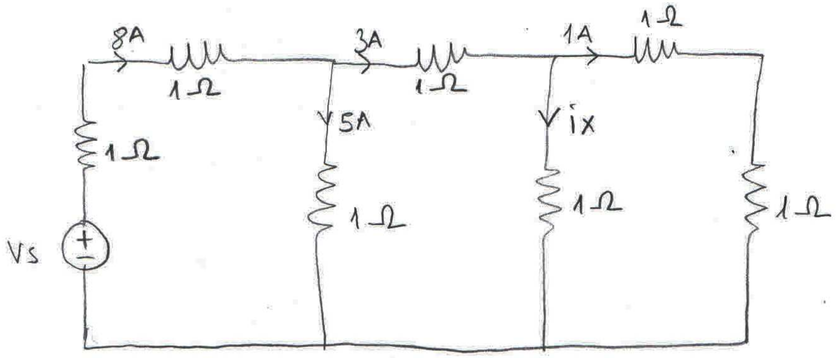
$x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

If linear $L\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t)$

$$L\{\alpha x_1(t)\} = \alpha y_1(t)$$

Ex :



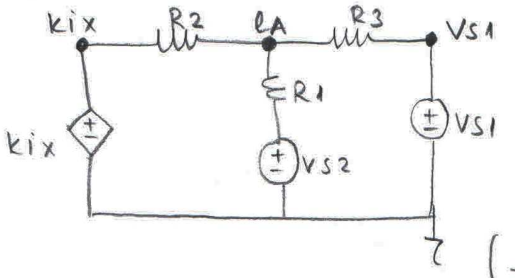
Ladder Network

Find ix

so let's apply the principle of linearity of soln

Assume ix = 2 A Vs = 21 V

Circuits with Multiple Sources



$$\frac{e_A - V_{S2}}{R_1} + \frac{e_A - k i_x}{R_2} + \frac{e_A - V_{S1}}{R_3} = 0$$

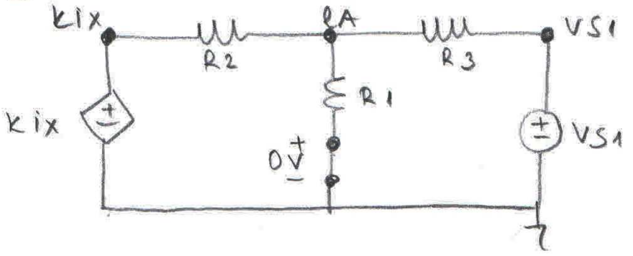
$$i_x = \frac{e_A - V_{S1}}{R_3}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} - \frac{k}{R_2 R_3}\right) e_A = \frac{V_{S2}}{R_1} + \frac{V_{S1}}{R_3} - \frac{k V_{S1}}{R_2 R_3}$$

e_A can be solved from these equations.

Solution by Superposition Principle (Linearity Rule)

(I) Take $V_{S2} = 0$ $V_{S1} \neq 0$

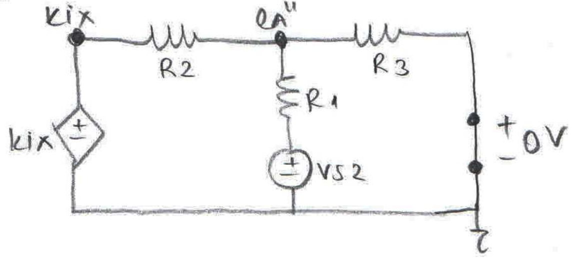


KCL; $\frac{e_A'}{R_1} + \frac{e_A' - k i_x'}{R_2} + \frac{e_A' - V_{S1}}{R_3} = 0$

solution for $V_{S2} = 0$

$$i_x' = \frac{e_A' - V_{S1}}{R_3}$$

(II) Take $V_{S1} = 0$ $V_{S2} \neq 0$



KCL: $\frac{e_A'' - k i_x''}{R_2} + \frac{e_A'' - V_{S2}}{R_1} + \frac{e_A''}{R_3} = 0$

$$i_x'' = \frac{e_A''}{R_3}$$

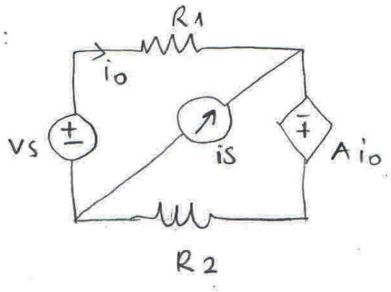
* Claim: e_A (solution when $V_{S1} \neq 0$ $V_{S2} \neq 0$) can be written as

$e_A = e_A' + c_A''$ superposition of two positions

* if $e_A = e_A' + e_A''$ then $i_x = i_x' + i_x''$

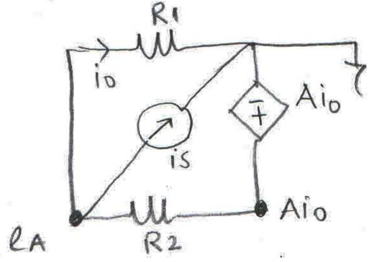
☆ Superposition is made only for independent sources.

Ex:



Find $i_o = ?$

for $V_s = 0$



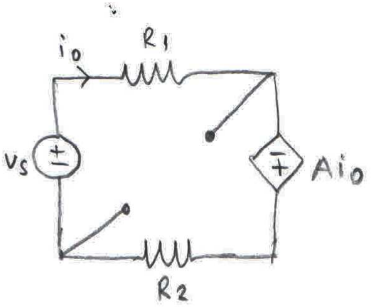
$$\frac{e_A}{R_1} + i_s + \frac{e_A - A i_o'}{R_2} = 0$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{A}{R_1 R_2}\right) e_A = -I_s$$

$$e_A = \frac{-I_s}{\frac{1}{R_1} + \frac{1}{R_2} - \frac{A}{R_1 R_2}}$$

$i_o' = \frac{\ell A}{R_1}$

for $i_s = 0$;



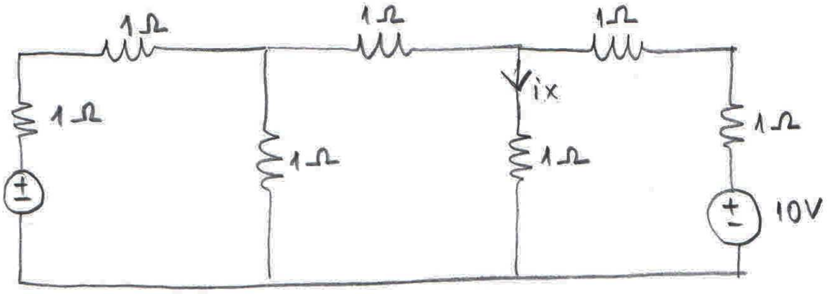
by KVL;

$-V_s + R_1 i_o'' - A i_o'' + R_2 i_o''$

$i_o'' = \frac{V_s}{R_1 - A + R_2}$

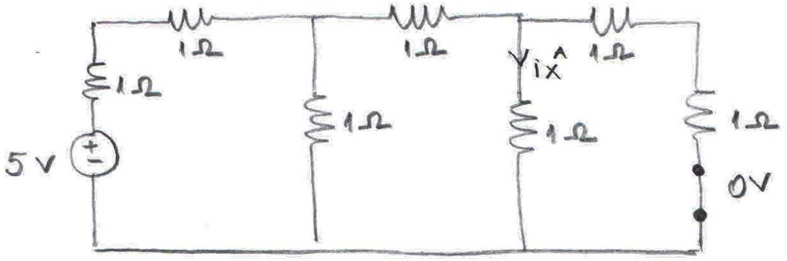
$$\boxed{I_o = i_o' + i_o''}$$

Ex:



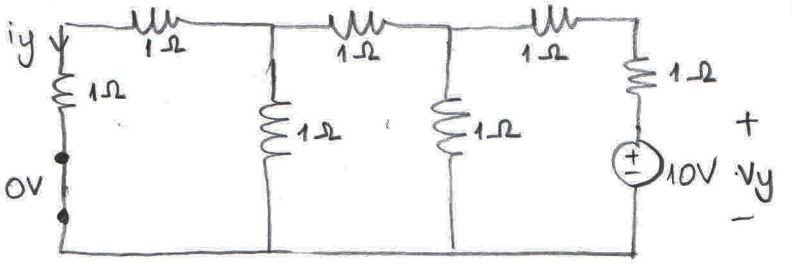
Find i_x?

A- kill (turn off) 10 V source.



$i_x^A = \frac{10}{21}$ A

B- kill (turn off) 5V source.



If $i_y = 1$ A then $V_y = 21$ V

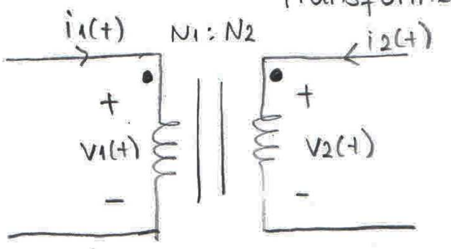
$i_y = ?$ if $V_y = 10V$

$i_y = \frac{10}{21}$ A if $V_y = 10V$

$i_x^B = 5 i_y$; $i_x^B = \frac{50}{21}$ A

$i_x = i_x^A + i_x^B = \frac{60}{21} = \frac{20}{7}$ A

Transformers



Dots are related to how turns are arranged in clockwise, counterclockwise direction

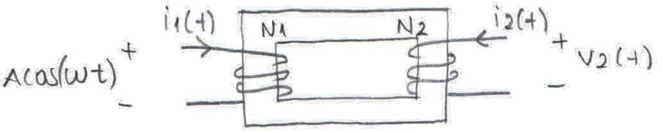
Two port circuit component

$\left[\frac{v_1(t)}{v_2(t)} = \frac{N_1}{N_2}\right]$ voltage ratio is directly proportional to turn ratio.

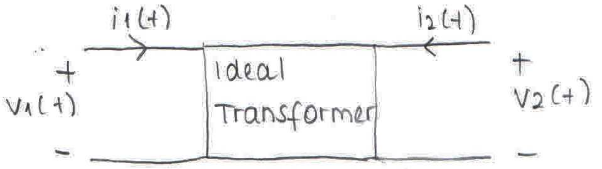
$\left[\frac{i_1(t)}{i_2(t)} = -\frac{N_2}{N_1}\right]$ current ratio is inversely proportional

transformer relations

N_1, N_2 : Turn ratio of transformer



Power consumed by Ideal Transformer

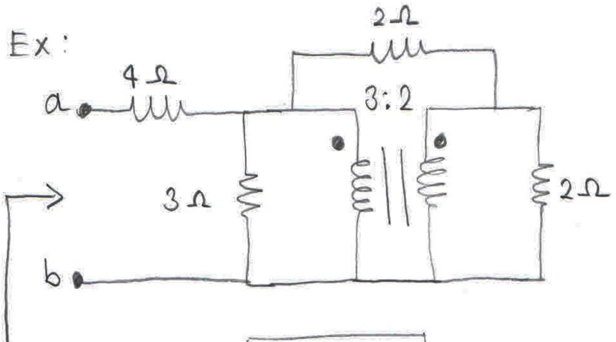


$$P_{consumed}(t) = v_1(t)\,i_1(t) + v_2(t)\,i_2(t)$$

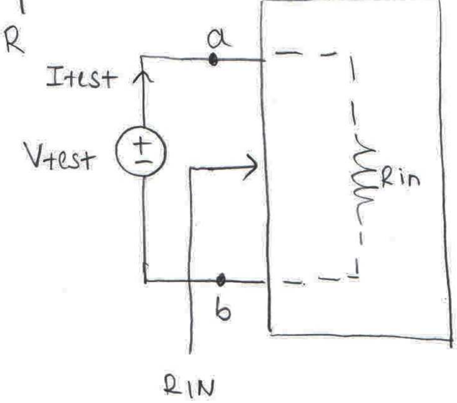
$$v_1(t)\,i_1(t) + \underbrace{\left(v_1(t)\frac{N_2}{N_1}\right)}_{v_2(t)}\underbrace{\left(-i_1(t)\frac{N_1}{N_2}\right)}_{i_2(t)} = 0 \text{ W}$$

* Ideal Transformer do not consume any power

Ex:

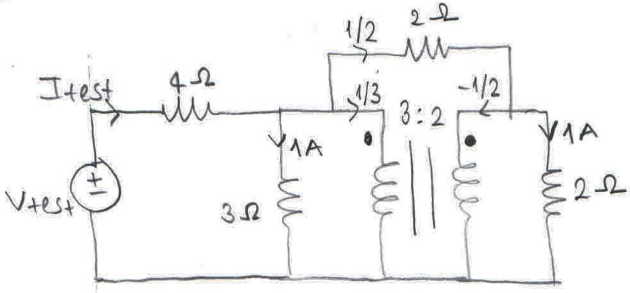


Find input resistance seen from the a-b terminals



RIN

$R_{IN} = \frac{V_{test}}{I_{test}}$

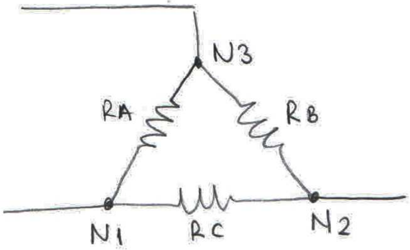


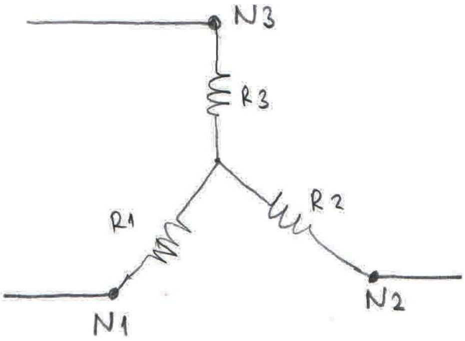
$I_{test} = \frac{11}{6}$ A

$V_{test} = 4\,\frac{11}{6} + 3 = \frac{62}{6}$

$$\boxed{R_{IN} = \frac{V_{test}}{I_{test}} = \frac{62}{11}\ \Omega}$$

Δ-Y Transformation





Δ

Y

Y → Δ

Δ → Y

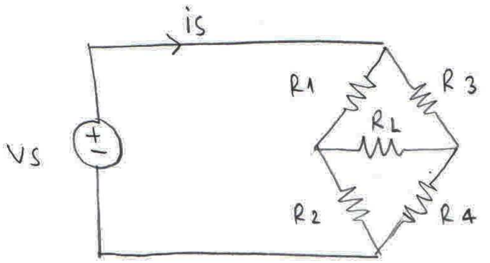
$$R_A = \frac{R_1R_2+R_1R_3+R_2R_3}{R_2} \qquad R_1 = \frac{R_AR_C}{R_A+R_B+R_C}$$

$$R_B = \frac{R_1R_2+R_1R_3+R_2R_3}{R_1} \qquad R_2 = \frac{R_BR_C}{R_A+R_B+R_C}$$

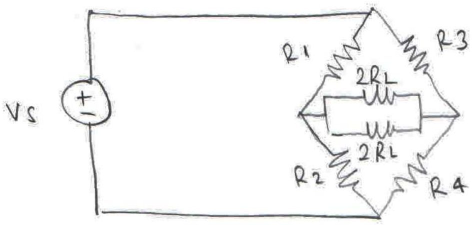
$$R_C = \frac{R_1R_2+R_1R_3+R_2R_3}{R_3} \qquad R_3 = \frac{R_AR_B}{R_A+R_B+R_C}$$

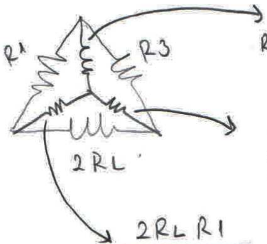
If (R_A, R_B, R_C) & (R_1, R_2, R_3) satisfy the equations then Δ and Y networks are equivalent

Ex :



Find is in terms of vs

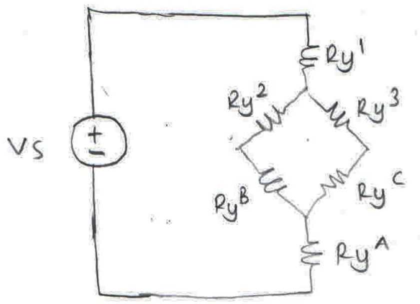




$$\frac{R_1R_3}{R_1+2R_L+R_3} = R_y^{1}$$

$$\frac{2R_LR_3}{R_1+2R_L+R_3} = R_y^{2}$$

$$\frac{2R_LR_1}{R_1+2R_L+R_3} = R_y^{3}$$



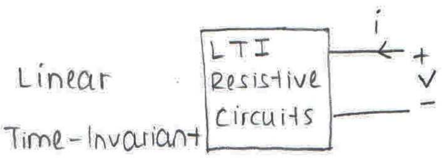
$$R_y^{A} = \frac{R_2R_4}{R_2+2R_L+R_4} \qquad R_y^{B} = \frac{2R_LR_2}{R_2+2R_L+R_4}$$

$$R_y^{C} = \frac{2R_LR_4}{R_2+2R_L+R_4}$$

$$R_{IN} = R_y^1 + R_y^A + \left[(R_y^2 + R_y^B) // (R_y^3 + R_y^C) \right]$$

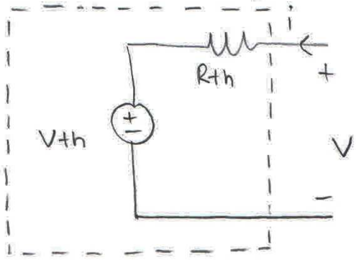
$$i_s = \frac{V_s}{R_{IN}}$$

Thevenin - Norton Equivalents



Finding (i-v) characteristic of a circuit
(seen from two terminals)

Thevenin Equivalent



V_{th} : Thevenin voltage

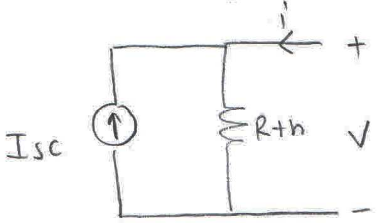
R_{th} : Thevenin Resistance

$V_{th} = V_{oc}$ open circuit voltage

$R_{th} \Rightarrow$ Input resistance seen from two terminals

equivalent circuit = thevenin equivalent
seen from two terminals

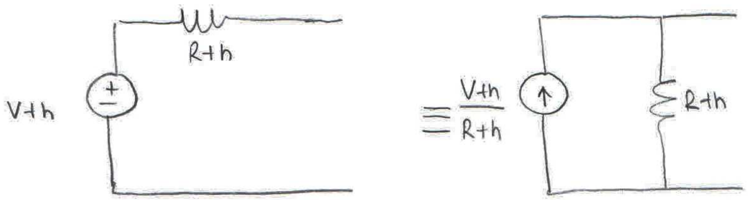
Norton Equivalent



I_{sc} : short circuit current

R_{th} : thevenin resistance

Note : If source transformation is applied to Thevenin Equivalent ;



$$\boxed{I_{sc} = \frac{V_{th}}{R_{th}}}$$

Procedure for finding Thevenin Equivalents

Procedure 1 Ⓐ Find V_{oc}

Ⓑ Find R_{th} ⟶ Turn off all Independent sources and find R_{IN} seen from two terminals.

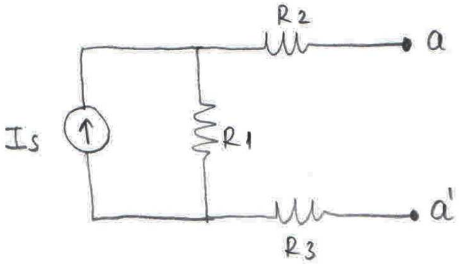
⟶ Turn off all independent sources, apply V_{test} and measure I_{test} then $R_{IN} = R_{th} = \frac{V_{test}}{I_{test}}$

Procedure 2: Ⓐ Find voc

Ⓑ Find Isc

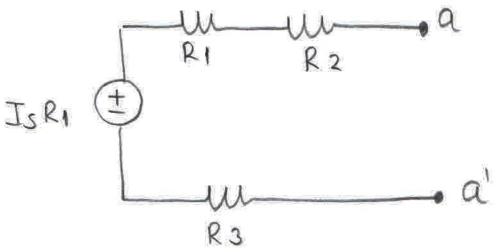
Ⓒ Calculate Rth by $\frac{voc}{Isc}$

EX:



Find an equivalent circuit for the LHS of aa' terminals

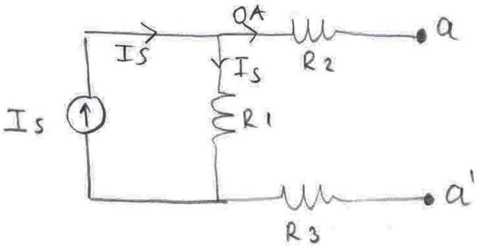
Solution: Apply source transformation



$V_{th} = I_s R_1$

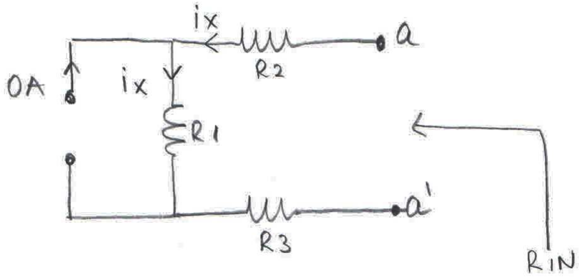
$R_{th} = R_1 + R_2 + R_3$

Procedure ① Ⓐ Voc = ? (open circuit terminals)



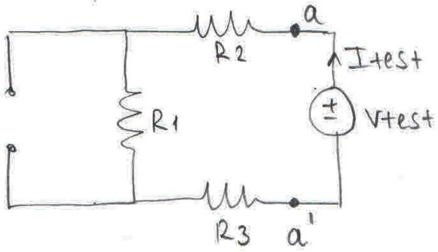
$V_{oc} = V_a - V_{a'} = I_s R_1$

Ⓑ①



$R_{IN} = R_1 + R_2 + R_3$

Ⓑ②



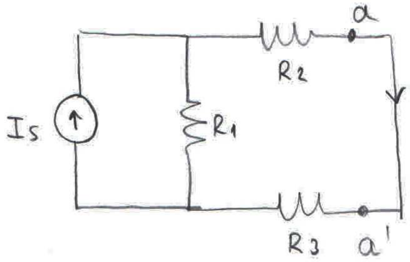
$$I_{test} = \frac{V_{test}}{R_1+R_2+R_3}$$

$$R_{IN} = \frac{V_{test}}{I_{test}} = R_1+R_2+R_3$$

Procedure ② Ⓐ Voc = Is R1

Ⓑ Isc = Short Circuit Current (terminals are short-circuited)

ⓒ $R_{th} = \frac{V_{oc}}{I_{sc}} = R_1 + R_2 + R_3$ ·

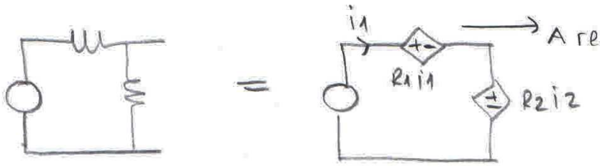


Isc flowing from a to a'

$$I_{sc} = I_s \cdot \frac{1/(R_2+R_3)}{(1/R_1) + \left(1/(R_2+R_3)\right)} = I_s \frac{R_1}{R_1+R_2+R_3}$$

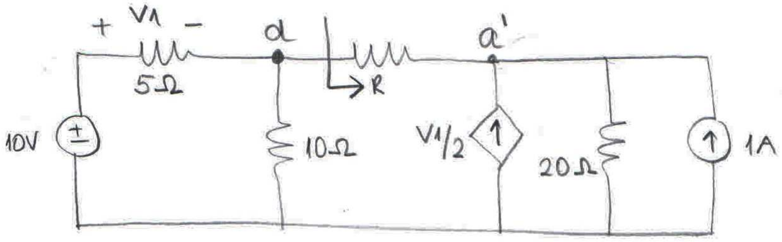
* Dependent sources are named as sources but we should interpret dependent sources as dependent components but not sources such as independent sources. So do not apply
- Source transformation
- Do not turn off dependent sources during R_{IN} calculations.

Remember

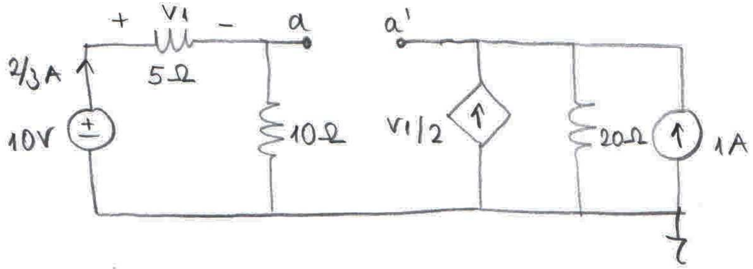


→ A resistor with R1 resistance

Ex:



Procedure 2: $V_{oc} = ?$ $V_{oc} = V_a - V_b$



$$V_1 = \frac{10V}{(5+10)\Omega} \cdot 10\Omega = \frac{10}{3} V$$

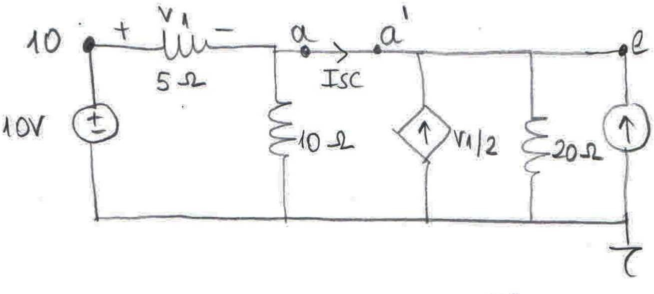
$$V_{20\Omega} = 20\Omega \left(1 + \frac{V_1}{2}\right) A$$

$$V_{20\Omega} = \frac{160}{3} V$$

$V_a = \frac{20}{3} V$ $V_{a'} = \frac{160}{3} V$

$V_{oc} = \frac{-140}{3} V$

Isc :



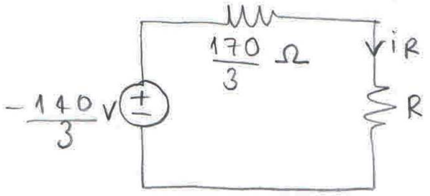
by KCL : $\frac{e-10}{5} - 1 + \frac{e}{20} - \frac{e}{10}$

$- \frac{V_1}{2} = 0$

$V_1 = 10 - e \; ; \qquad 17e = 160 \; ; \qquad e = \frac{160}{7}$ V

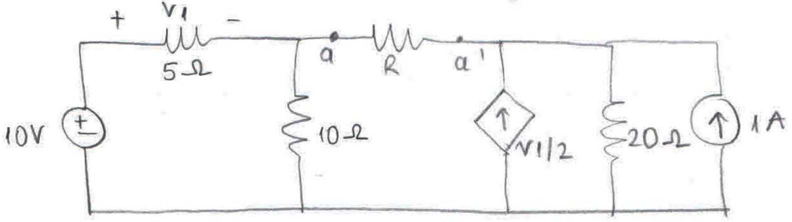
$$I_{SC} = \frac{10-e}{5} - \frac{e}{10} = \frac{20-3e}{10} = -\frac{14}{17} \text{ A}$$

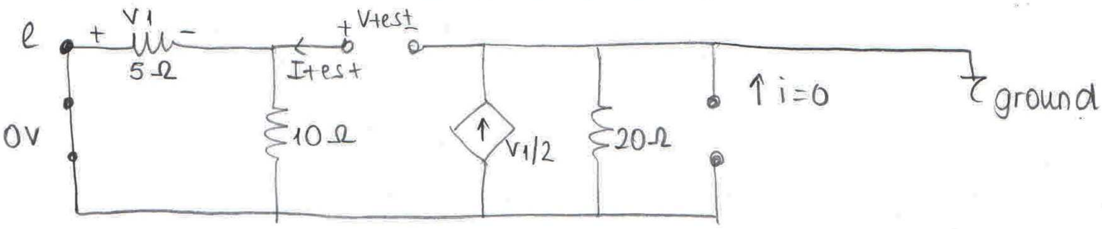
$$R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{-140/3}{-14/17} = \frac{170}{3} \; \Omega$$



$$i_R = \frac{-140/3}{(170/3)+R} \quad \text{for any } R$$

Same Circuit : Apply test voltage method for Rth calculation.





$$\text{by KCL} \quad \frac{e}{20} + \frac{\overset{e-V_{test}}{V_1}}{2} + \frac{e-V_{test}}{10} + \frac{e-V_{test}}{5} = 0$$

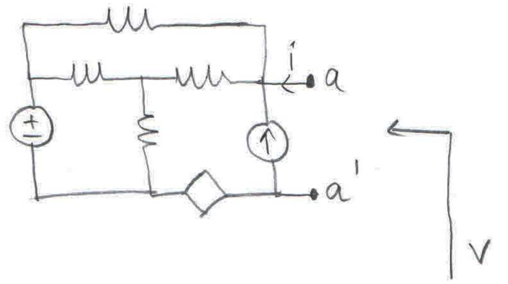
$17e = V_{test} \cdot 16$

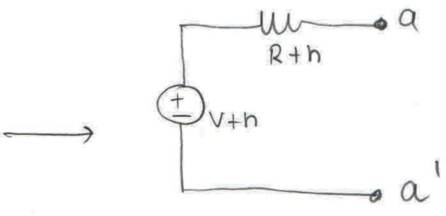
$e = \frac{16}{17} V_{test}$

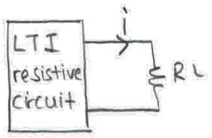
$$i_{test} = \frac{V_1}{2} + \frac{e}{20} = \frac{11}{20} e - \frac{V_{test}}{2} = \frac{3}{170} V_{test}$$

$$R_{IN} = \frac{V_{test}}{I_{test}} = \frac{170}{3} \; \Omega$$

Maximum Power Transfer

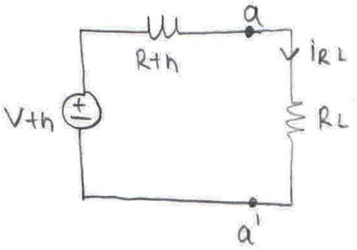






Max power transfer question: Select RL such that power on RL is maximum

Replace the equivalent circuit seen by RL



$$P_{RL} = i_{RL}^2 R_L = \left(\frac{V_{th}}{R_L+R_{th}}\right)^2 R_L$$

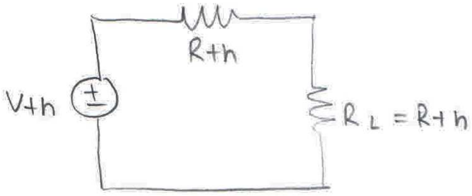
$$\frac{\partial P_{RL}}{\partial R_L} = 0 \longrightarrow V_{th}^2 \left[\frac{(R_{th}+R_L)^2 - 2(R_{th}+R_L)R_L}{(R_{th}+R_L)^4}\right]$$

$(R_{th}+R_L)^2 = 2(R_{th}+R_L)R_L$

$(R_{th}+R_L) = 2R_L$ $\quad R_{th}+R_L \neq 0$

$R_{th}+R_L = 0$ $\quad R_L = \{R_{th}, -R_{th}\}$ $\quad R_L = R_{th}$

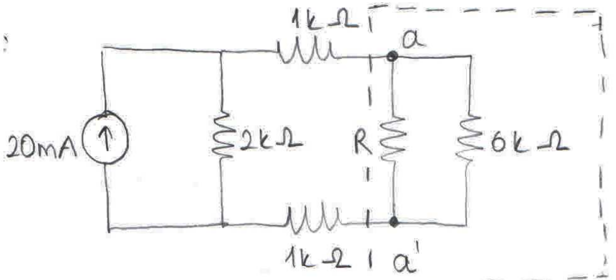
At maximum power transfer condition;



$$P_{max} = \left(\frac{V_{th}}{2R_{th}}\right)^2 R_{th} = \frac{V_{th}^2}{4R_{th}}$$

Note: Power delivered RL is equal to the power dissipated over Rth so efficiency of max power transfer condition is 50 %

Ex:



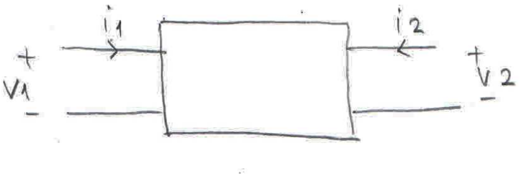
Find R such that power delivered to load is maximized.

Thevenin equivalent of LHS of a-a'

$R_{th} = 4k\Omega$ $\quad \longrightarrow$ for max power transfer

$V_{oc} = 40\ V$ $\quad 4k\Omega = R // 6k\Omega$ $\quad R = 12k\Omega$

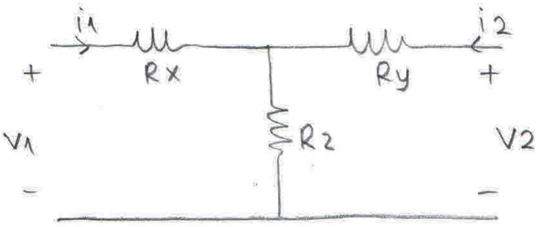
Two-Ports



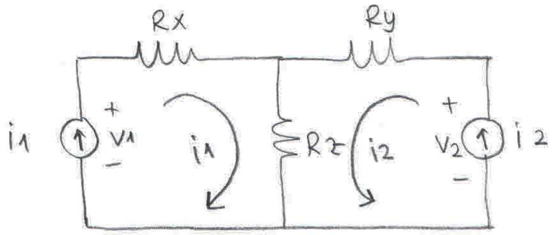
2 port circuit

Goal: Represent two port circuit with some system parameters

Ex:



R parameters
$$\underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\text{output/unknown}} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \underbrace{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}_{\text{consider this part input}}$$



$$V_1 = i_1 R_x + R_z(i_1 + i_2)$$
$$V_2 = i_2 R_y + R_z(i_1 + i_2)$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_x + R_y & R_y \\ R_y & R_y + R_z \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

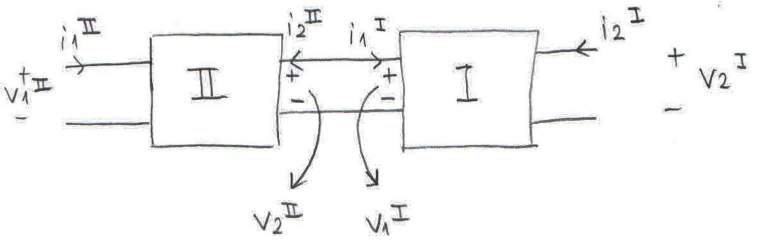
G conductance parameters
$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad \underline{\underline{G}} = \underline{\underline{R}}^{-1}$$

Hybrid - I parameters
$$\begin{bmatrix} V_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H} \begin{bmatrix} i_1 \\ V_2 \end{bmatrix}$$

Hybrid - II parameters
$$\begin{bmatrix} i_1 \\ V_2 \end{bmatrix} = \underset{\substack{\downarrow \\ \text{hybrid II}}}{H^{-1}} \begin{bmatrix} V_1 \\ i_2 \end{bmatrix}$$

Transfer Parameters = ABCD parameters = Chain parameters

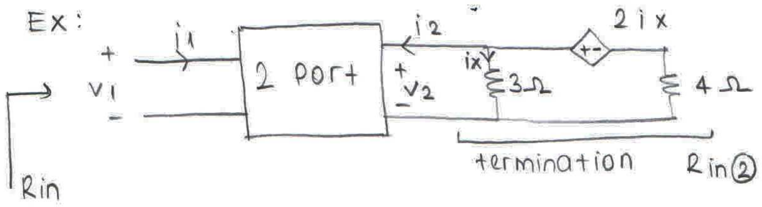
$$\begin{bmatrix} V_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -i_2 \end{bmatrix}$$



$$(ABCD)_{\text{cascode}} = (ABCD)_{II}\,(ABCD)_{I}$$

* ABCD parameters of the cascode is the matrix multiplication of two ports I & II

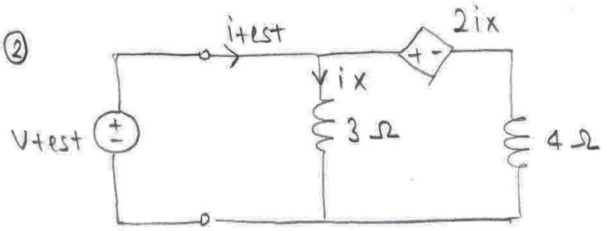
Ex:



find Rin = ?

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Rin ②

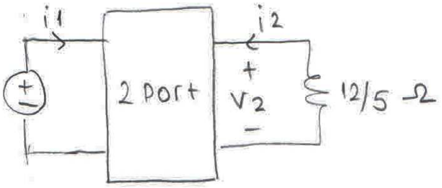


$$i_x = \frac{V_{test}}{3}$$

$$i_{test} = i_x + \frac{i_x}{4} = \frac{5}{4} i_x = \frac{5}{12} V_{test}$$

$$R_{in} ② = \frac{V_{test}}{i_{test}} = \frac{12}{5} \ \Omega$$

Vtest = V1



Let's apply Vtest and measure Itest

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

I need $\frac{V_1}{i_1}$

$$V_1 = 3i_1 + 2i_2$$

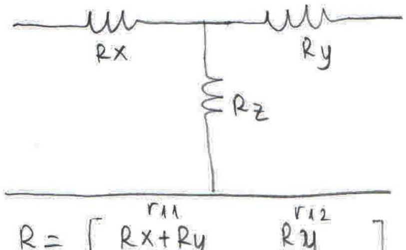
$$V_2 = 2i_1 + 4i_2$$

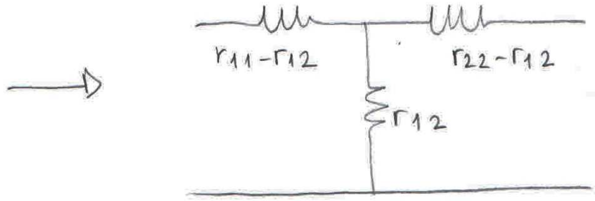
$$V_2 = \frac{12}{5}(-i_2) = -\frac{12}{5} i_2 = 2i_1 + 4i_2 \ ; \qquad i_1 = -\frac{16}{5} i_2$$

$$V_1 = 3i_1 + 2i_2 = 3i_1 - \frac{5}{8} i_1 = \frac{19}{8} i_1$$

$$\boxed{R_{in} = \frac{19}{8} \ \Omega}$$

T network

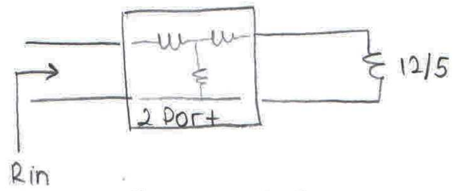




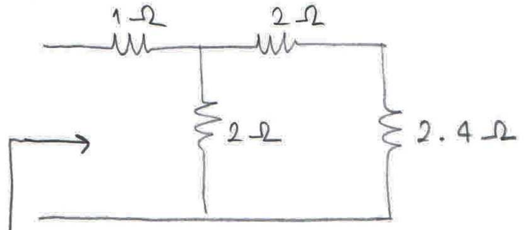
$$R = \begin{bmatrix} \overset{r_{11}}{R_X + R_y} & \overset{r_{12}}{R_y} \\ \underset{r_{21}}{R_y} & \underset{r_{22}}{R_y + R_z} \end{bmatrix}$$

$r_{12} = r_{21}$ (T network)

Let's go back previous problem



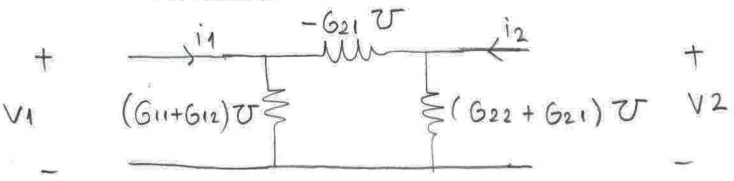
$$\underline{\underline{R}} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$



Rin $(2 + 12/5) // 2 + 1$

Note : T network requires a symmetric Rmatrix

π Network

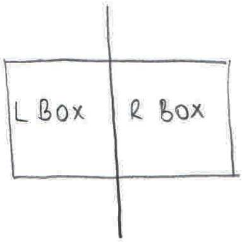


$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

conductance matrix

(requires symmetric $\underline{\underline{G}}$ matrix)

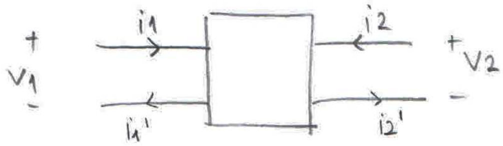
*



To apply Thevenin Equivalent, all circuit components in the port of the circuit whose equivalent is shought should be well defined, that is dependent sources should have their dependent variable branch in the port of the circuit where Thevenin is shought

Similarly, the both sides of the transformer should be in the same box.

Two Ports



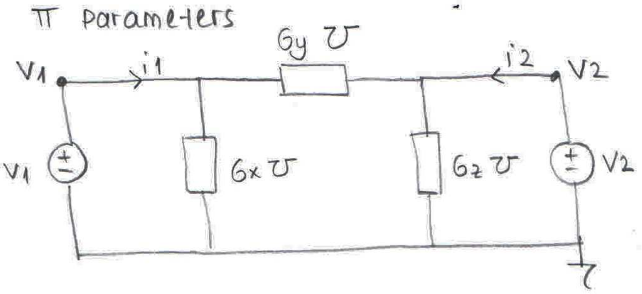
validity conditions: $i_1 = i_1'$

$i_2 = i_2'$

Main assumption : $\{ i_1, V_1, i_2, V_2 \}$ 4 variables

2 out of these 4 variables are inputs (independent variable)

The rest is determined from the outputs (so a function of inputs) linear mapping of inputs π parameters



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

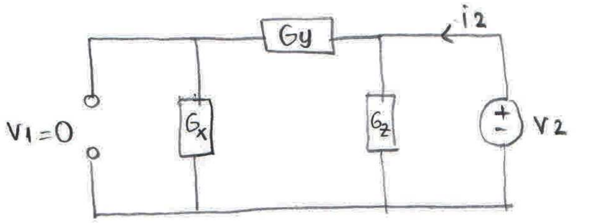
$$i_1 = G_x V_1 + G_y (V_1 - V_2)$$

$$i_2 = G_z V_2 + G_y (V_2 - V_1)$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} G_x + G_y & -G_y \\ -G_y & G_y + G_z \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

If you want to find only G_{22} then.

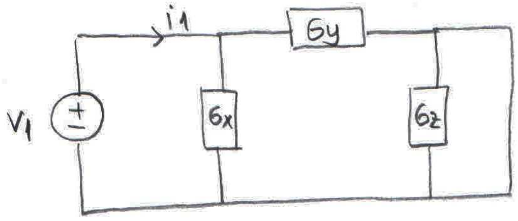
$$i_2 = G_{21} V_1 + G_{22} V_2 \underset{V_1 = 0}{\downarrow} \qquad i_2 = G_{22} V_2$$



$$\frac{V_2}{i_2} = R_{in} = R_y // R_z$$

$$G_{22} = G_y + G_z$$

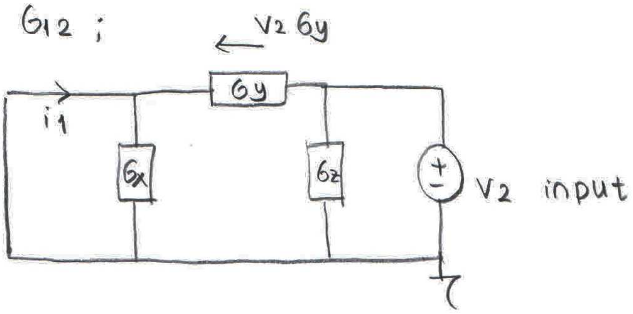
for G_{11} ; $i_1 = G_{11} V_1 + G_{12} V_2 \underset{V_2 = 0}{\downarrow}$



$$\frac{V_1}{i_1} = R_x // R_y$$

$$G_{11} = G_x + G_y$$

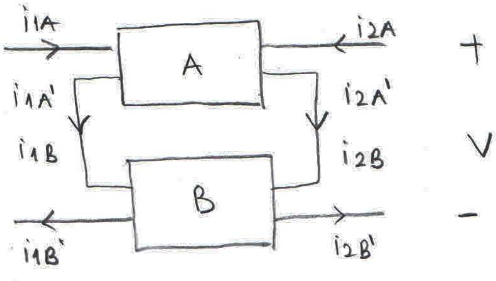
for G_{12} ;



$$i_1 = -G_y V_2$$

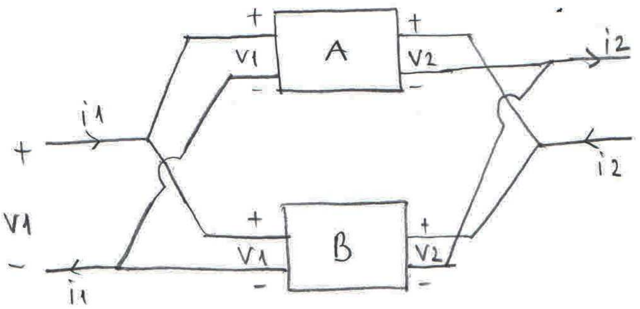
$$G_{12} = -G_y$$

Interconnection of Two-Ports



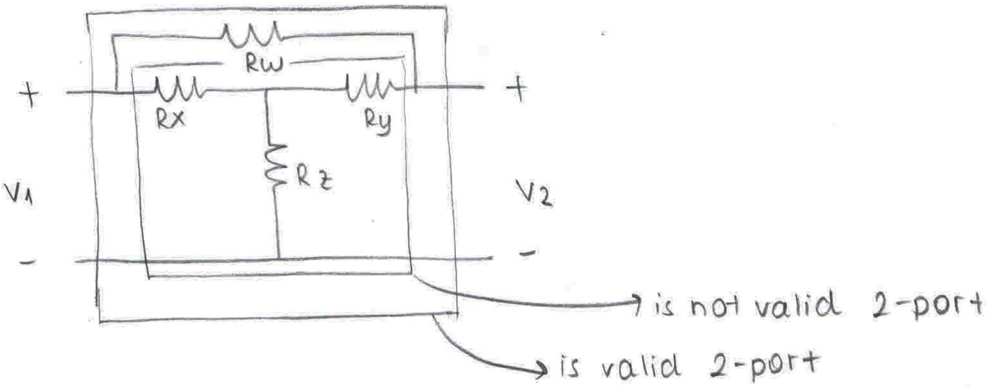
R parameters of the connected 2 ports;

$$R = R_A + R_B$$

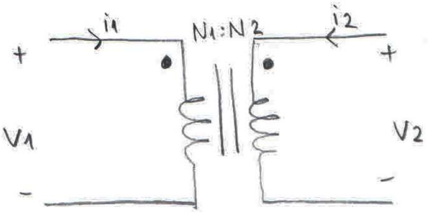


$G = G_A + G_B$

Two-Ports with Feedback



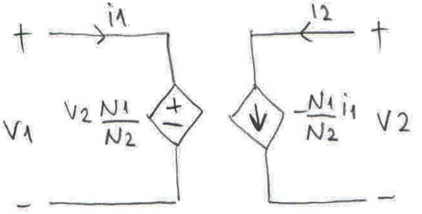
Transformers

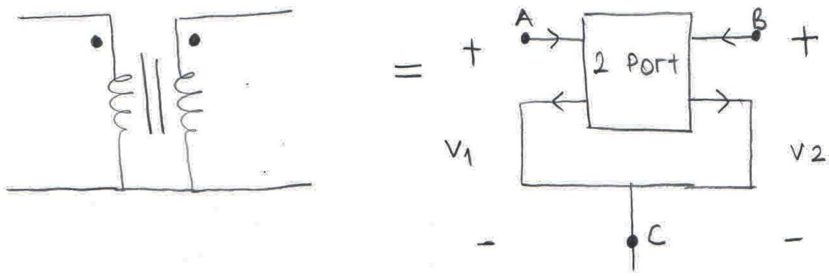


$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = -\frac{i_2}{i_1}$$

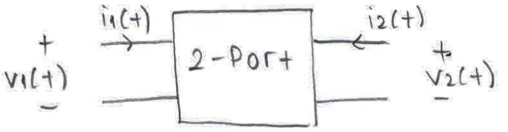
$$\begin{bmatrix} V_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} N_1/N_2 & 0 \\ 0 & N_2/N_1 \end{bmatrix} \begin{bmatrix} V_2 \\ -i_2 \end{bmatrix}$$

ABCD parameters





ABC, three terminal two port



$P_{2port} = V_1(t)\, i_1(t) + V_2(t)\, i_2(t)$ (Absorbed power) by two port

$= 0$

$P(t)$ of ideal transformer is zero (Lossless component)

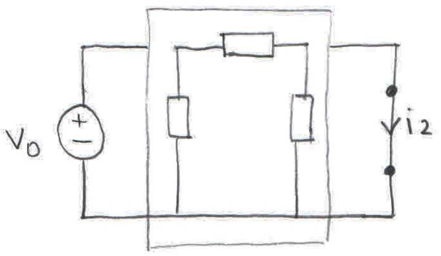
Reciprocity

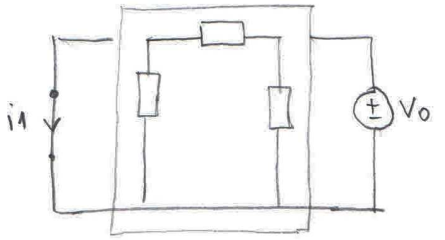
If a 2port doesnot contain a dependent component (source), then $\underline{\underline{R}}$ & $\underline{\underline{G}}$ matrices are guaranteed to be symmetric ($r_{12} = r_{21}$, $g_{12} = g_{21}$)

Under the same conditions; (no dependent source)

$h_{12} = -h_{21}$ (skew symmetric)

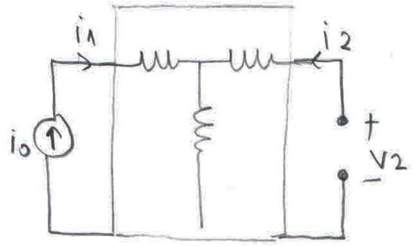
1-)

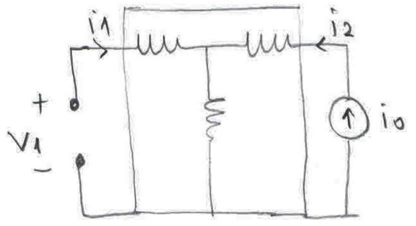




$i_1 = i_2$ since $g_{12} = g_{21}$

2-)

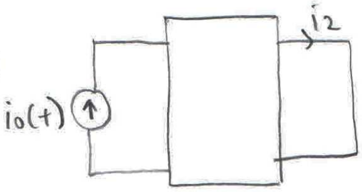


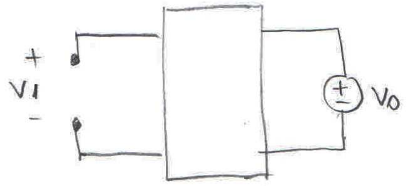


$V_1 = V_2$ since $r_{12} = r_{21}$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

3-)





$V_1 = i_2$ (numerically) since $h_{21} = -h_{12}$

Proof of 3rd case;

$V_1 = i_2$

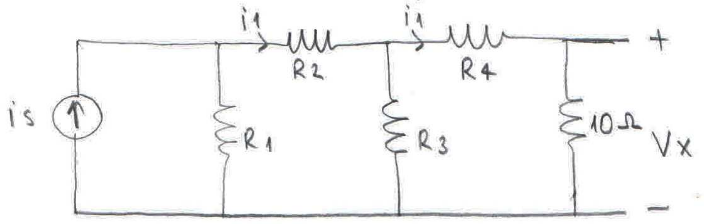
$\sum V_k \hat{i}_k = \sum \hat{V}_k i_k$ (2 port contain resistors)

$$V_1 \hat{i}_1 + V_2 \hat{i}_2 + \underbrace{\sum_{k=3} V_k \hat{i}_k}_{\text{for the branches in 2-port}} = \hat{V}_1 i_1 + \hat{V}_2 i_2 + \sum_{k=3} \hat{V}_k i_k$$

$$\cancel{V_1}\overset{0}{\hat{i}_1} + \cancel{V_2}\overset{0}{\hat{i}_2} = \hat{V}_1 i_1 + \hat{V}_2 i_2$$

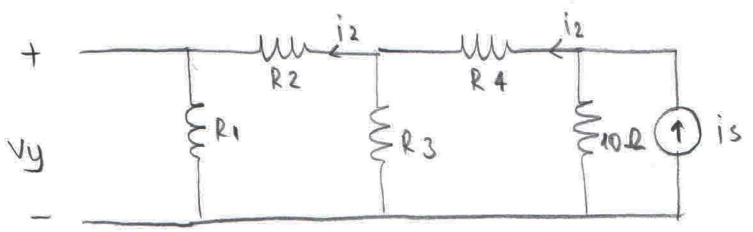
$$= \hat{V}_1(-i_0) + (-i_2) i_0 \quad ; \qquad \hat{V}_1 = i_2$$

EX :



Calculate R_1

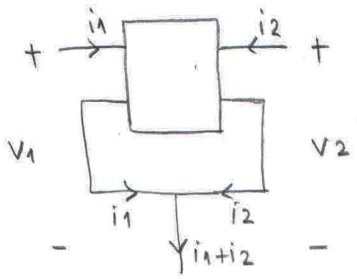
$i_1 = 0.6\, i_s$
$\hat{i}_1 = 0.3\, i_s$

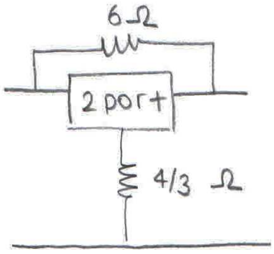


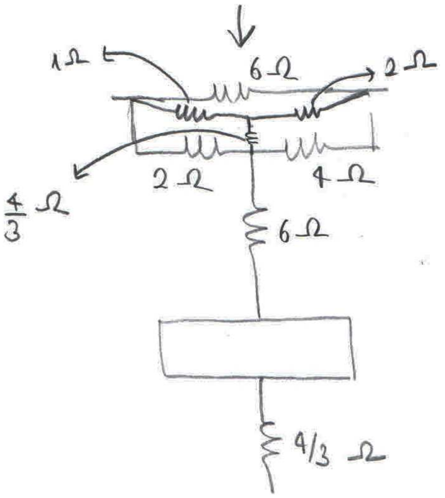
$i_2 = 0.2 i_s$
$\hat{i}_2 = 0.5 i_s$

Vx = Vy from reciprocity $\quad 10\, \hat{i}_1 = R_1 i_2 \quad R_1 = \dfrac{10\,\hat{i}_1}{i_2} = 15\,\Omega$

2PS II ; 10-) $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$







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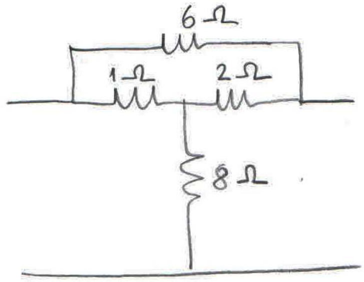
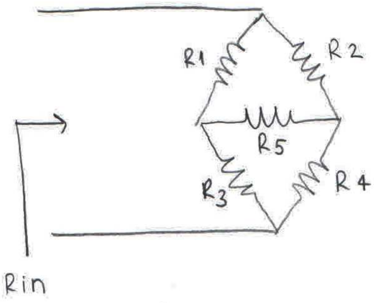


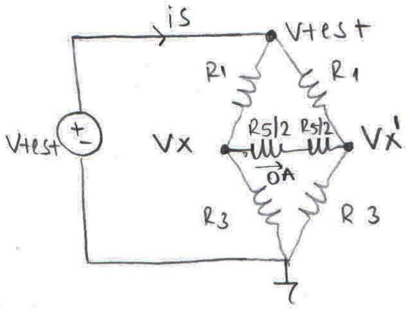
$$R = \begin{bmatrix} 9 & 8 \\ 8 & 10 \end{bmatrix}$$

Symmetric Circuits



1_ $R_1 = R_2$;

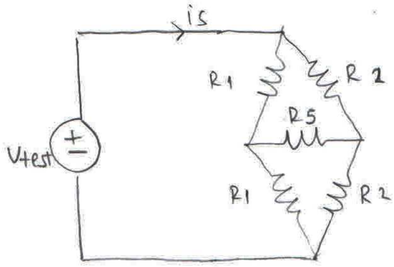
$R_3 = R_4$

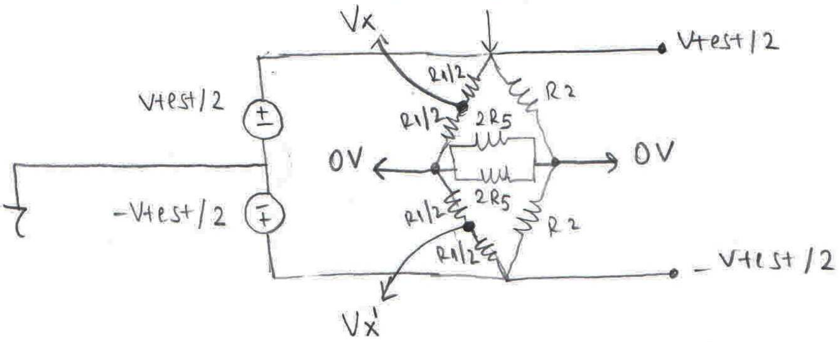


$$R_{in} = \left(\frac{R_1 + R_3}{2} \right)$$

2_ $R_1 = R_3$,

$R_2 = R_4$

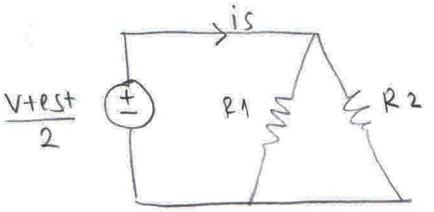




Due to symmetry $V_x = -V_x'$

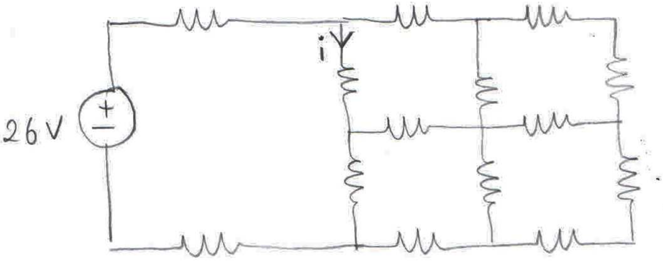
so symmetry axis   has 0V potentia

due to symmetry



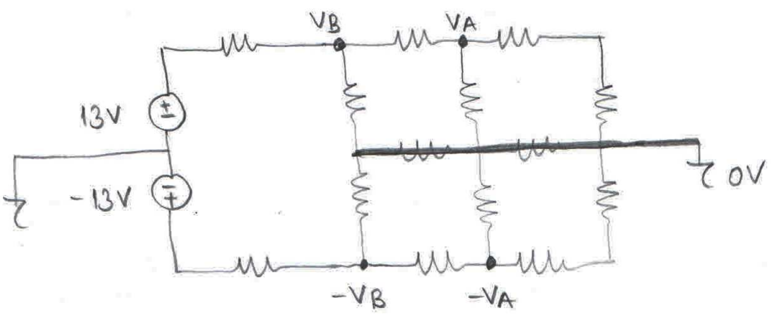
$R_{in} = \frac{V_{test}}{i_s} = 2(R_1 // R_2)$ since $i_s = \frac{V_{test}}{2(R_1//R_2)}$

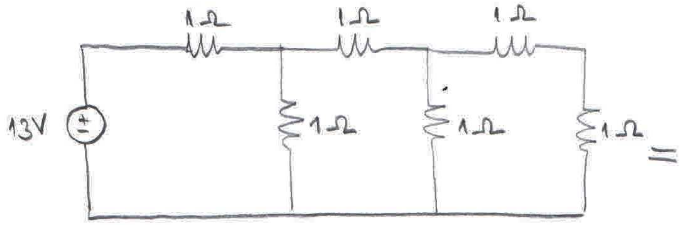
EX :



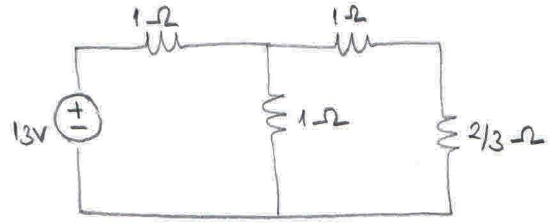
All resistors are 1Ω

Find i = ?





=



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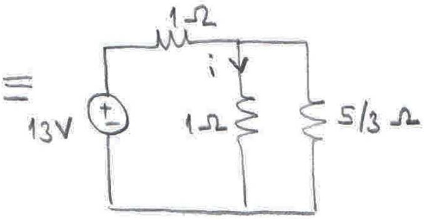

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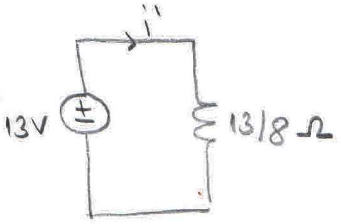
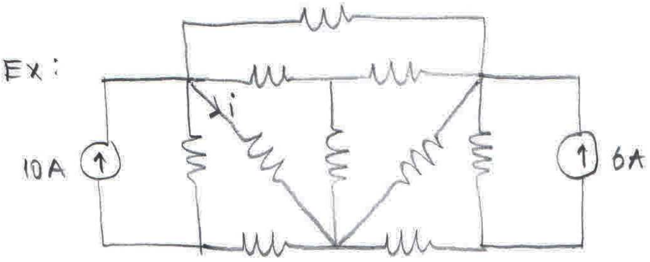


i' = 8A

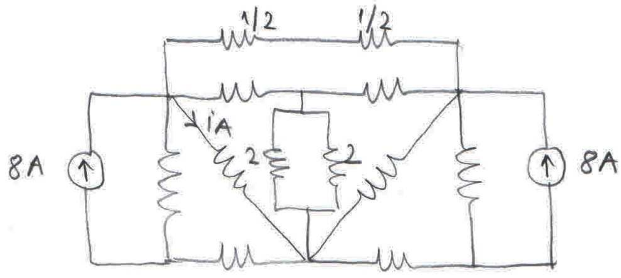
i = 5A

Ex:



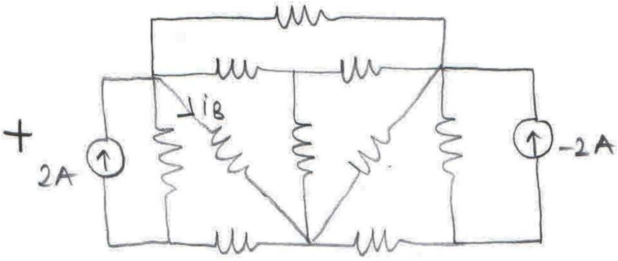
All resistors are 1 Ω Find i = ?

$$\begin{bmatrix} r_{11} & & r_{12} \\ & \swarrow\!\!\nearrow & \\ r_{21} & & r_{22} \end{bmatrix}$$ reciprocial



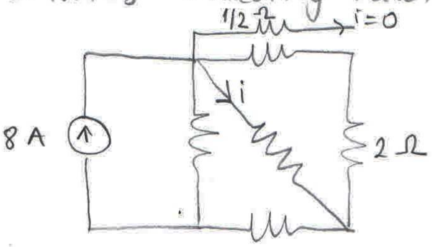
symmetric excitation

+

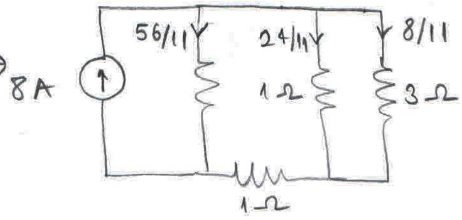


anti-symmetric excitation

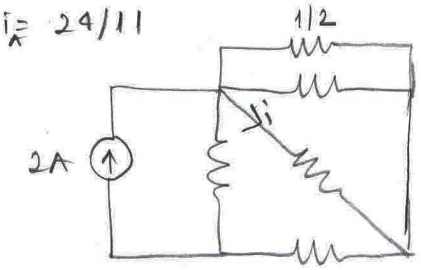
$i = i_A + i_B$ (linearity rule)

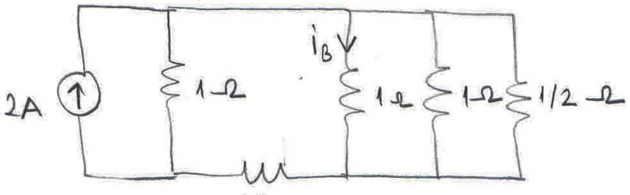


⟶



$i_A = 24/11$





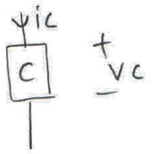
$i_B = 2/9$

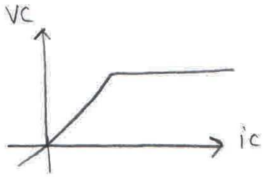
$i = i_A + i_B$

Diodes → Nonlinear Resistors

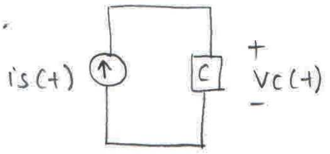
current controlled component $V_c = f(i_c)$

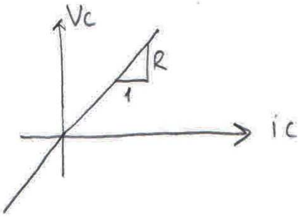
voltage controlled component $i_c = f(V_c)$



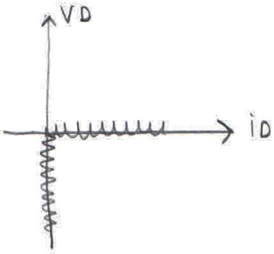


Current controlled: every current value, there is a voltage value.



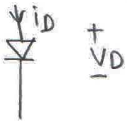


Both current and voltage controlled

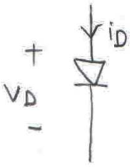


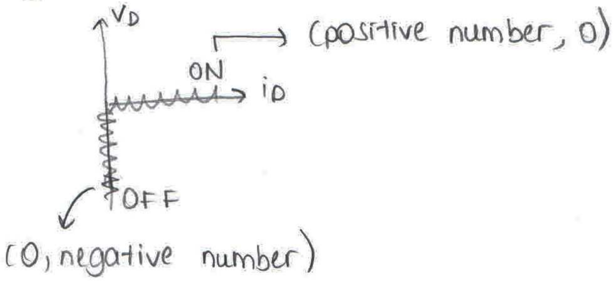
Neither current nor voltage controlled

Nonlinear



Operation Regions:





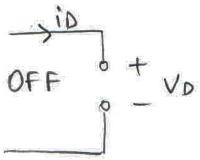
OFF → $i_D = 0$

ON → $V_D = 0$

for ideal diode

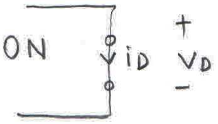
OFF → $i_D = 0$

provided that $V_D \leq 0$

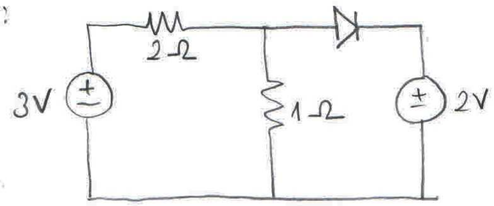


ON → $V_D = 0$

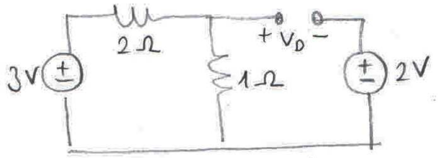
provided that $i_D \geq 0$



Ex:



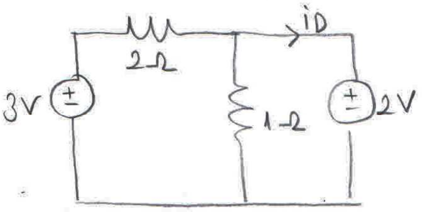
Assume OFF;



$V_D = -1 < 0$

correct assumption

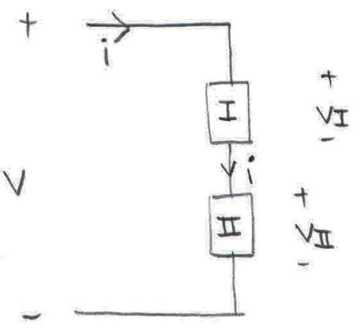
Assume ON;



$i_D = -3/2 < 0$ wrong assumption!

Series and parallel combination

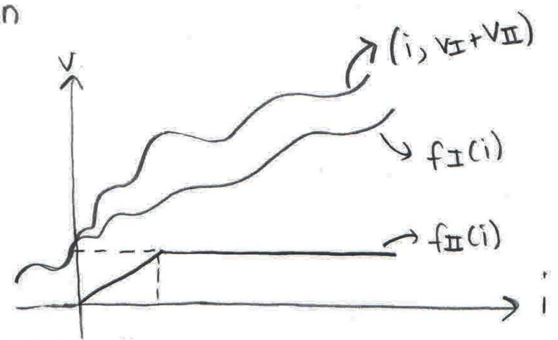
series :



$V = V_I + V_{II}$

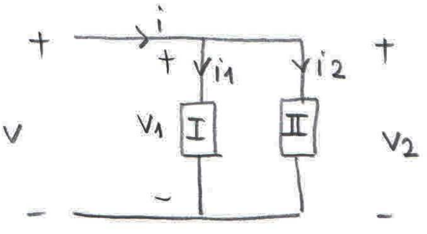
$V_I = f_I(i)$

$V_{II} = f_{II}(i)$

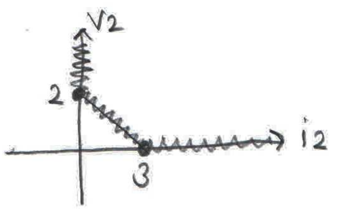


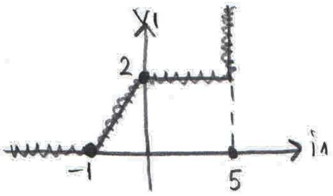
graphs are added vertically

Parallel :



constraints : $V = V_1 = V_2$ $i = i_1 + i_2$

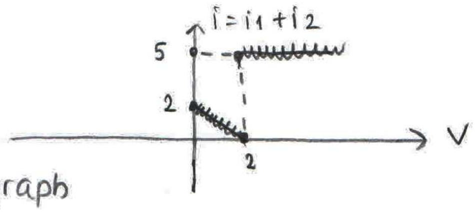




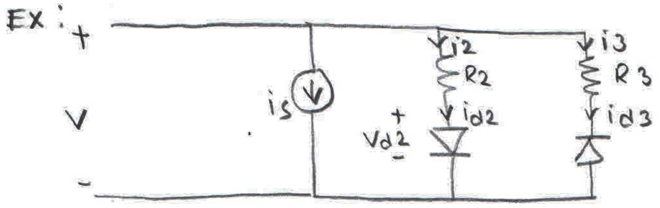
$i_2 = g_2(V_2)$ $i_1 = g_1(V_1)$

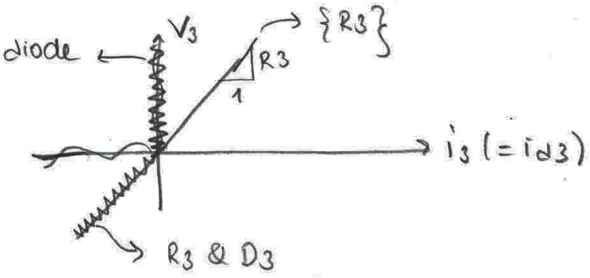
Graphs are added horizontally

Note : From these 2 graphs, this is parallel combination i − v is calculated by vertically adding the graph

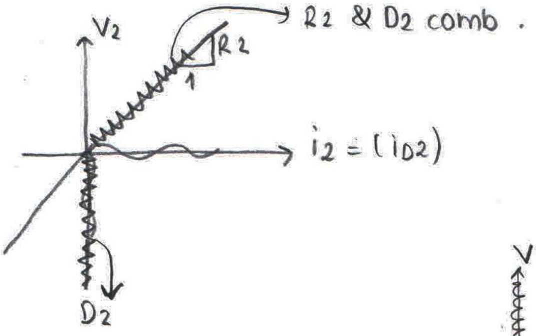


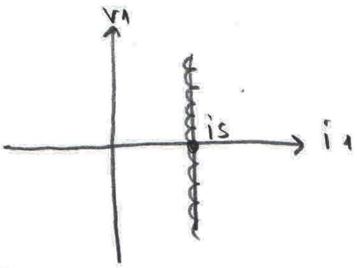
EX :





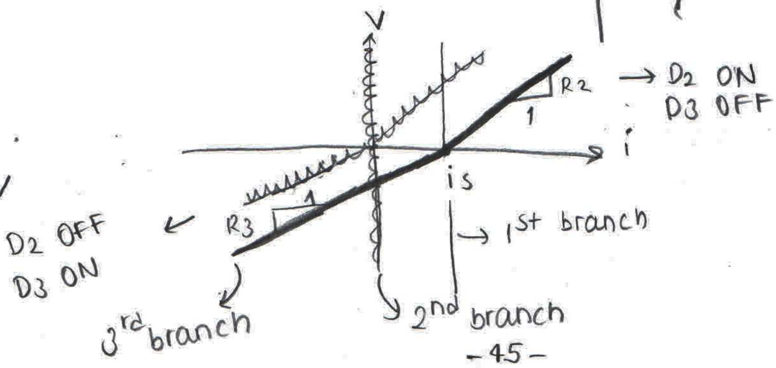
Diodes are ideal



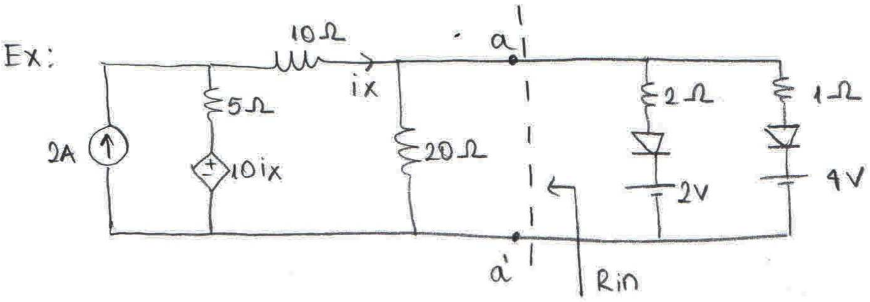


combine all .

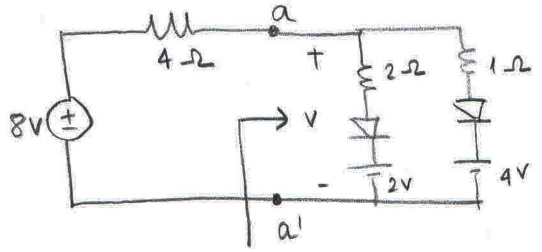
$V_1 = V_2 = V_3 = V$



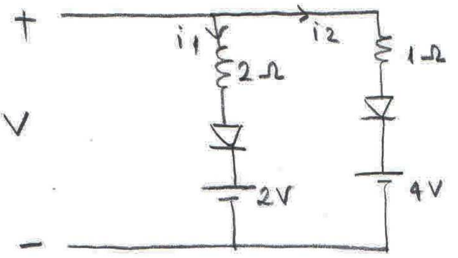
Ex:

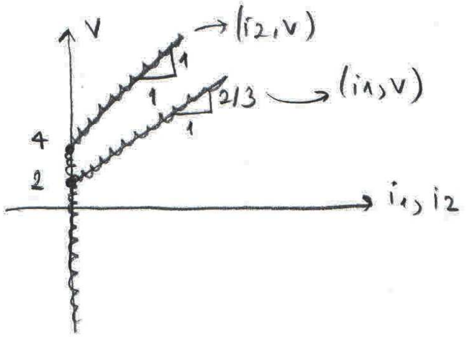


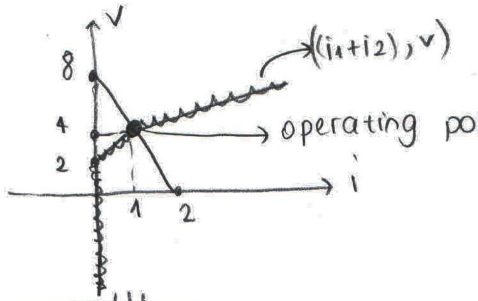
Thevenin equivalent of the LHS of a-a'



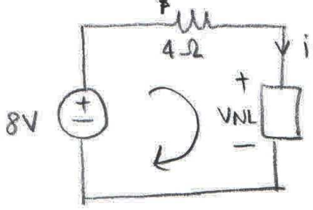
So let's find i-v characteristic of RHS of a-a'







→ operating point $(i_{NL}, V_{NL}) = (1A, 4V)$

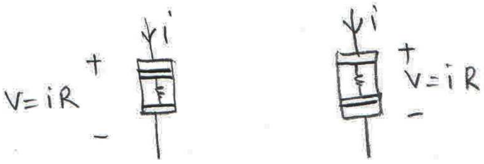


$$-8 + 4i_{NL} + V_{NL} = 0$$

$$V_{NL} = 8 - 4i_{NL}$$

Bilateral Component

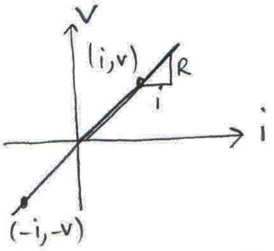
Simple LTI resistors are bilateral. That is if you flip the resistor, you get the same i-v characteristic



so if you change $i \to -i$

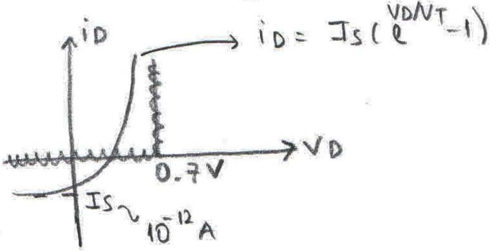
$V \to -V$

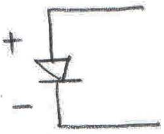
In other words, both i-v and (-i)-(-v) should be on the i-v characteristics for bilateral component



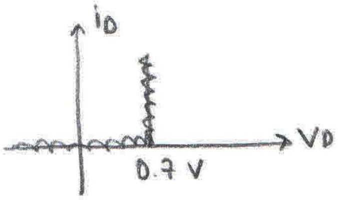
So; bilateral components has symmetry across the origin

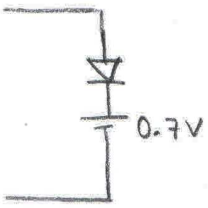
Practical Diody



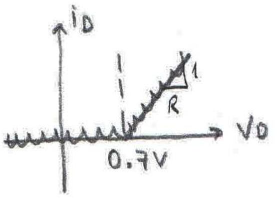


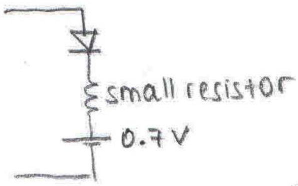
practical diody





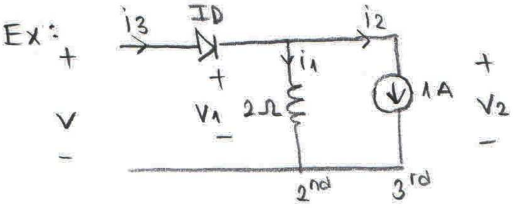
practical diode

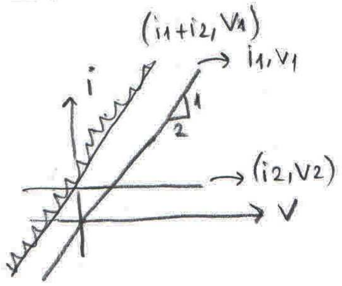




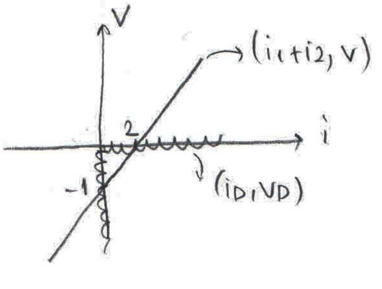
Another practical diode

Ex:





for 2nd and 3rd branches



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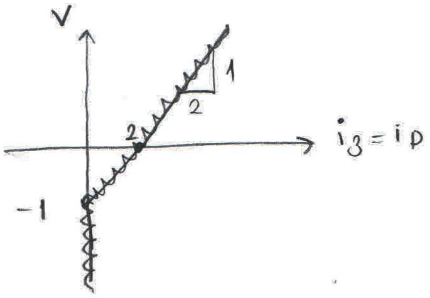
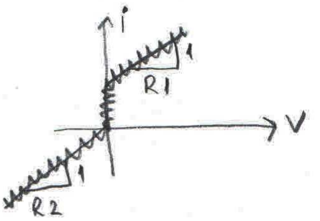
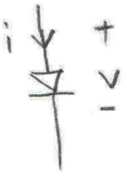


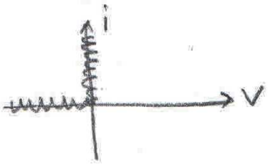
Synthesis

Design a one port with R's and independent sources and ideal diodes such that we have the desired i-v characteristics .

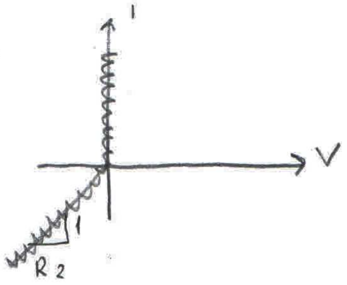
EX :



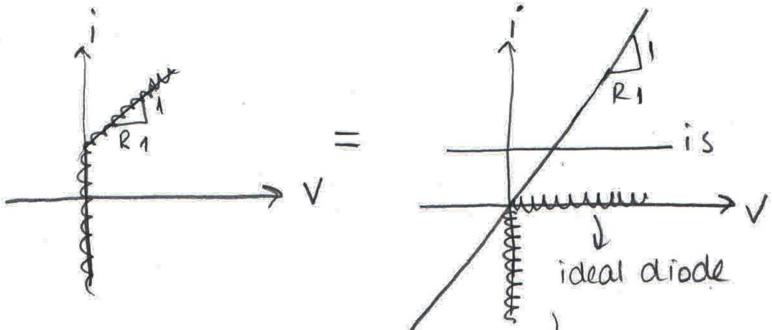




1_

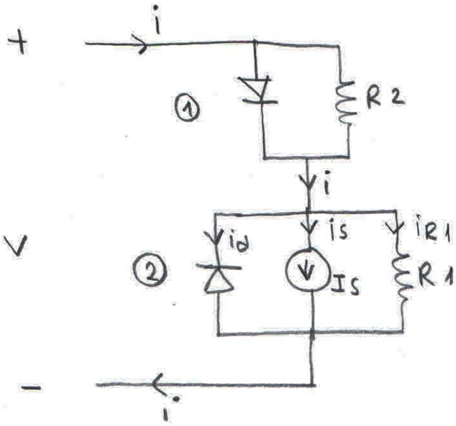


2-

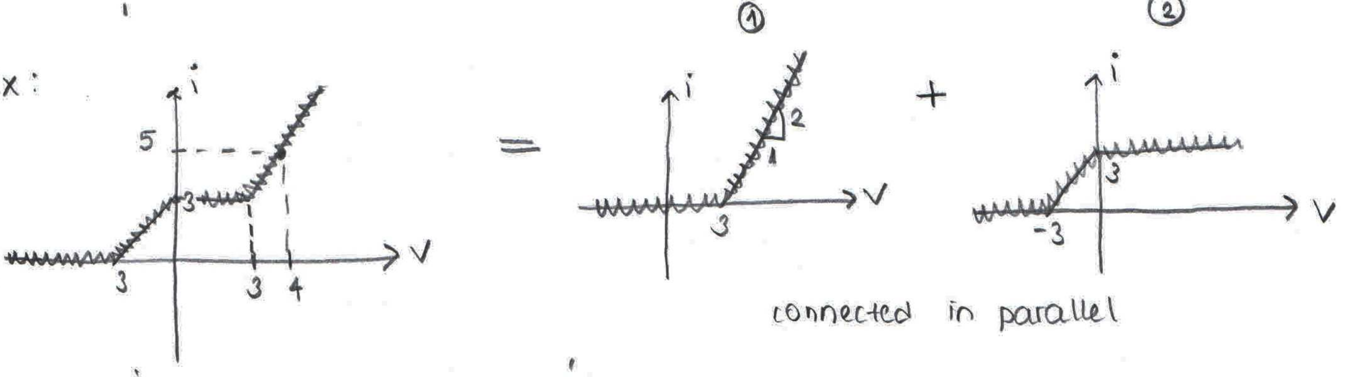


1 & 2 are connected in series.

parallel connected

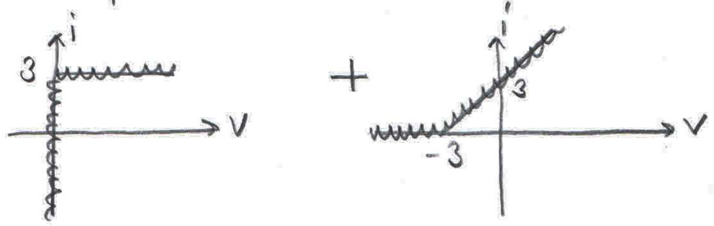


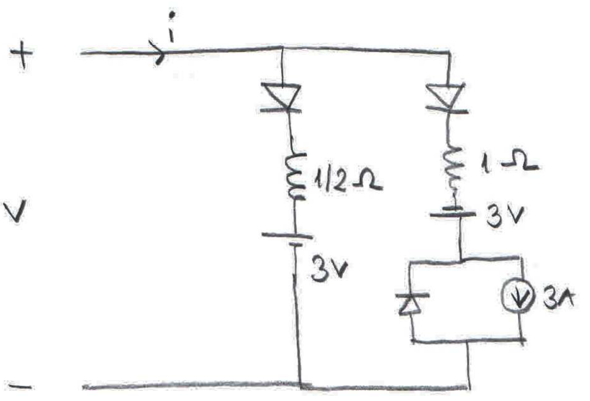
Ex :



connected in parallel

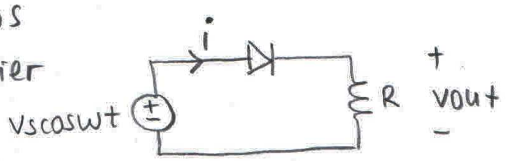
② =

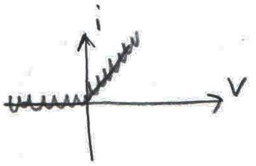


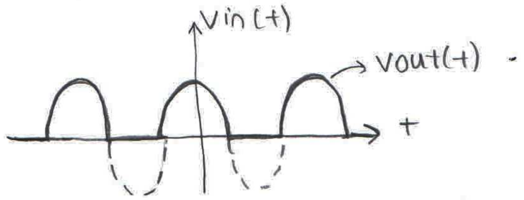


Diode Applications

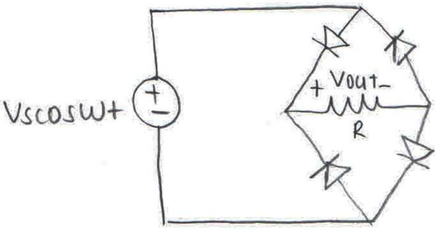
Half-Wave rectifier

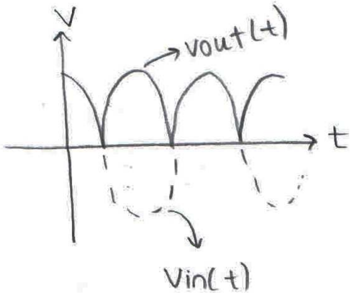




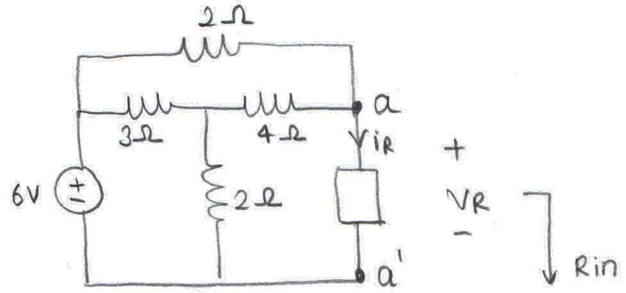


Full wave rectifier:





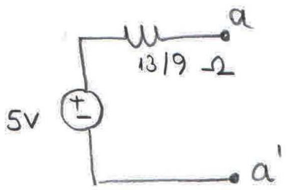
Ex:



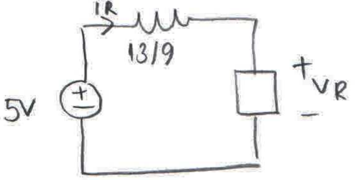
$i_R = \begin{cases} 0.03\, V_R^2, & V_R \geq 0 \\ 0, & V_R < 0 \end{cases}$

Find Thevenin equivalent and then solve for i_R using load lines or algebraically.

Thevenin equivalent:



Algebraic method:



by KVL; $-5 + 13/9\ i_R + V_R = 0$

Let's assume $V_R > 0$

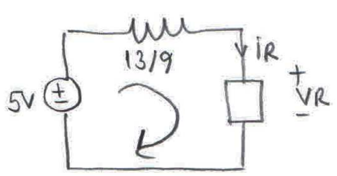
$-5 + 13/9\,(0.03 V_R^2) + V_R = 0$

$V_R = \{$ 4.22V; a negative number $\}$

$V_R = 4.22$ V is a solution

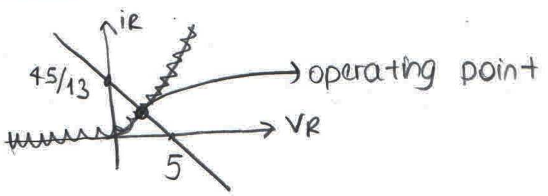
$i_R = 0.03\,(4.22)^2$

Load line Method:



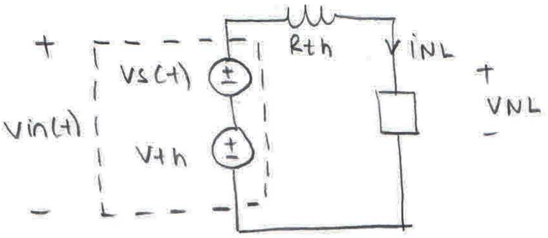
$-5 + (13/9)\, i_R + V_R = 0$ 2 equations, 2 unknowns

$i_R = \begin{cases} 0.03\, V_R^2, & V_R \geq 0 \\ 0 & V_R < 0 \end{cases}$



} load line technique

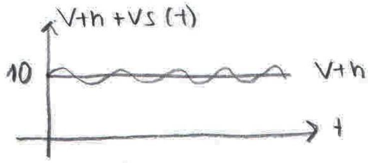
Small-Signal Analysis



Vth is a DC voltage

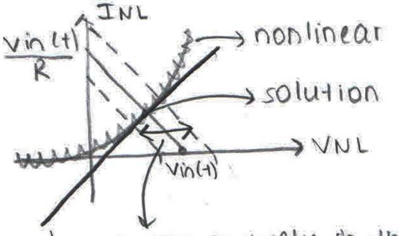
vs(t) is a time function

we assume that $V_{th} \gg v_s(t) \quad \forall t$



$V_{th} = 10V$, $v_s(t) = 0.1 \sin(\omega t)$

As in previous problem,



nonlinear component

solution @ t=t1

solution is surely in this segment (since vin changes over t)

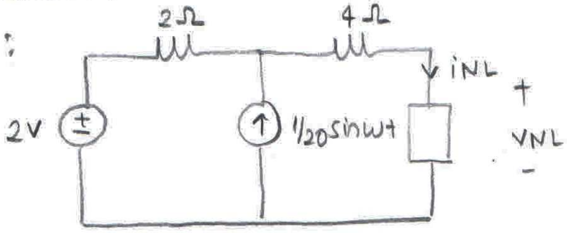
approximation of NL component around DC operating point

1- Find DC operating point

2- Express Taylor series expansion of NL function around DC operating point and take the linear term only

3- Using the slope in the Taylor series expansion, find the solution for alternati component.

Ex :

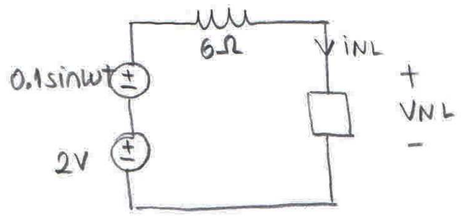


$$i_{NL}(t) = \begin{cases} V_{NL}^2, & V_{NL} \geq 0 \\ 0 & V_{NL} < 0 \end{cases} = f(V_{NL})$$

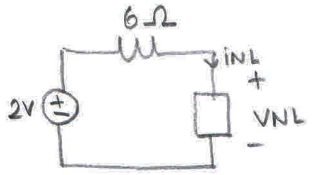
$$i_{NL}(t) = f(V_{NL}) = \sum_{k=0}^{\infty} \frac{f^k(t_0)}{k!} (t-t_0)^k$$

$$f(V_{NL}) = V_{NL}^2 = \begin{cases} 1 + \frac{2}{1!}(V_{NL}-1) + \frac{2}{2!}(V_{NL}-1)^2 & \text{expansion around } V_0 = 1 \\ 4 + \frac{4}{1!}(V_{NL}-2) + \frac{4}{2!}(V_{NL}-2)^2 & \text{expansion around } V_0 = 2 \end{cases}$$

1-Find DC operating point



for DC operating point;



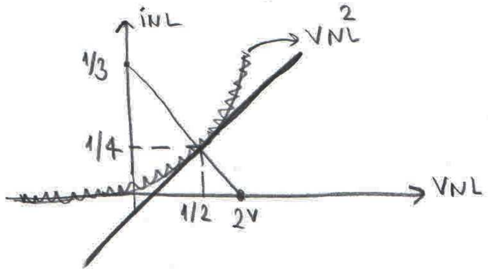
$-2+6I_{NL}+V_{NL}=0 \qquad V_{NL}^2=I_{NL}$ for $V_{NL}\geqslant 0$

$V_{NL}=\{1/2,\ -2/3\}$ for $V_{NL}\geqslant 0$

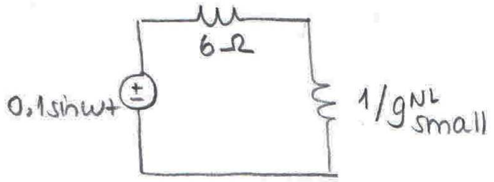
then $V_{NL}=1/2$ is the solution

2- Then expand nonlinearity around $V_o=1/2$ (DC operating point)

$$V_{NL}=\frac{1}{4}+\frac{1}{1!}(V_{NL}-1/2)+\frac{2}{2!}(V_{NL}-1/2)^2$$

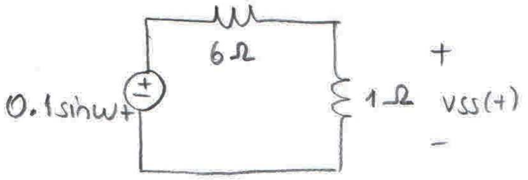


Then, small signal model;



$$g^{NL}_{small}=f'(V_o)=2V_o\Big|_{V_o=1/2}=1\ \mho$$

small signal conduction parameter

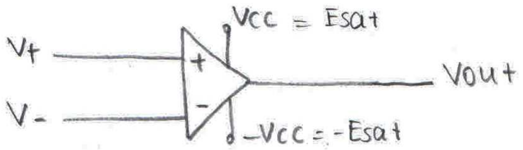


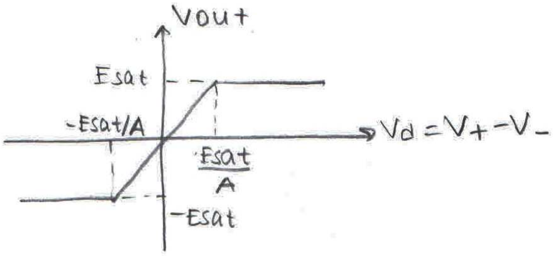
$v^{ss}(t)=1/7\ (0.1\sin wt)$

$V_{NL}(t)=1/2+1/70\ \sin wt$ → solution due to alternat input

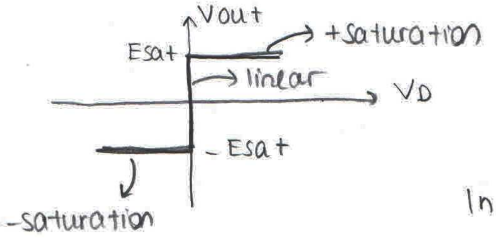
↑ DC operating point

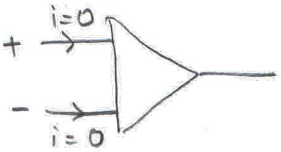
Operational Amplifier





Ideal Op-Amp Model $A=\infty$

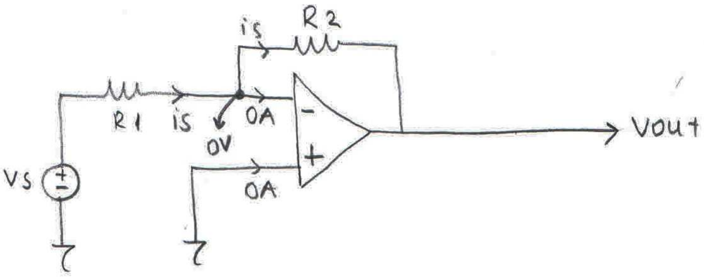




$[R_{in}=\infty,\ r_{out}=0]$

In linear region; $V_D=0$, $V_+=V_-$ and $|V_{out}|<E_{sat}$

Inverting Amplifier

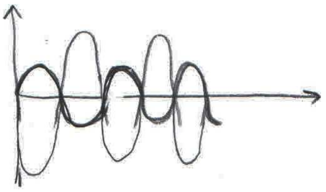


Assume that Op-Amp is in linear region. Vout= ?

$i_s = \frac{V_s - 0}{R_1}$ $V_{R2} = i_s R_2 = \frac{R_2}{R_1} V_s$

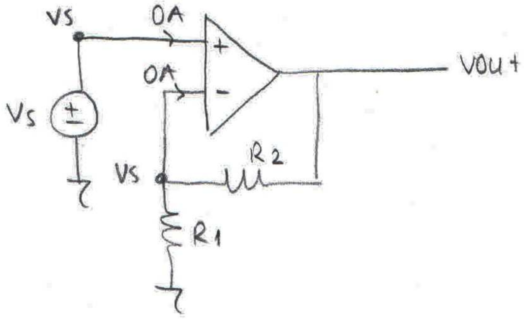
$V_{out} = 0 - V_{R2} = -\frac{R_2}{R_1} V_s$

$\frac{V_{out}}{V_s} = -\frac{R_2}{R_1}$ inverting ($R_2/R_1 = 2$)



The analysis (Vout) is correct if $|V_{out}| < E_{sat} \frac{R_2}{R_1}$ (Validity condition)

Noninverting Amplifier

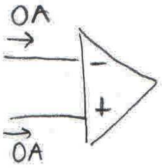


Assume that ideal Op-Amp in linear region Find vout

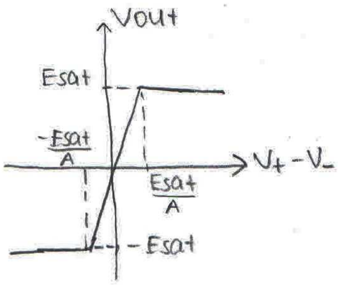
$\frac{-V_s}{R_1} = \frac{V_s - V_{out}}{R_2}$; $V_{out} = \frac{R_1 + R_2}{R_1} V_s$

noninverting amplifier

Improved Model



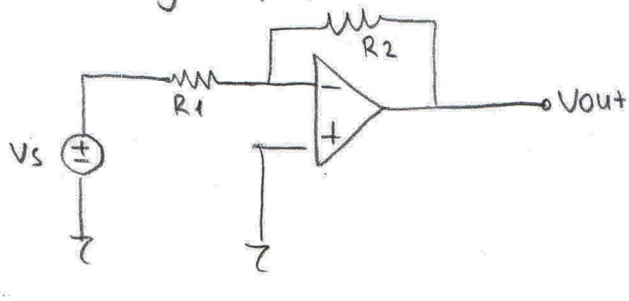
$R_{in} = \infty$ $r_o = 0$



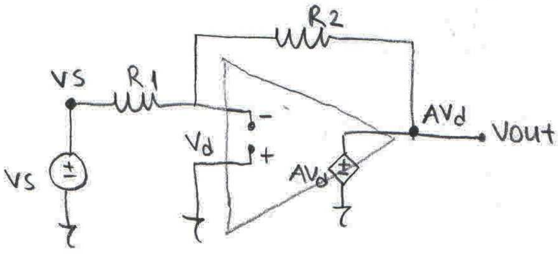
Finite gain (A) model

A: open loop gain

Inverting Amplifier



1- Assume Op-Amp in linear region



$\frac{Vs+Vd}{R_1} = -\frac{Vd-AVd}{R_2}$, $R_2Vs+R_2Vd = -R_1Vd-R_1AVd$

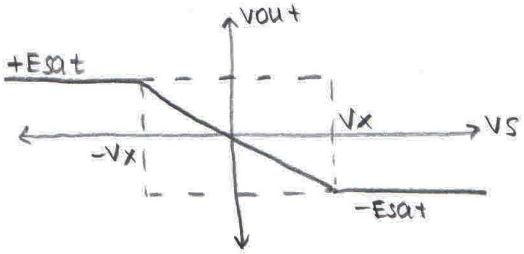
$(R_1A+R_2+R_1)V_d = -R_2Vs$; $V_d = \frac{-R_2}{R_1A+R_1+R_2}Vs$ in linear region

$Vout = AVd = \frac{-AR_2}{R_1A+R_1+R_2}Vs$

as $A \to \infty$ $Vout \longrightarrow -\frac{R_2}{R_1}Vs$ (as previously found)

Linear region assumption is valid.

$|Vout| < Esat \longrightarrow |Vin| < \frac{R_2+(1+A)R_1}{AR_2}Esat$

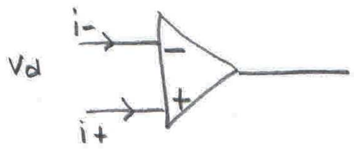


$Vx = \frac{R_1(A+1)+R_2}{AR_2}Esat$

$Vout = \frac{-AR_2}{R_1(A+1)+R_2}Vs$

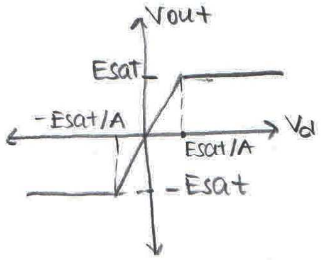
2- Assume $-Esat$;

$Vout = -Esat$

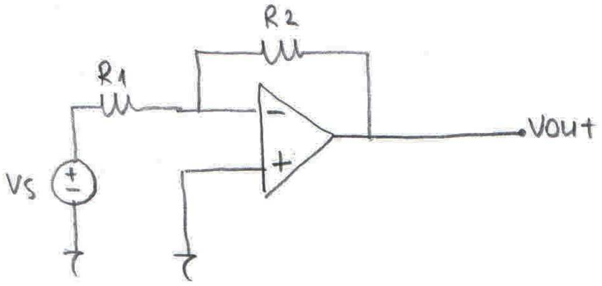


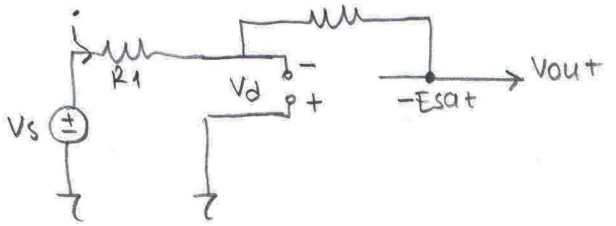
$i_- = i_+ = 0$ A

Validity condition $Vd < \frac{-Esat}{A}$



Assume $-Esat$





$Vs - iR_1 = -Vd$

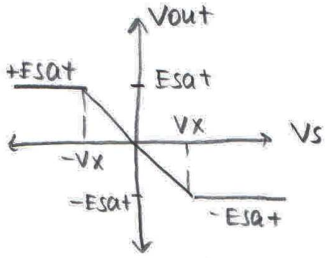
$i = \frac{Vs+Esat}{R_1+R_2}$

$Vd = \frac{R_1(Vs+Esat)}{R_1+R_2} - Vs$

So $-Esat$ region is valid when $Vd < -\frac{Esat}{A}$

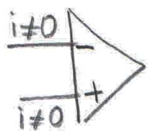
$\frac{R_1(Vs+Esat)}{R_1+R_2} - Vs < -\frac{Esat}{A} \Rightarrow AR_1(Vs+Esat) - A(R_1+R_2)Vs < -(R_1+R_2)Esat$

$V_s > \frac{E_{sat}(R_1+R_2)+AR_1E_{sat}}{AR_2}$. $V_s > \frac{R_1(1+A)+R_2}{AR_2} E_{sat}$ $v_{out} = -E_{sat}$ if condition is satisfi



Transfer (input-output) characteristic.

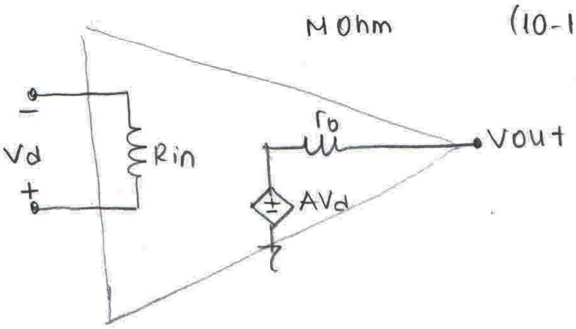
Further improved Model



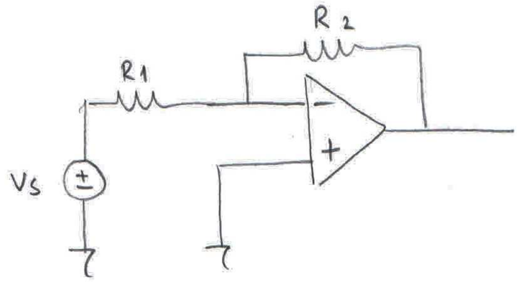
Linear Region

typical values: Rin: finite   ro: finite   A: finite

M Ohm   (10-100Ω) 10^6

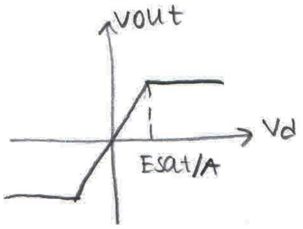


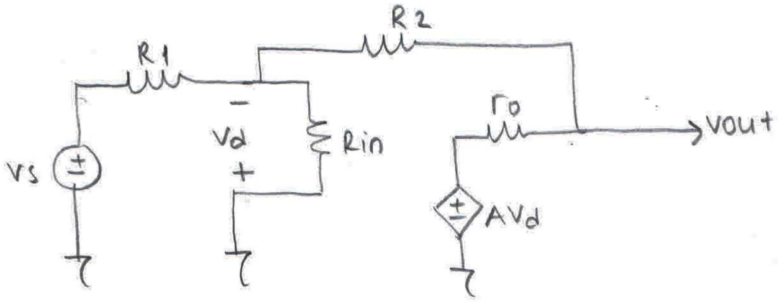
Inverting Amplifier (nonideal)



1- Assume linear (Rin, ro, A : finite) and

$|V_d| < \frac{E_{sat}}{A}$





For no load condition that at the output of Op-Amp there is nothing connected

Then $i_{load} = 0$

Write KCL at $-V_d$ node
at Vout node } solve for V_d, Vout

$$v_{out} = \frac{-A + r_o/R_2}{\frac{R_1}{R_2}\left(1+A+\frac{r_o}{R_{in}}\right)+\left(\frac{R_1}{R_{in}}+1\right)+\frac{r_o}{R_2}} \cdot V_s \quad *$$

In (*) as $A \to \infty$ $V_{out} = -\frac{R_2}{R_1} V_s$, $V_{out} = \frac{-A}{\frac{R_1}{R_2}(1+A)+1}$ as A finite V

$(r_o = 0,\ R_{in} \to \infty)$ result for 1st Model   (result for 2nd Model)

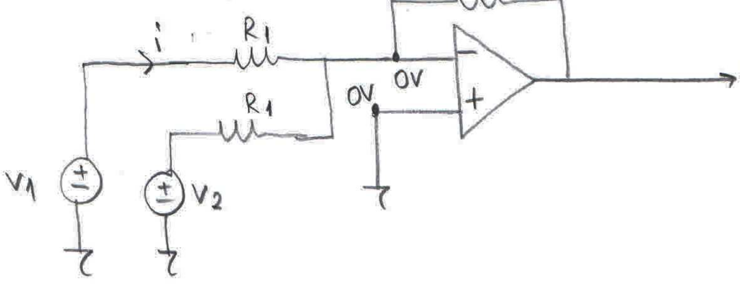
So if R_1, R_2 in the inverting amplifier $\{R_1, R_2\} \gg r_o$

then $\{R_1, R_2\} \ll R_{in}$

Operational Amplifier Applications

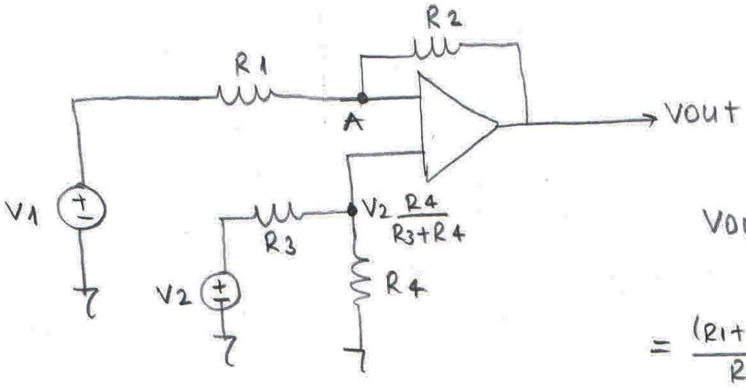
Assume linear region for Op-Amp and take $A \to \infty$ i.e. the ideal Op-Amp model in the following circuits.

1- Summer Amplifier



$$V_{out} = -\frac{R_2}{R_1}V_1 - \frac{R_2}{R_1}V_2 = \frac{-R_2}{R_1}(V_1+V_2)$$

2. Difference Amplifier



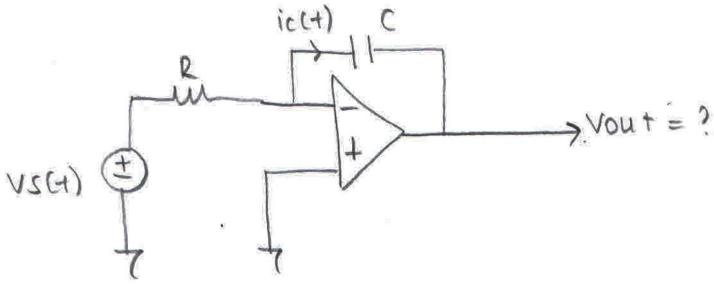
KCL at A ;

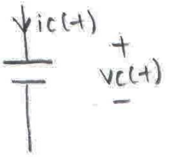
$$\frac{V_A - V_1}{R_1} + \frac{V_A - V_{out}}{R_2} = 0$$

$$V_{out} = \frac{(R_1+R_2)V_A - R_2V_1}{R_1}$$

$$= \frac{(R_1+R_2)}{R_1}\frac{R_4}{R_3+R_4}V_2 - \frac{R_2}{R_1}V_1 = V_{out}$$

3- Integrator Amplifier





$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$i_c(t) = V_s/R$

$v_c(t) = -V_{out}$; $\frac{V_s(t)}{R} = C\frac{d}{dt}(-V_{out})$

$$\frac{d}{dt}V_{out}(t) = -\frac{1}{RC}V_s(t)$$

Integrate this equation to find Vout

1- Assume $V_c(0^-) = V_o$ is given

$$\int_{0^-}^{t}\frac{d}{dt}v_{out}(t)\,dt = -\frac{1}{RC}\int_{0^-}^{t}V_s(\tau)d\tau \quad ; \quad v_{out}(t) - v_{out}(0^-) = -\frac{1}{RC}\int_{0^-}^{t}V_s(\tau)d\tau$$

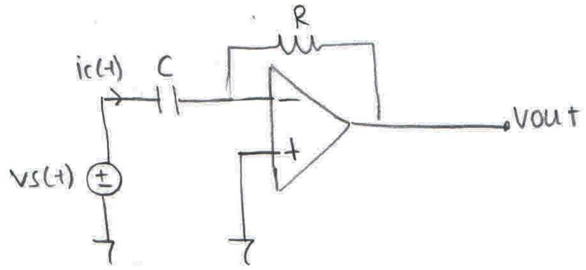
$V_{out}(t) = V_{out}(0^-) - \frac{1}{RC}\int_{0^-}^{t} V_s(\tau)\,d\tau$ $V_{out}(0^-) = -V_{cap}(0^-) = -V_0$

2- An initial condition is not provided $V_{cap}(-\infty) = 0\,V$

Integrate between $(-\infty)$ and (t)

$V_{out}(t) = V_{out}\underset{0}{\cancel{(-\infty)}} - \frac{1}{RC}\int_{-\infty}^{t} V_s(\tau)\,d\tau = \frac{-1}{RC}\int_{-\infty}^{t} V_s(\tau)\,d\tau$

4- Differentiator



$i_c(t) = C\,\frac{dV_{c}(t)}{dt}$ $\xrightarrow{} V_s(t)$

$V_{out} = -R\,i_c(t) = -RC\,\frac{dV_c(t)}{dt}$

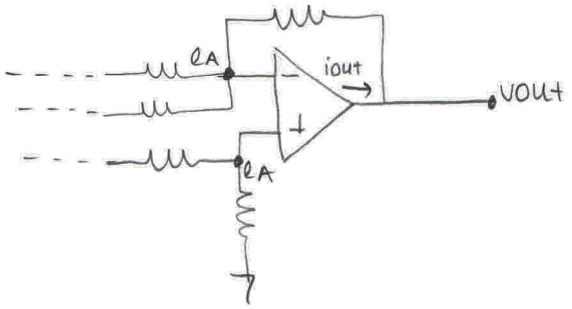
$V_{out}(t) = -RC\,\frac{d}{dt}V_s(t)$

Node Analysis with Op-Amps

Important notes: Never write a KCL equation at the output of the Op-Amp

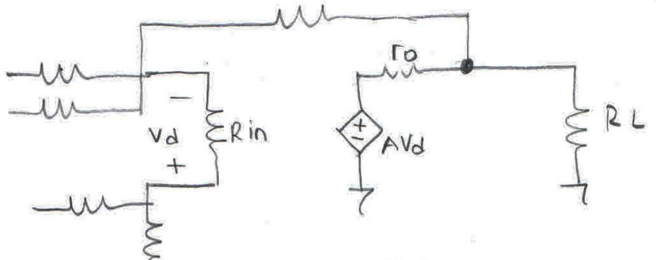
$i_{out}(t)$ = Op-Amp's output current is an unknown. so KCL at Vout can't be writte

1-



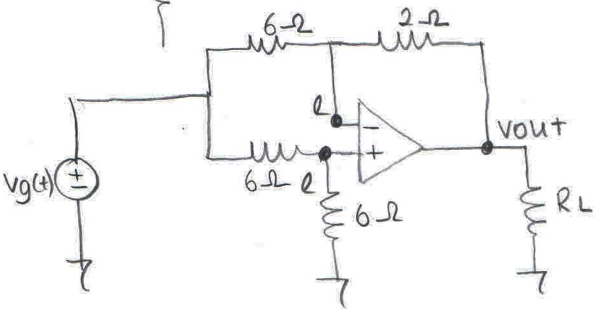
Linear Region $A \rightarrow \infty$

2- If A is finite and r_o is given



KCL at output can be written

Ex:



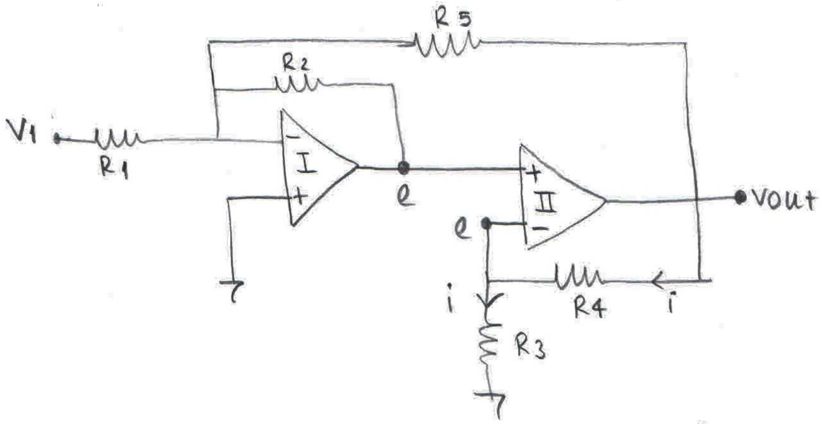
Rin, ro, AVd given

Find Vout. assume in linear region (A→

KCL @ $V_- = \frac{e - V_g(t)}{6} + \frac{e - V_{out}}{2} = 0$

KCL at V+; $\frac{e}{6} + \frac{e-V_g(t)}{6} = 0$ $e = V_g(t)/2$ $V_{out}(t) = 1/3\ V_g(t)$

Ex :



KCL @ V- of II : $\frac{e}{R_3} + \frac{e - V_{out}}{R_4} = 0$ $e = \frac{R_3}{R_3+R_4} V_{out}$

KCL @ V- of I $-\frac{V_1}{R_1} - \frac{e}{R_2} - \frac{V_{out}}{R_5} = 0$ $V_{out} = \frac{-R_5R_2(R_3+R_4)V_1}{[R_2(R_3+R_4)+R_3R_5]R_1}$

For the same problem; find input range (in Volts) so that both Op-Amps guaranteed to be in linear region

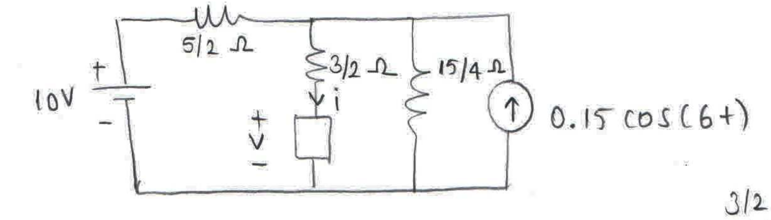
Op-Amp I $\xrightarrow{\text{linear}}$ $|e| < E_{sat}$ $|V_1| < Y$] number

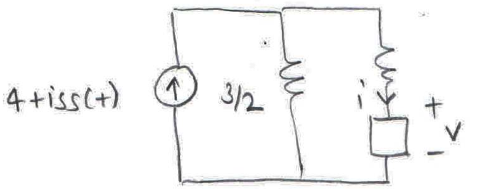
Op-Amp II $\xrightarrow{\text{linear}}$ $|V_{out}| < E_{sat}$ $|V_1| < X$

That intersection of two intervals gives me the input range : $|V_1| < \min(X, Y)$

Ex : ZPS III , 4a ;

$$i = \begin{cases} \frac{1}{9}v^2 & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$





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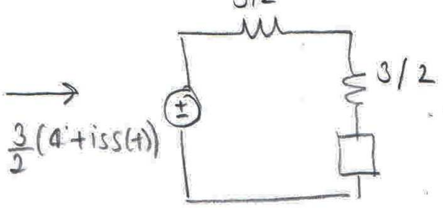

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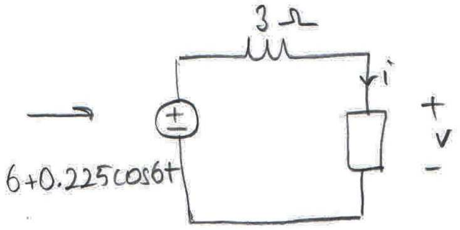
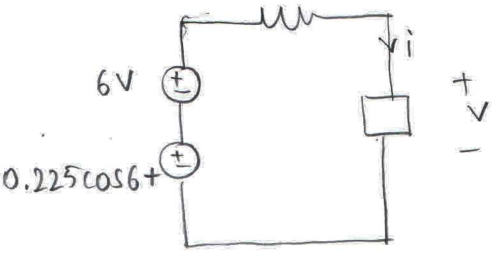


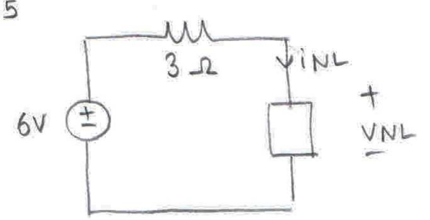


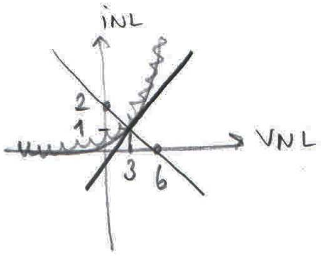
$-v_s(t) + 3i + v = 0$

$v = -6.3\ V$

AC component is sufficiently small in comparison the DC components

$\frac{6}{0.225} \sim 30$

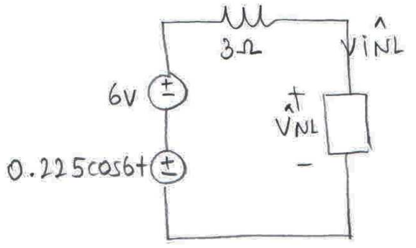




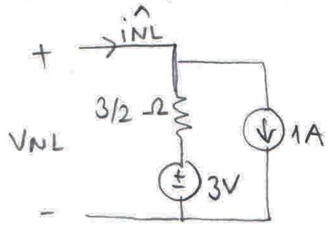
Slope = 2/3

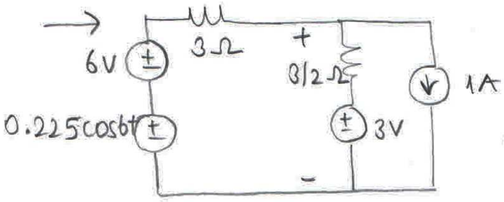
$y = 1+2/3\,(x-3)$ eqn of this lin

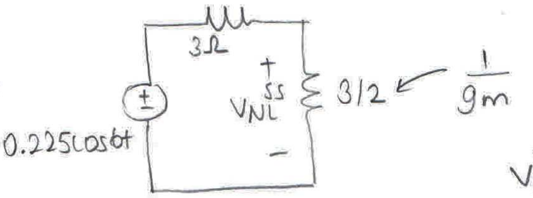
$i_N = 1+2/3\,(V_{NL}-3)$



$\hat{i}_{NL} = 1+2/3\,(\hat{V}_{NL}-3)$





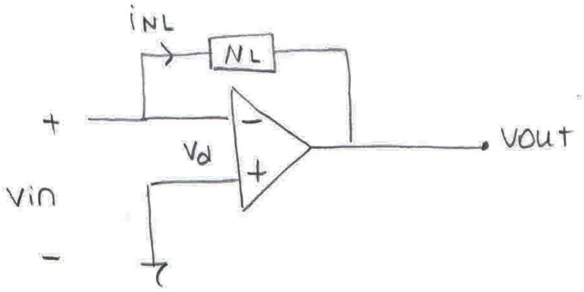


$V_{NL}^{SS} = \frac{0.225}{3}\cos 6t$

slope at operating point

$\hat{V}_{NL} = \hat{V}_{NL}^{DC} + \hat{V}_{NL}^{SS}$

Op-Amps with nonlinear Components



Assume Op-Amp in +Sat

Vout = Esat $V_{NL} = V_- - Esat \rightarrow V_{NL} < -Esat$

$V_d > 0 \quad -V_- > 0 \quad V_- < 0 \longrightarrow V_{in} < 0$

Assume −Esat

$V_{out} = -Esat \quad V_{NL} = V_- - (-Esat); \quad V_{NL} > Esat$

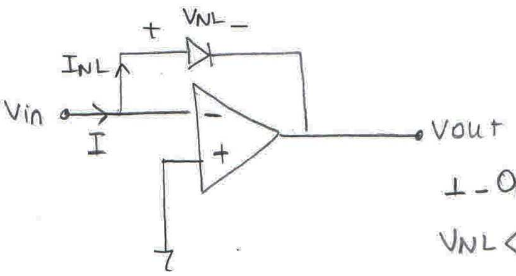
$V_d < 0 \,; \quad V_- > 0 \quad V_{in} > 0$

Assume linear region;

$-Esat < V_{out} < Esat \qquad V_{NL} = V_- - V_{out} \longrightarrow -Esat < V_{NL} < Esat$

$V_d = 0 \rightarrow V_- = 0 \qquad V_{in} = 0$

Ex:



a.) Find I vs Vin (input characteristic)

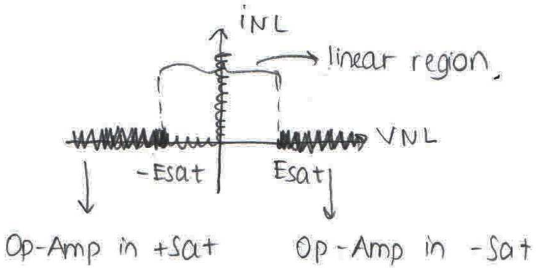
b.) Find Vout vs Vin (transfer characteristic)

1 - Op-Amp in +Sat ;

$V_{NL} < -Esat \quad V_{in} < 0 \quad V_{out} = +Esat$

2 - Op-Amp in −Sat ; $V_{NL} > Esat \quad V_{in} > 0 \quad V_{out} = -Esa$

Then, Op-Amp does not enter into −Sat, since V_{NL} can not positive



3- Op-Amp in linear region;

$-E_{sat} \leqslant V_{NL} \leqslant E_{sat}$

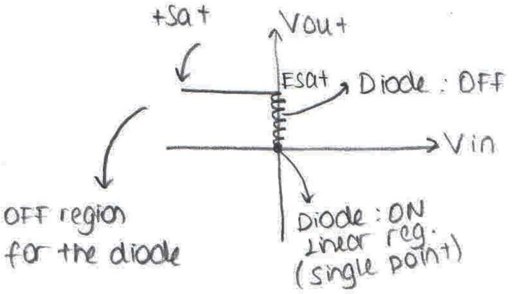
$-E_{sat} \leqslant V_{out} \leqslant E_{sat}$

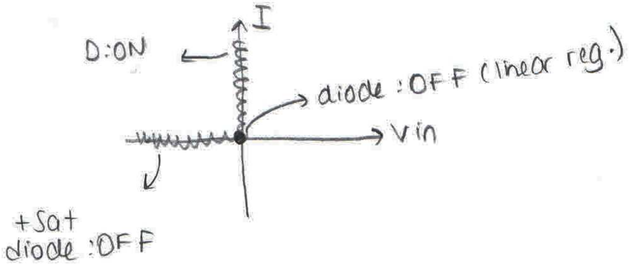
$V_{in} = 0$ $\quad 0 \leqslant V_{NL} < E_{sat}$ $\quad -E_{sat} \leqslant V_{NL} \leqslant 0$

only possible $\quad$ Diode: OFF

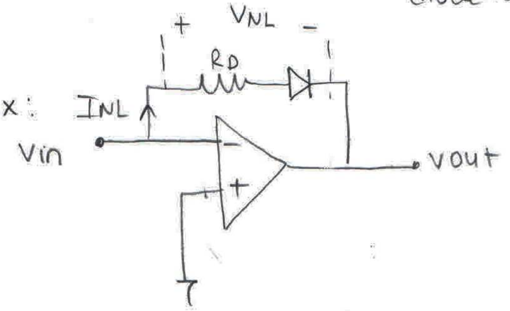
$V_{NL} = 0$ $\quad V_{out} = -V_{NL}$

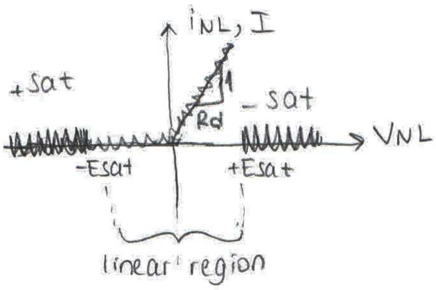
Diode: ON





Ex:

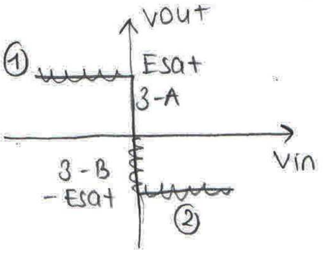


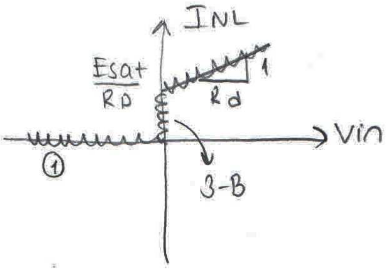


1- +Sat; $V_{NL} < -E_{sat}$

$V_{in} < 0$

$V_{out} = +E_{sat}$





2- -Sat; $V_{NL} > 0$, diode: ON

$V_{out} = -E_{sat}$

$V_{in} > 0$

3- In linear Region;

$-E_{sat} < V_{NL} < +E_{sat}$

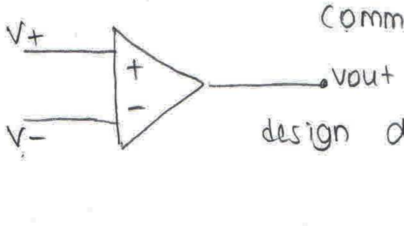
A- $V_{in} = 0 \rightarrow -E_{sat} < V_{NL} < 0$

$V_{out} = -V_{NL}$ $\quad 0 < V_{out} < E_{sat}$ $\quad I = 0$ since diode is OFF

B- $0 < V_{NL} < E_{sat}$

$-E_{sat} < V_{out} < 0$; $I = \frac{V_{NL}}{R_D}$ diode is ON [from (INL, VNL)]

Common Node Rejection Ratio



In linear region; "A" open loop gain can be due to design different for + and - terminals so we have A+ and A-by definition

Ideal case $A_+ = A_- \overset{\Delta}{=} A$

Practical case $V_{out} = A_+V_+ - A_-V_-$

$V_{common} \overset{\Delta}{=} V_c = \frac{V_+ + V_-}{2}$ $\Big]$ $V_+ = V_c + V_d/2$

$V_- = V_c - V_d/2$

$V_d \overset{\Delta}{=} V_+ - V_-$

$$V_{out} = A_+V_+ - A_-V_- = A_+(V_c + V_d/2) - A_-(V_c - V_d/2)$$

$$= \underbrace{(A_+ - A_-)}_{A_c}V_c + \underbrace{\frac{A_+ + A'_-}{2}}_{A_d} V_d$$

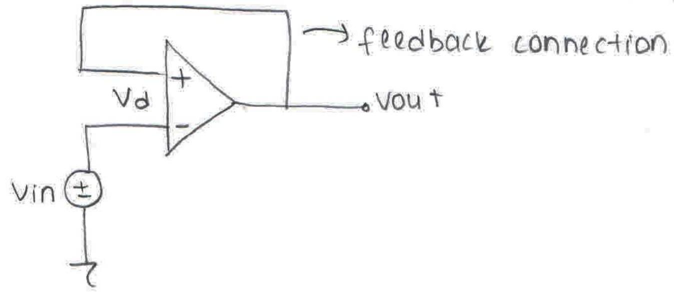
Ideally $A_+ = A_- = A$

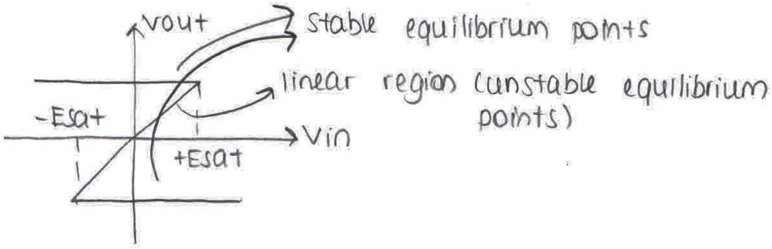
$V_{out} = AV_d$

$20\log_{10} CMRR \overset{\Delta}{=} (CMRR)_{dB}$ since $CMRR \overset{\Delta}{=} \left|\frac{A_d}{A_c}\right|$ → Ideally $CMRR \longrightarrow \infty$

linear scale.

Op-Amps with Positive Feedback





1- Op-Amp in +Sat;

$V_d > 0 \longrightarrow V_{in} < E_{sat+}$

$V_{out} = +E_{sat+}$

$V_d = V_{out} - V_{in}$

2- Op-Amp in −Sat;

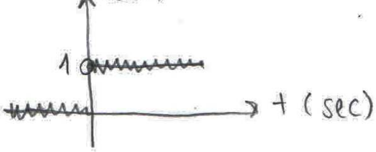
$V_d < 0$

$V_{out} = -E_{sat+}$

$V_{in} > -E_{sat+}$

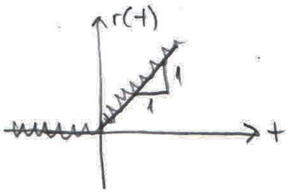
Wave forms

1- Unit-step function



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

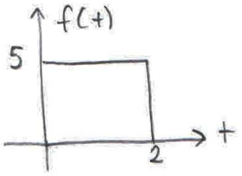
2- Ramp function



$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

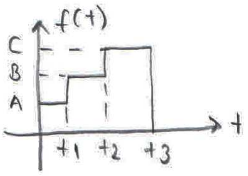
Superposition of $u(t)$ and $r(t)$

Ex :



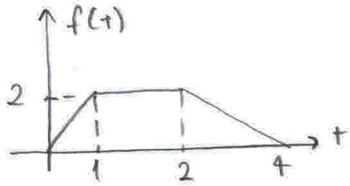
$$f(t) = 5u(t) - 5u(t-2)$$

Ex :



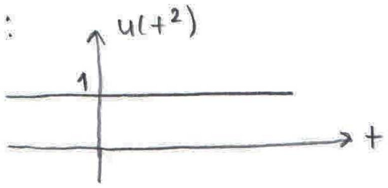
$$f(t) = Au(t) + (B-A)u(t-t_1) + (C-B)u(t-t_2) - C(t-t_3)$$

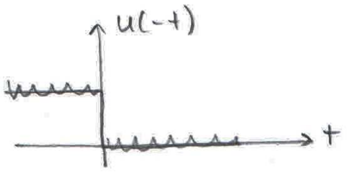
Ex :

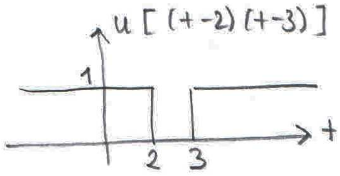


$$f(t) = 2r(t) - 2r(t-1) - r(t-2) + r(t-4)$$

Note :

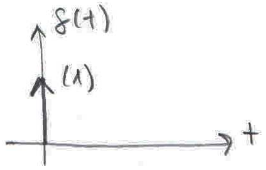


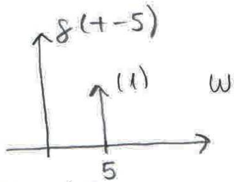




Impulse Function (Generalized Function Distribution)

$\delta(t)$ = Impulse function





weight of Impulse

unit weight impulse function /unit impulse

Properties :

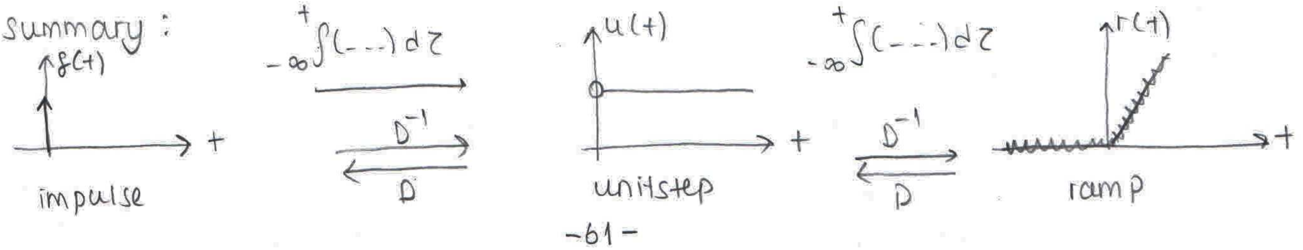
1_ $\int_{0^-}^{0^+} \delta(t)dt = 1$

2_ $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

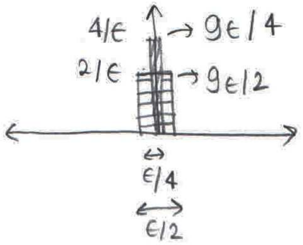
3_ $\delta(t) = \frac{d}{dt}u(t)$

4_ $\int_{0^-}^{0^+} f(\tau)\delta(\tau)d\tau = \int_{0^-}^{0^+} f(0)\delta(\tau)d\tau = f_0 \int_{0^-}^{0^+} \delta(\tau)d\tau = f_0$

Summary :



Definition of impulse



$\int_{-\infty}^{\infty} g_{\epsilon}(t) = 1$

$\int_{-\infty}^{\infty} g_{\epsilon/2}(t) = 1$

$\int_{-\infty}^{\infty} g_{\epsilon/4}(t) = 1$

$g_{\epsilon}(t) \xrightarrow[\epsilon \to 0]{} \delta(t)$

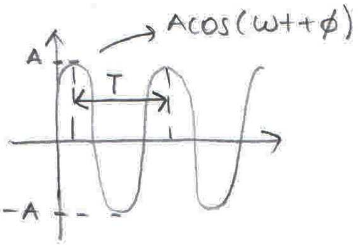
So property 1 follows the definition $\int_{0^-}^{0^+} \delta(t)dt = 1$

Sinusoidal Functions

$f(t) = A\cos(\omega t + \phi)$

A: amplitude ω: radial frequency (rad/sec)

t: second ϕ: phase

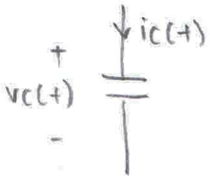


$\omega = 2\pi f$

f: frequency (1/sec, Hz): number of oscillations per second

$T = 1/f$ time for one oscillation.

capacitors



$Q(t) = CV(t)$

C: capacitance (Farad)

V(t): voltage applied to capacitor

Q(t): charge stored in the capacitor

If capacitance is time varying ; $Q(t) = C(t)V(t)$

* If $Q(t) = CV(t)$; $\frac{d}{dt}Q(t) = i(t) = C\frac{d}{dt}V(t)$

* If $Q(t) = C(t)V(t)$: $\frac{d}{dt}Q(t) = i(t) = [\frac{d}{dt}C(t)]V(t) + C(t)\frac{d}{dt}V(t)$

* $P_{cap}(t) = V_c(t)\, i_c(t)$ Watts

Energy work done by capacitor $\int_{-\infty}^{t} P_{cap}(\tau)d\tau = \Delta E$

$$\underset{E_{cap}(t) - E_{cap}(-\infty)}{\Delta E}$$

$$= \int_{-\infty}^{t} V_c(\tau)\, C\frac{dV_c(\tau)}{d\tau} d\tau = C\int_{-\infty}^{t} V_c(\tau)dV_c(\tau) = \frac{C}{2}[V_c(\tau)\Big|_{-\infty}^{t}]^2 = \frac{C}{2}[V_c^2(t) - V_c^2(-\infty)]$$

$= \Delta E$

Assume that at $t \rightarrow -\infty$, capacitor does not have any energy and $V_{cap}(-\infty) = 0V$

$$E(t) = \frac{1}{2} C V_c^2(t)$$

* $i_c(t) = C \frac{d}{dt} v_c(t)$

$$\int_{-\infty}^{t_0} i_c(\tau) d\tau = C \int_{-\infty}^{t_0} \frac{dv_c(\tau)}{d\tau} d\tau = C \left[v_c(t_0) - V_c(-\infty) \right]$$

$$V_c(t_0) = \frac{1}{C} \int_{-\infty}^{t_0} i_c(\tau) d\tau + \cancelto{0}{V_c(-\infty)}$$

$$V_c(t_1) = \frac{1}{C} \int_{-\infty}^{t_1} i_c(\tau) d\tau = \frac{1}{C} \int_{t_0}^{t_1} i_c(\tau) d\tau + \underbrace{\frac{1}{C} \int_{-\infty}^{t_0} i_c(\tau) d\tau}_{v_c(t_0)}$$

$$V_c(t_1) = V_c(t_0) + \frac{1}{C} \int_{t_0}^{t_1} i_c(\tau) d\tau$$

Important Note: $v_c(t)$, capacitor voltage is a continuous function of time (unless its current contan impulses)

To show this note

$$V_c(t+\epsilon) = V_c(t) + \frac{1}{C} \int_{t}^{t+\epsilon} i_c(\tau) d\tau$$

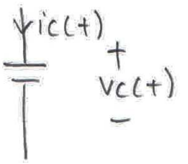
as $\epsilon \rightarrow 0$ $\quad \lim_{\epsilon \to 0} V_c(t+\epsilon) = V_c(t) + \frac{1}{C} \cancelto{0}{\lim \int_{t}^{t+\epsilon} i_c(\tau) d\tau}$

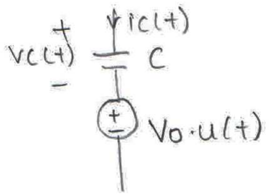
$V_c(t^+) = V_c(t)$ continuous.

* $D = \frac{d}{dt}$ $\quad D^{-1} = \int_{-\infty}^{t}$

$$V_c(t) = \underset{\substack{\uparrow \\ \text{initial condition}}}{V_c(t_0)} + \frac{1}{C} \int_{t_0}^{t} i_c(\tau) d\tau \qquad t > t_0$$

· Initial Condition Models for Capacitor





→ empty capacitor

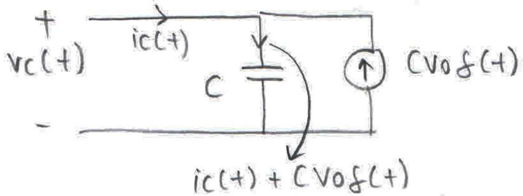
zero volts at $t=0$

zero energy

$v_c(0^-) = V_0$

$$V_c(t) = V_0 + \frac{1}{C} \int_{0}^{t} i(\tau) d\tau \qquad t > 0$$

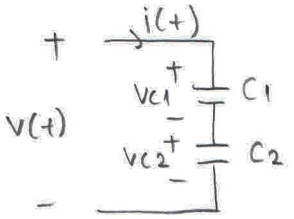
$$v_c(t) = \overset{empty}{v_c(t)} + V_0\, u(t) = V_0 + \frac{1}{C}\int_0^t i_c(\tau)\,d\tau = \frac{1}{C}\int_0^t i(\tau)\,d\tau + V_0 \qquad \text{for } t>0$$



$$v_c^{empty}(t) = v_c(0^-) + \frac{1}{C}\int_{0^-}^t i_c^{empty}(\tau)\,d\tau = \frac{1}{C}\int_{0^-}^t i(\tau)\,d\tau + C\,\frac{V_0}{C}\underbrace{\int_{0^-}^t \delta(\tau)\,d\tau}_{1} \qquad t>0$$

$$= V_0 + \frac{1}{C}\int_{0^-}^t i_c(\tau)\,d\tau$$

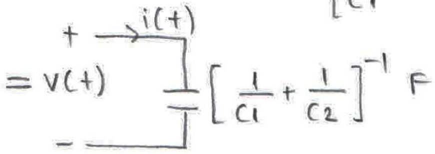
Series and Parallel Connection



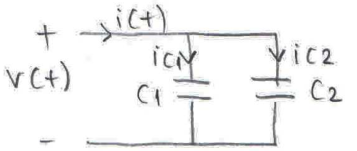
C1 and C2 have any energy at t=0

$$v(t) = v_{c1}(t) + v_{c2}(t) = \frac{1}{C_1}\int_0^t i_{c1}(\tau)\,d\tau + \frac{1}{C_2}\int_0^t i_{c2}(\tau)\,d\tau$$

$$= \left[\frac{1}{C_1} + \frac{1}{C_2}\right]\int_0^t i_c(\tau)\,d\tau$$

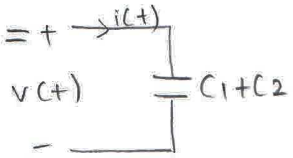


Parallel

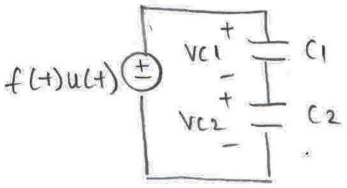


$$i(t) = i_{c1}(t) + i_{c2}(t)$$

$$= C_1\frac{dv(t)}{dt} + C_2\frac{dv(t)}{dt} = (C_1+C_2)\frac{dv(t)}{dt}$$



Voltage Division for Capacitors



$v_{c1}(0) = v_{c2}(0) = 0$

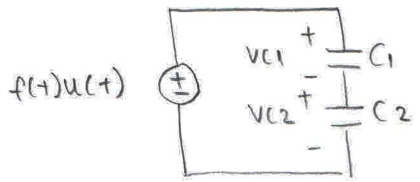
they are connected in series

$$i(t) = \frac{C_1C_2}{C_1+C_2}\frac{d}{dt}\{f(t)\,u(t)\} = \frac{C_1C_2}{C_1+C_2}\left[\frac{df(t)}{dt}\,.\,u(t) + \underbrace{\frac{du(t)}{dt}}_{\delta(t)}\,.\,f(t)\right]$$

$$v_{c1} = \frac{1}{C_1}\int_{0^-}^t i(\tau)\,d\tau = \frac{C_2}{C_1+C_2}\left[\int_{0^-}^t f'(\tau)u(\tau)\,d\tau + f(0)\int_{0^-}^t \delta(\tau)\,d\tau\right]$$

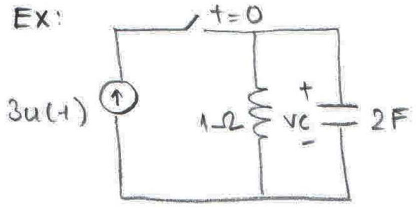
$$= \frac{C_2}{C_1+C_2}\left[f(t) - f(0) + f(0) - 1\right]$$

$$= \frac{C_2}{C_1+C_2}\,f(t) = \frac{1/C_1}{1/C_1 + 1/C_2}\,f(t)$$



vc1 and vc2 are proportional with $\frac{1/C_1}{1/C_1+1/C_2}$ and $\frac{1/C_2}{1/C_1+1/C_2}$

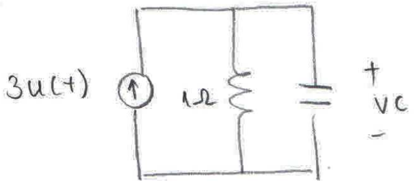
Ex:



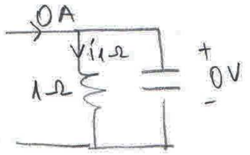
$V_c(0^-)=0$

Find $t=0^+$ solution

what happened just after switching?

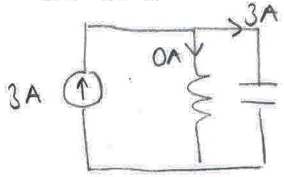


at $t=0^-$



$i_{1\Omega}(0^-) = 0/1 = 0A$

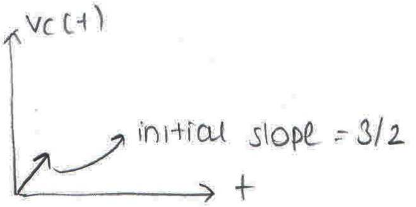
at $t=0^+$



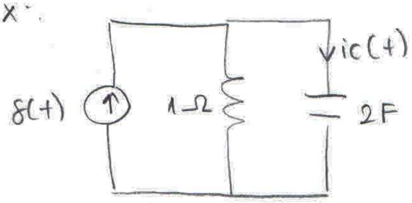
$V_c(0^-)=V_c(0^+)=0$ [voltage is continuous if no impulses in the system]

$i_c(0^+)=3A$ $\quad C\frac{dv_c(t)}{dt}\Big|_{t=0^+}$ $\quad 2\dot{V_c}(0^+)=3$

$\dot{V_c}(0^+)=3/2$



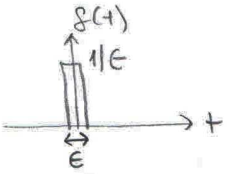
Ex:



$V_c(0^-)=0V$

at $t=0^-$

$0^- < t < 0^+$

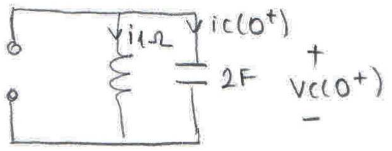


The same result of previous example

$i_c(t)=\delta(t)$

$$V_c(0^+) = \frac{1}{2}\underbrace{\int_{0^-}^{0^+} i_c(\tau)\,d\tau}_{1} = 1/2\ V$$

At $t=0^+$

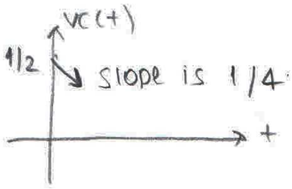


$V_c(0^+)=1/2\ V$

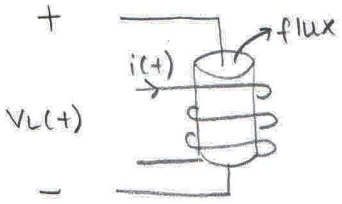
$i_1(0^+)=1/2\ A$

$i_c(0^+)=-1/2A = 2\dot{V_c}(0^+)$;

$\dot{V_c}(0^+)=-1/4$

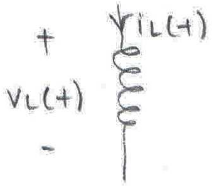


Inductors



$\phi(t) = L(t)\, i(t)$ similar $(Q = CV)$

$$V_L(t) = \frac{d\phi(t)}{dt} = \frac{d}{dt}\{L(t)\, i(t)\}$$



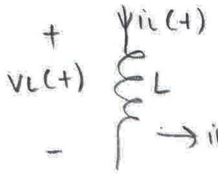
for LTI inductors $V_L(t) = L\frac{d}{dt} i_L(t)$

$$\int_{-\infty}^{t} V_L(\tau)d\tau = L\int_{-\infty}^{t}\frac{d}{dt} i_L(\tau)d\tau = L\left[i_L(t) - i_L(-\infty)\right]$$

$i_L(t) = i_L(-\infty) + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$ integral form

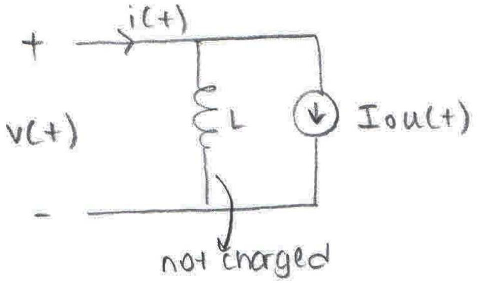
$v_L(t) = L\frac{di_L(t)}{dt}$ differential form

Initial condition Models

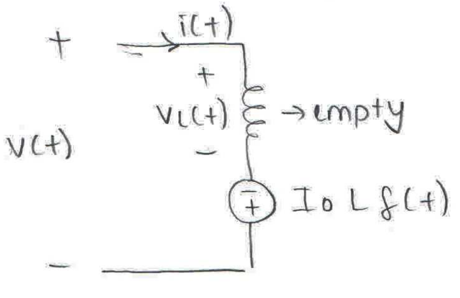


→ initially choged inductor

$i_L(0^-) = I_0 \longrightarrow i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} V_L(\tau)d\tau$ for $t > 0$



$i(t) = I_0 + \frac{1}{L}\int_{0^-}^{t} V_L(\tau)d\tau$ $t > 0$

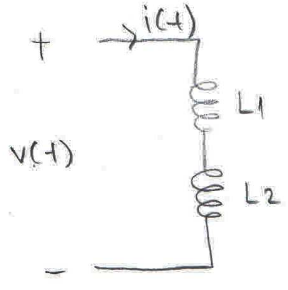


$V_L(t) = V(t) + I_0 L\delta(t)$

$i_L(t) = \frac{1}{L}\int_{0^-}^{t}\left[v_L(\tau) + I_0 L\delta(\tau)\right]d\tau$

$= I_0 + \frac{1}{L}\int_{0^-}^{t} v(\tau)d\tau$ $t > 0$

Inductors in series and parallel



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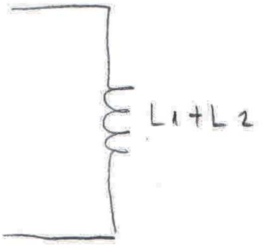
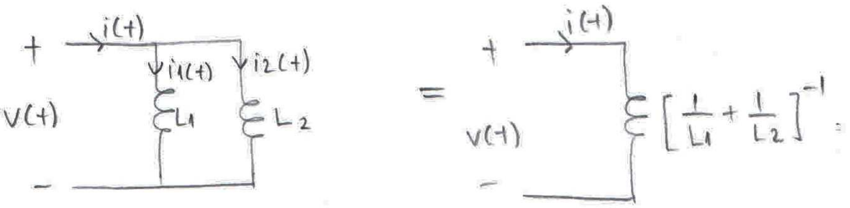


connected in series

$V_{L1}(t) = L_1 \frac{di(t)}{dt}$ $V_{L2}(t) = L_2 \frac{di(t)}{dt}$ $V(t) = V_{L1}(t) + V_{L2}(t) = (L_1+L_2)\frac{di(t)}{dt}$

parallel



$V(t) = L_1 \frac{di_1(t)}{dt} = L_2 \frac{di_2(t)}{dt}$

$i(t) = i_1(t) + i_2(t)$

$i_1 = \frac{1}{L_1}\int_0^t v(\tau)d\tau$ $i_2 = \frac{1}{L_2}\int_0^t v(\tau)d\tau$

$i = i_1 + i_2 = \left[\frac{1}{L_1} + \frac{1}{L_2}\right]\int_0^t v(\tau)d\tau = \frac{1}{L_{eq}}\int_0^t v(\tau)d\tau$

then $L_{eq} = \left[\frac{1}{L_1} + \frac{1}{L_2}\right]^{-1}$

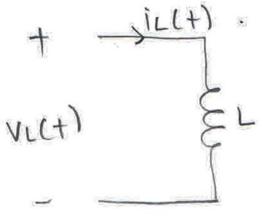
* $D^{-1}\{v_L(t)\} = L D^{-1} D i_L(t)$

$\int_{-\infty}^{t} v_L(\tau)d\tau = L i_L(t)$

$i_L(t) = \underbrace{\frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau}_{i_L(t_0)} + \frac{1}{L}\int_{t_0}^{t} v_L(\tau)d\tau$

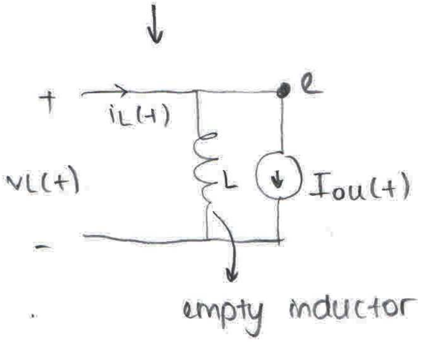
$i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} v_L(\tau)d\tau$ Assume no initial energy for L1, L2

$\frac{1}{2} L I_L^2(t) = E$; $E(t) = 0$; $i_L(t) = 0$



$i_L(0^-) = I_0$ A

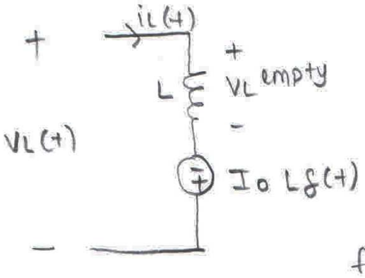
$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v(\tau)d\tau$



$i_L^{empty}(t) = \frac{1}{L}\int_{0}^{t} v_L(\tau)d\tau$ t>0 then

KCL at a ; $i_L(t) = i_L^{empty}(t) + I_{ou}(t)$

$= \frac{1}{L}\int_{0^-}^{t} v_L(\tau)d\tau + I_0 u(t)$



$$v_L^{\text{empty}}(t) = v_L(t) + I_o L\delta(t)$$

$$i_L(t) = \frac{1}{L}\int_{0^-}^{t}\Big[\underbrace{v_L(\tau)+I_o L\delta(\tau)}\Big]d\tau$$

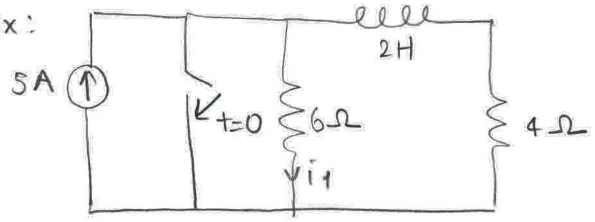
empty inductors voltage

$$\text{for } t>0 \quad i_L(t) = \frac{I_o L}{L}\underbrace{\int_{0^-}^{t}\delta(\tau)d\tau}_{1} + \frac{1}{L}\int_{0^-}^{t}v_L(\tau)d\tau$$

NOTE! Inductors current is a continuous function of time unless there is an impulse in the circuit or there is a pathological connection.

$$i_L(t_o^+) = i_L(t_o) + \frac{1}{L}\int_{t_o}^{t_o^+}v_L(\tau)d\tau \qquad i_L(t_o^+) = i_L(t_o)$$

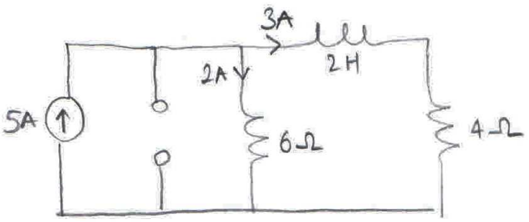
Ex:



$i_1(0^-) = 2A$

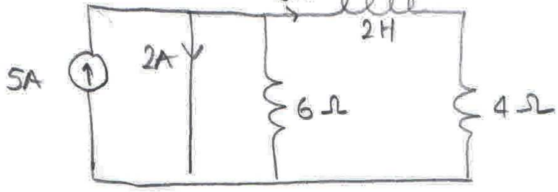
Find $i_L(0^+)$, $i_1(0^+)$, $\frac{d}{dt} i_L(t)\Big|_{t=0^+}$

At $t=0^-$



$i_L(0^-) = 3A$

At $t=0^+$

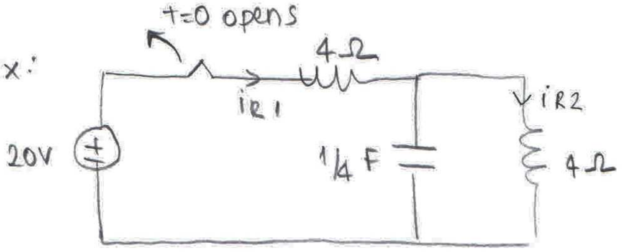


$i_1(0^+) = 0$

$$v_L(0^+) = -12V = L\frac{di(t)}{dt}$$

$$\frac{di(t)}{dt}\Big|_{t=0^+} = -6$$

Ex:



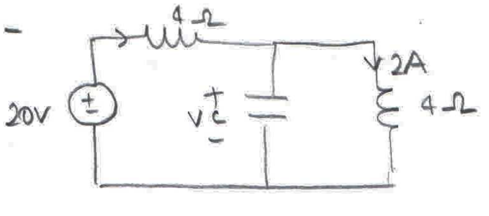
$i_{R2}(0^-) = 2A$

Find $Q(0^-)$ and $Q(0^+)$

$i_R(0^-)$ and $i_{R1}(0^+)$

$i_C(0^-)$, $i_C(0^+)$, $v_C'(0^+)$

At $t=0^-$



$V_C(0^-) = 8V$

$Q(0^-) = 2$ C

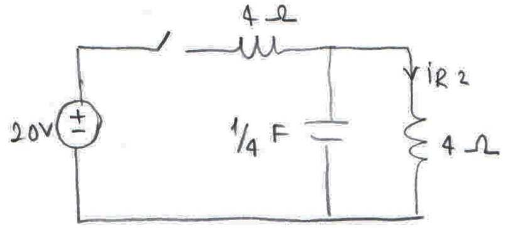
$i_{R1}(0^-) = 3A$

$i_C(0^-) = 1A$

Capacitors: Voltage is continuous

Inductors: Current is continuous

At $t=0^+$



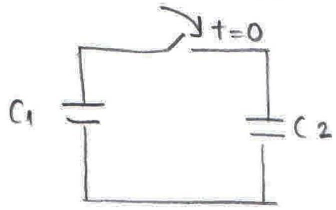
$V_c(0^+) = 8V \quad Q(0^+) = 2\ C$

$i_{R2}(0^+) = 8/4 = 2A \quad i_R(0^+) = 0A$

$i_c(0^+) = -2\ A = C\dot{V}_c(t)$

$\dot{V}_c(t) = -8$

Some Pathological Circuits



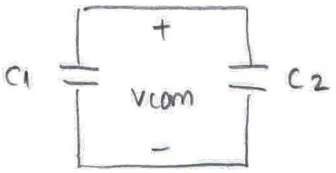
$V_{c1}(0^-) = V_1 \qquad V_{c2}(0^-) = V_2$

$Q_{c1}(0^-) = C_1V_1 \qquad Q_{c2}(0^-) = C_2V_2$

$Q^{total}(0^-) = C_1V_1 + C_2V_2$

$Q^{total}(0^+) = C_1V_1 + C_2V_2$

At $t=0^+$; switch is closed



After switching the capacitance changes to $C_1 // C_2 = C_1 + C_2$

$Q_1(0^+) = C_1 V_{com}$

$Q_2(0^+) = C_2 V_{com}$

$Q(0^+) = Q_1 + Q_2 = V_{com}(C_1 + C_2)$

$$V_{com}(0^+) = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

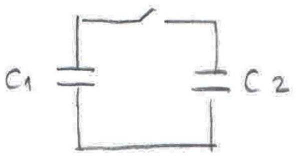
Energy at $t=0^-$; $E_{total}(0^-) = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$

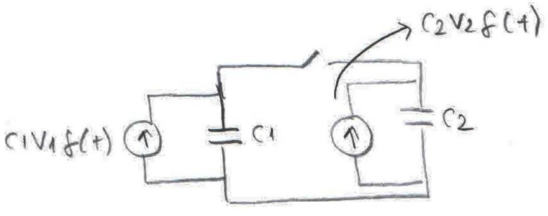
$$E_{total}(0^+) = \frac{1}{2}(C_1 + C_2)\frac{(C_1V_1 + C_2V_2)^2}{(C_1 + C_2)^2}$$

$E_{total}(0^-) > E_{total}(0^+)$

In EE201, we say that energy is loss during switching (More on this provided EM courses)

At $t=0^-$

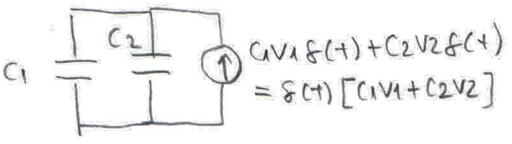




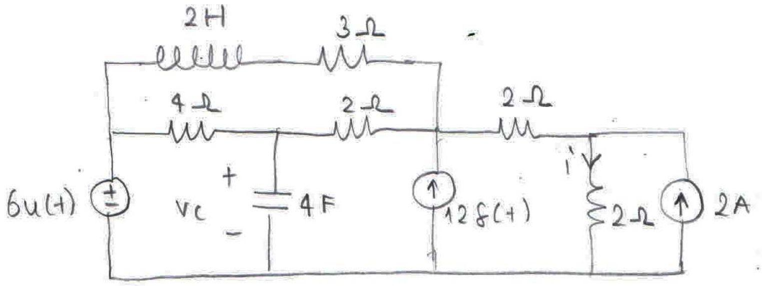
$V_{c1}(0^-) = V_1$

$V_{c2}(0^-) = V_2$

At $t=0^+$



$$V_{eq}(0^+) = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$



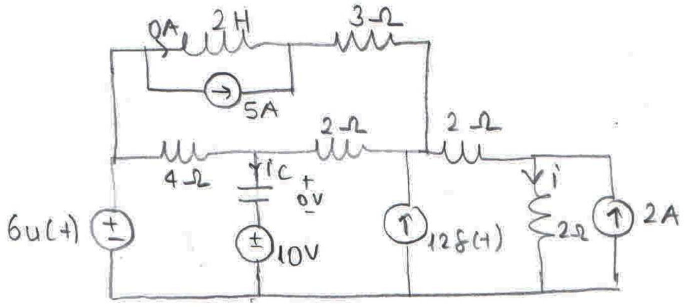
$V_C(0^-) = 10V$

$i_L(0^-) = 5A$

Find $V_C(0^+)$ $i_L(0^+)$

$0^- < t < 0^+$ during the application of impulse

$0-\epsilon < t < 0+\epsilon$



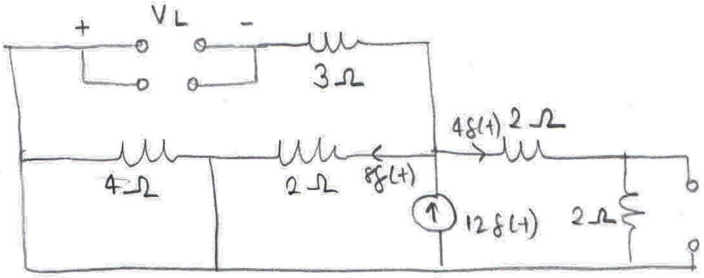
I'd like to find $i_C(t)$ & $V_L(t)$

during $0^- < t < 0^+$

$V_C(0^+) = V_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_C(\tau)\,d\tau$

$\searrow$ I_{C} by superpositio

$I_L(0^+) = I_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} V_L(\tau)\,d\tau$



then $i_{C1} = 8\delta(t)$

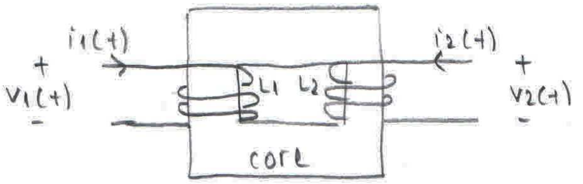
$V_{L1} = -16\delta(t)$

Then $V_C(0^+) = 10 + \frac{1}{C}\int_{0^-}^{0^+} (8\delta(t') + k)dt'$

$I_L(0^+) = 5 + \frac{1}{L}\int_{0^-}^{0^+} (-16\delta(t') + k)\,dt' = -3A$

$= 12V$

Mutual inductor

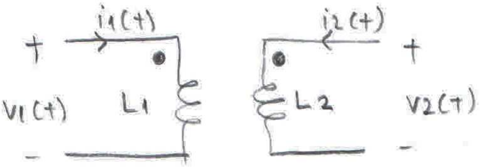


$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

special cases:

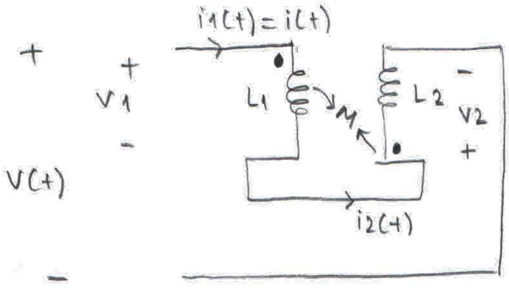
1. $M = 0 \longrightarrow L_1, L_2$ two regular inductance, (only self inductance)
2. $M = \sqrt{L_1 L_2}$ $\quad k = \frac{M}{\sqrt{L_1 L_2}} = 1$ coupling coefficient $= 1$ No loss

ideal transformer



with dot convention following terminal equation results

$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} d/dt\ i_1(t) \\ d/dt\ i_2(t) \end{bmatrix}$$



$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} di_1(t)/dt \\ di_2(t)/dt \end{bmatrix}$$

$$\left.\begin{aligned} i(t) &= i_1(t) \\ i(t) &= i_2(t) \end{aligned}\right] \; i_1(t) = i_2(t)$$

$$v_1(t) + v_2(t) = v(t).$$

$$\left.\begin{aligned} v_1(t) &= L_1\, di/dt + M\, di/dt \\ v_2(t) &= L_2\, di/dt + M\, di/dt \end{aligned}\right] \; v(t) = di/dt\, (L_1 + L_2 + 2M)$$

First Order Circuits

$(D+\gamma)\, x(t) = f(t)$ satisfies I.C. and differential equation for $t>0$

$f(t)$: forcing term - external input

$x(t)$: scalar unknown function $x(0^-) = x_0$ initial condition

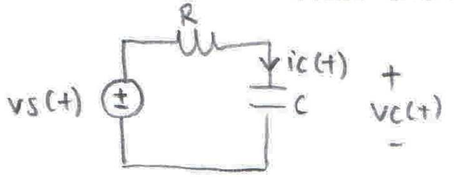
D: d/dt

Types of solutions

1- Homogenous - Particular solution 2- zero input - zero state solution

3- Transient - Steady State Solution 4- Complete solution

First Order RC, RL Circuits

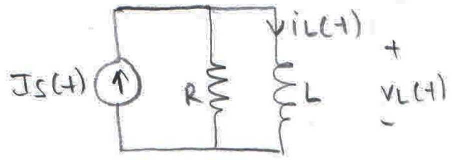


$$-v_s(t) + i_c(t)\cdot R + v_c(t) = 0$$

$$RC\, \dot{v_c}(t) + v_c(t) = v_s(t)$$

$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{v_s(t)}{RC} \qquad v_c(0^-) = V_0$$

$\downarrow$ dual



$$\frac{v_L}{R} + i_L - i_s(t) = 0$$

$$\frac{L}{R}\, \dot{i_L}(t) + i_L(t) = i_s(t)$$

$$\left(D + \frac{R}{L}\right) i_L(t) = \frac{R}{L}\, i_s(t) \qquad i_L(0^-) = I_0$$

1- Homogenous - Particular Solutions

$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{v_s(t)}{RC} \qquad v_c(0^-) = V_0$$

$\swarrow$ homogenous soln. $\searrow$ particular soln.

$$\left(D + \frac{1}{RC}\right) v_c^h(t) = 0 \qquad \left(D + \frac{1}{RC}\right) v^P(t) = \frac{v_s(t)}{RC}$$

CCDE (Constant Coefficient Differential Equation)

$v_c^h(t) = a\, e^{\lambda t}$ a, λ : generic unknowns

Substitude $v_c^h(t)$ in D.E. $(D+\frac{1}{RC})\,\alpha\, e^{\lambda t} = 0 \rightarrow (D+\frac{1}{RC})\,\alpha = 0$

Then $\rightarrow \alpha = 0$, $v_c^h(t) = 0$ trivial and not useful solution

$\rightarrow \lambda = -\frac{1}{RC}$ $v_c^h(t) = \alpha e^{-t/RC}$ solution is valid for any $\alpha \in R$

Particular solution

1- $v_s(t) = B u(t) \longrightarrow (D+\frac{1}{RC})\, v_c(t) = \frac{v_s(t)}{RC}$ case 1

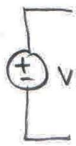
$(D+\frac{1}{RC})\, v_c(t) = \frac{B}{RC}$ t>0

Guess $v_c^p(t) = A \longrightarrow (D+\frac{1}{RC})\, A = B/RC$ t>0 then A=B

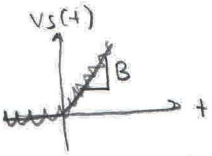
$v_c^p(t) = B$ → particular solution $v_s(t) = Bu(t) \longrightarrow \overset{\text{complete}}{v_c(t)} = \underbrace{\alpha e^{-t/RC}}_{v_c^h(t)} + \underbrace{B}_{v_c^p(t)}$ t>0

α : unknown : will be found using initial condition

2- case 2



$v_s(t) = B t u(t) = B r(t)$



$(D+\frac{1}{RC})\, v_c^p(t) = \frac{Bt}{RC}$ t>0

Guess $v_c^p(t) = A_0 + A_1 t + \cancel{A_2 t^2}$ $(D+\frac{1}{RC})\, v_c^p(t) = \frac{Bt}{RC}$ t>0

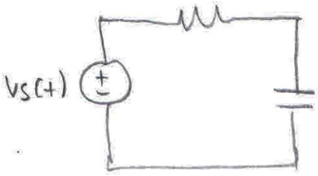
$A_1 + \frac{A_0 + A_1 t}{RC} = \frac{Bt}{RC}$ $A_1 = B$ $A_0 = -BRC$ $v_c^p(t) = -BRC + Bt$ t>0

$v_c^{complete}(t) = \alpha e^{-t/RC} - BRt + Bt$ t>0

α is adjusted so that $V_0 = v_c(0^-) = v_c(0^+)$

2- zero input / zero state solutions

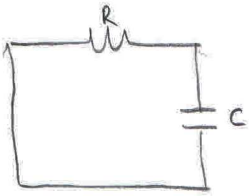
zero input solution



$v_c(0^-) = V_0$

vs(t) > 0

zero input



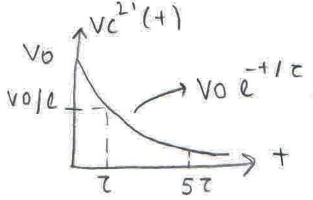
$v_c(0^-) = V_0$

$v_c(0^-) = V_0$

$(D+\frac{1}{RC})\, v_c^{zi}(t) = 0$ $v_c^{zi}(0) = V_0$ $v_c^{zi}(t) = \alpha e^{-t/RC}$ t>0

$v_c(0^-) = v_c(0^+) = V_0$

$\boxed{v_c^{zi}(t) = V_0 e^{-t/RC}}$



$\tau = RC$ in seconds

5τ can be sufficient for decay of IC

$$\frac{V}{I}\cdot\frac{Q}{V} = \frac{Q}{\frac{\Delta Q}{\Delta t}} = \Delta t \text{ seconds} \qquad \Omega\, F = sec$$

zero state solutions

State concept, states describe the dynamics system completely and sufficient to determine the output form input and the states

Circuit theory states are $V_{cap}(t)$ and $I_L(t)$

$V_c(0^-)$, $I_L(0^-)$ external input $f(0^-) = f_0$

$V_c(0^+)$ $I_L(0^+)$ can be found

$$\left(D+\frac{1}{RC}\right) v_c^{2s}(t) = \frac{V_s(t)}{RC} \qquad V_{cap}^{2s}(0^-) = 0 \text{ volts}$$

Case 1 : $V_s(t) = Bu(t)$

$$\left(D+\frac{1}{RC}\right) v_c^{2s}(t) = \frac{B}{RC} \quad t>0 \qquad v_c^{2s}(0^-)=0 \qquad v_c^{2s}(t) = \alpha e^{-t/RC} + B \text{ from earlier results}$$

$$v_c^{2s}(0^-) = v_c^{2s}(0^+) = 0 \qquad \alpha = -B$$

$$v_c^{2s}(t) = B(1-e^{-t/RC}) \quad t \geq 0$$

$$v_c^{com}(t) = v_c^{2i}(t) + v_c^{2s}(t) = \underset{\substack{\downarrow \\ \text{zero input}}}{V_o e^{-t/RC}} + B(1-e^{-t/RC}) \qquad t>0$$

Case 2 : $V_s(t) = Br(t)$

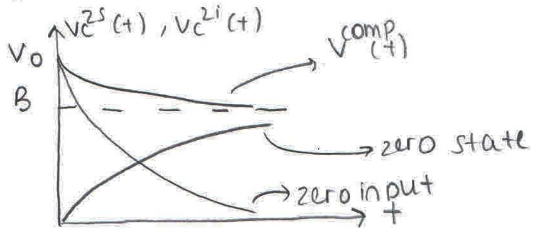
$$\left(D+\frac{1}{RC}\right) v_c^{2s}(t) = \frac{Bt}{RC} \quad t>0 \qquad v_c^{2s}(0^-)=0$$

$$v_c^{2s}(t) = BRC\, e^{-t/RC} + Bt - BRC$$

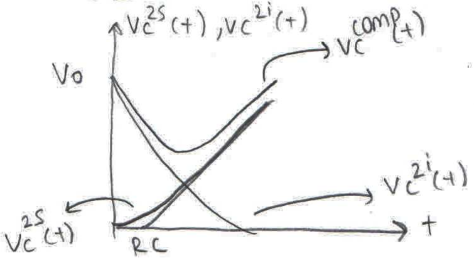
$$V^{comp}(t) = \underbrace{V_o e^{-t/RC}}_{\text{due to I.C.}} + \underbrace{(BRC\, e^{-t/RC} + Bt - BRC)}_{\text{zero state}}$$

Sketching the curves

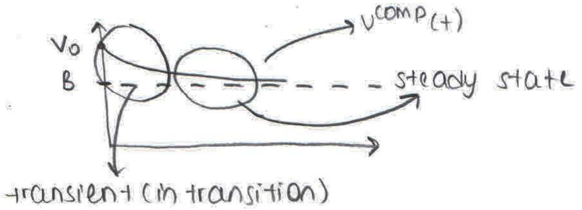
Case 1 :



Case 2 :



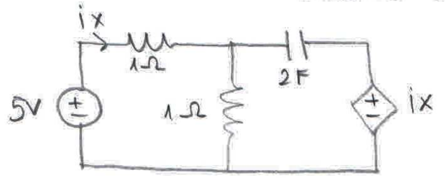
3- Transient and Steady State



In transient part, the component of the solution due to IC dominates the complete solutions. For RC/RL circuits (5τ) can be considered as the external of transient region $\tau = RC$ or $\tau = L/R$ sec

Steady state soln is the part which is free of initial conditions. Initial conditions are decayed to zero.

Circuits with DC inputs



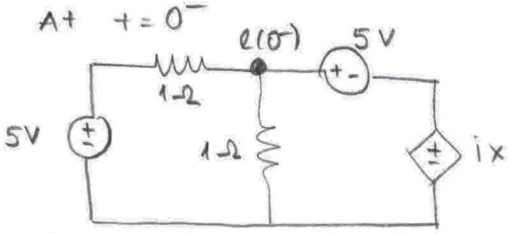
$v_c(0^-) = 5V$ Find $v_c(t)$ $i_x = 5-e$

$$\frac{e-5}{1} + \frac{e}{1} + C\frac{dV}{dt} + 2e - 5 + 2(e-(5-e)) = 0$$

$$2e(t) - 5 + 4e'(t) = 0 \; ; \quad (4D+2)\,e(t) = 5 \; ;$$

$$(D + 1/2)\,e(t) = 5/4 \qquad e(t) = \alpha e^{-t/2} + 5/2$$

At $t = 0^-$



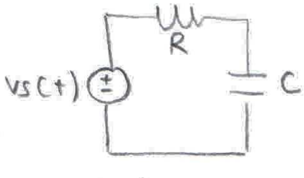
$i_x = 0$

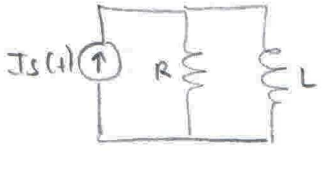
$e(t) = 5/2(1 + e^{-t/2})$

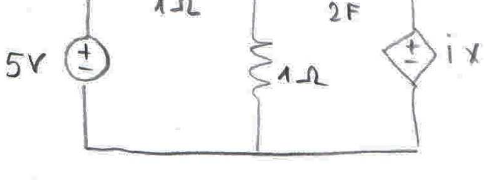
$v_c(t) = e(t) - (5-e) = 2e(t) - 5$

$v_c(t) = e(t) = 5e^{-t/2}$ $t \geq 0$

First (1st) Order circuits with DC inputs



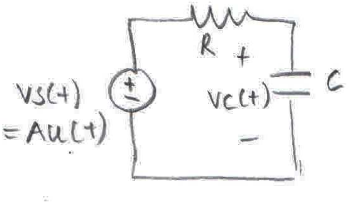




$v_c(0^-) = 5V$

$v_c(t) = 5e^{-t/2}\,u(t)$ $t \geq 0$

DC input case:



$v_c(0^-) = V_0$ $\quad v_s(t) = V_R + V_C = i_C R + V_C = RC\dot{V}_c(t) + V_C$

$$= (RCD + 1)\,v_c(t)$$

$$\left(D + \frac{1}{RC}\right) v_c(t) = \frac{V_s(t)}{RC} = \frac{Au(t)}{RC}, \quad v_c(0^-) = V_0$$

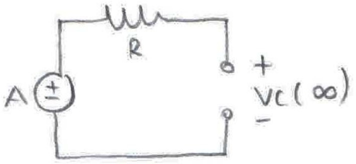
$$V_{cap}(t) = V^p_{cap}(t) + V^h_{cap}(t) = A + \alpha e^{-t/RC}\,u(t) = \underset{\uparrow \text{Algebraic solution}}{A} + (V_0 - A)\,e^{-t/RC}\,u(t)$$

So that the initial condition is satisfied $v_{cap}(t) = A + (V_0 - A)\, e^{-t/RC} u(t)$

as $t \to \infty$ capacitor gets fully charged

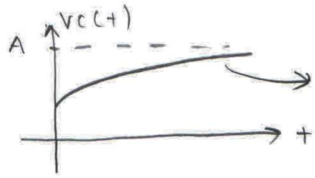
For DC input capacitors will be eventually charged and its current will be zero

So capacitor acts like open circuit as $t \to \infty$



$V_c(\infty) = A$

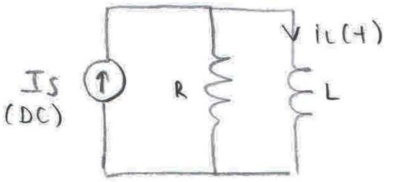
$V_c(0^-) = V_0$ $\quad V_c(0^+) = V_0$



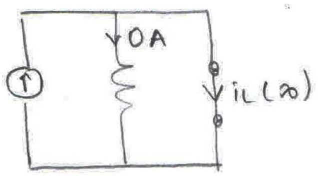
exponential function with $\tau = RC$

$$V_c(t) = V_c(\infty) + [V_c(0^+) - V_c(\infty)]\, e^{-t/\tau} u(t)$$

First Order RC circuits with DC inputs.



as $t \to \infty$

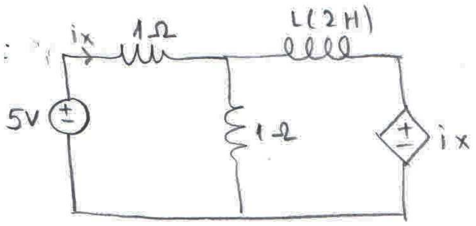


L acts as short circuit

$i_L(\infty) = I_S$

$$i_L(t) = I_L(\infty) + (I_0 - I_L(\infty))\, e^{-t/\tau} u(t) \qquad \tau = L/R$$

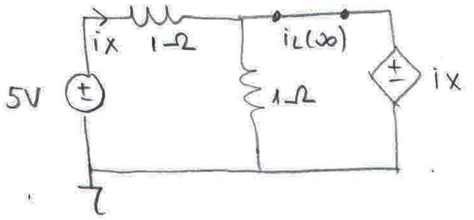
Ex:



Rseen by inductor = 1Ω

$\tau = L/R = 2$ sec's

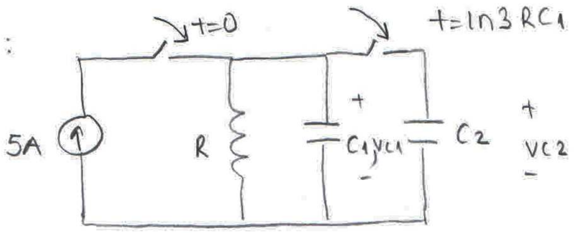
$I_L(\infty) = ?$ $\quad I_0 = 5A$ $\quad I_L(t) = ?$



$i_L(\infty) = 0A$

$i_L(t) = 5 e^{-t/2} u(t)$

Ex:



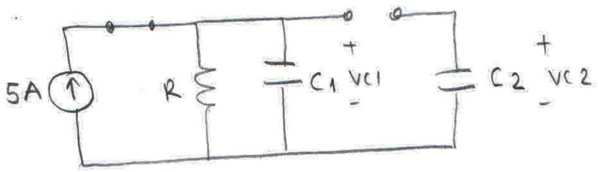
$V_{c1}(0^-) = 2R$ $\quad V_{c2}(0^-) = 6R$

At t=0 switch 1 closes

At t=ln3RC1 switch 2 closes

Find $V_{c1}(t)$ and $V_{c2}(t)$

$0 < t < \ln 3RC_1$



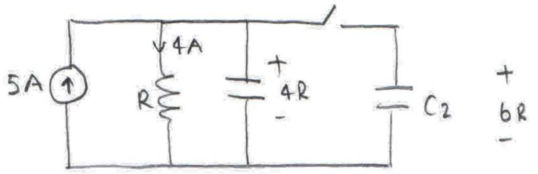
$V_{c2}(t) = V_{c2}(0^+) = V_{c2}(0^-) = 6R$

$V_{c1}(t) = V_{c1}(\infty) + [V_{c1}(0^+) - V_{c1}(\infty)]\, e^{-t/2}$ u

$V_{c1} = 5R - 3R\, e^{-t/\tau} u(t)$

$V_{C1}(RC_1\ln 3) = 5R - 3R\, e^{-\ln 3} = 4R$

$\ln 3RC_1^- < t < \ln 3RC_1^+$



$V_{C1}(\text{before}) = 4R$

$V_{C2}(\text{before}) = 6R$

$\searrow t = \ln 3RC_1^-$

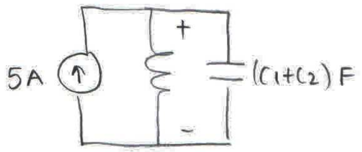
Vcom(after) they are in parallel

$Q^{\text{before}} = C_1V_1^{\text{before}} + C_2V_2^{\text{before}} = 4RC_1 + 6RC_2$

$Q^{\text{after}} = Q^{\text{before}} = (C_1+C_2)V_{com} = \dfrac{4RC_1+6RC_2}{C_1+C_2}$ Vcom is after $t = \ln 3RC_1^+$

for $t > \ln 3RC_1^+$

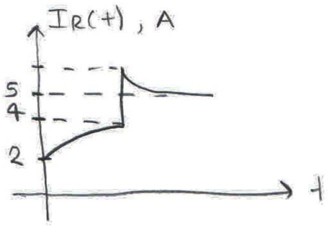
$V_{C1}(\ln 3RC_1^+) = V_{C2}(\ln 3RC_1^+) = V_{com}$



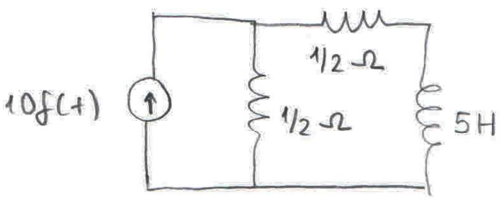
$V_{com}(\ln 3RC_1^+) = \dfrac{4RC_1+6RC_2}{C_1+C_2}$

$V_{com}(t) = V_C(\infty) - [V_C(\infty) - V_{com}(\ln 3RC_1^+)]\, e^{-(t-\ln 3RC_1)/\tau}$ $t > \ln 3RC$

$V_{com}(t) = 5R - (5R - R\dfrac{4C_1+6C_2}{C_1+C_2})\, e^{-(t-\ln 3RC_1)/RC(C_1+C_2)}$ $t > \ln 3RC_1$

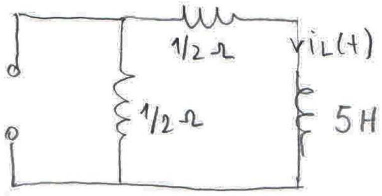


EX.: 2PS VI , 1-



$i_L(0^-) = -2A$

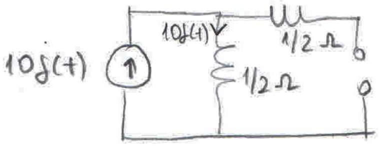
$i_L(0^+) = ?$ for $t > 0^+$



$i_L(0^+) = I_0$ $i_L(t) = I_0\, e^{-t/\tau}\, u(t)$ $t > 0^+$

$\tau = L/R_{seen} = 5/1 = 5$ secs

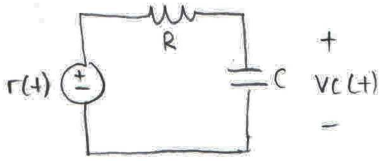
for $0^- < t < 0^+$



$I_0 = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v_L(\tau)\,d\tau = -2 + \frac{1}{5}\int_{0^-}^{0^+} 5\delta(t')\,dt' = -1A$

a-) Unit Step Response

b-) Ramp Response

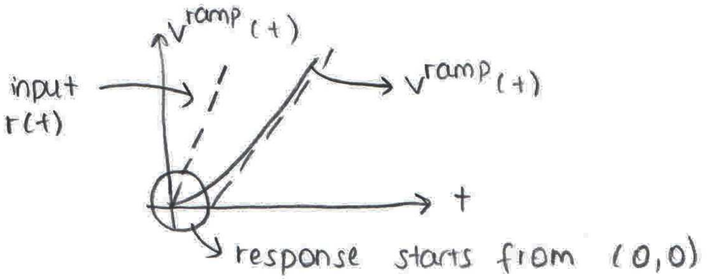


$$\left(D+\frac{1}{RC}\right) v^{ramp}(t) = \frac{t}{RC} \quad t \geq 0 \qquad v^{ramp}(0^-) = 0V$$

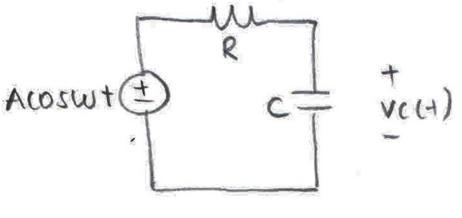
$$v^{ramp}(t) = \underbrace{A + Bt}_{particular} + \underbrace{\alpha e^{-t/RC}}_{homognous} \quad t \geq 0$$

$$v^{ramp}(t) = \left\{ t - RC\left(1 - e^{-t/RC}\right)\right\} u(t) \quad t \geq 0$$

$$v^{ramp}(0^-) = v^{ramp}(0^+) = -RC + \alpha = 0 \; ; \quad \alpha = RC$$



c-) Sinusoidal Response



$$V_C(0^-) = 0\,V$$

$$\left(D+\frac{1}{RC}\right) V_C^{sin} = \frac{A\cos wt}{RC}$$

$$V_C^{sin}(t) = \underbrace{B_1\cos wt + B_2 \sin wt}_{particular} + \underbrace{\alpha e^{-t/RC}}_{homogenous}$$

$$\left(D+\frac{1}{RC}\right)\left(B_1\cos wt + B_2 \sin wt\right) = \frac{A}{Rc}\cos wt$$

$$\cos wt\left(\frac{B_1}{Rc} + B_2\right) + \sin wt\left(\underbrace{-wB_1 + \frac{B_2}{RC}}_{0}\right) = \frac{A}{Rc}\cos wt \quad \forall t \geq 0$$

$$\frac{B1}{RC} + B2 = \frac{A}{RC} \qquad -\omega B1 + B2/RC = 0 \qquad B2 = \omega RC B1$$

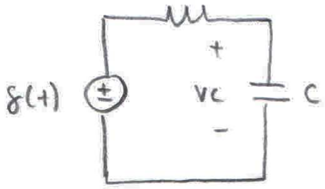
$$\begin{bmatrix} 1/RC & \omega \\ -\omega & 1/RC \end{bmatrix} \begin{bmatrix} B_1 \\ B2 \end{bmatrix} = \begin{bmatrix} A/RC \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} B_1 \\ B2 \end{bmatrix} = \frac{1}{(RC)^2 \omega^2} \begin{bmatrix} A/(RC)^2 \\ \frac{A}{RC}\ \omega \end{bmatrix}$$

$$V^{sin}(t) = \underbrace{\frac{A/RC}{(RC)^2+\omega^2}\left[\frac{1}{RC}\cos\omega t + \omega \sin\omega t\right]}_{\text{particular}} + \underbrace{\overset{}{\alpha} \xrightarrow{} -B1 \; e^{-t/RC}}_{}$$

$V^{sin}(0^+) = 0V$

$$\alpha = \frac{-A/(RC)^2}{(RC)^2+\omega^2} = -B1$$

d.Impulse Response



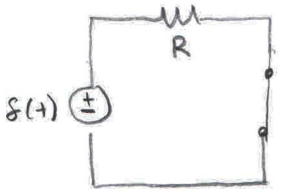
$V_c(0^-) = 0V$

$$\left(D+\frac{1}{RC}\right) V_c^{imp}(t) = \frac{\delta(t)}{RC} \qquad t \geqslant 0$$

$$\left(D+\frac{1}{RC}\right) V_c^{imp}(t) = 0 \qquad t \geqslant 0 \qquad V_c^{imp}(0^+) = ?$$

$$V_c^{imp}(t) = V_c(0^+)\, e^{-t/\tau}$$

$0^- < t < 0^+$

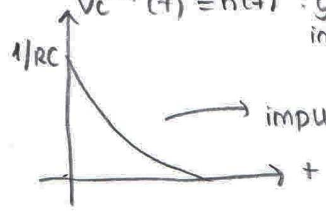


$$i_c(t) = \frac{\delta(t)}{R} = C\dot{V}_c(t)$$

$$V_c(0^+) = V_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)\,d\tau = 0 + \frac{1}{C}\int_{0^-}^{0^+} \frac{\delta(t')}{R}\,dt' = \frac{1}{RC}$$

$$V_c^{imp}(t) = \frac{1}{RC}\, e^{-t/RC}\, u(t) \qquad t \geqslant 0$$

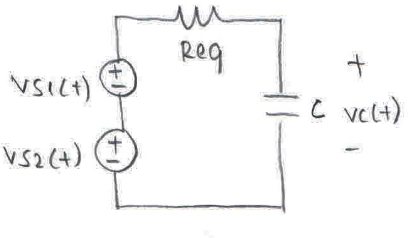
$V_c^{imp}(t) = h(t)$: general rotation for impulse response for LTI system



→ impulse response of RC circuit

Responses are calculated when the system does not contain any initial energy and the response is the reaction of the system to an input.

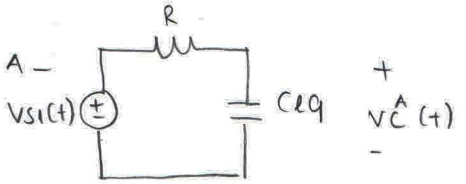
Linearity of zerostate Responses.

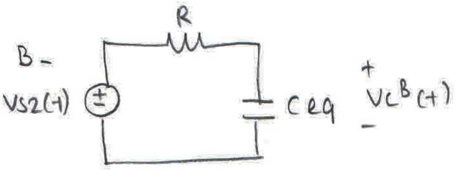


$V_c(0^-) = 0\ V$

$$\left(D+\frac{1}{RC}\right) V_c^{A+B}(t) = \frac{V_{S1}(t)+V_{S2}(t)}{RC}$$

$$V_c^{A+B}(0^-) = 0V \longrightarrow V_c^{A} + V_c^{B} = V_c^{A+B} \quad \text{satisfied}$$

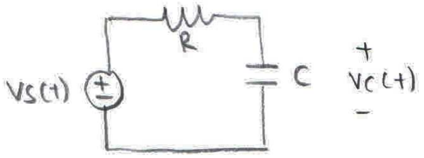




$v_C^A(t)$ is the response to V_{s1} $v_C^B(t)$ is the response to V_{s2}

so linearity of zerostate responses follows from the discussion

EX : $v_s(t) = 3u(t) + \delta(t)$



$v_C(0^-) = 0$

Find $v_C(t)$?

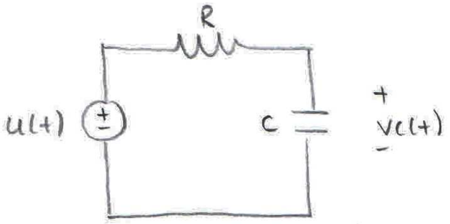
$$v_C(t) = 3v_C^{step}(t) + h(t)$$

$$v_C^{step}(t) \rightarrow (1 - e^{-t/RC})$$

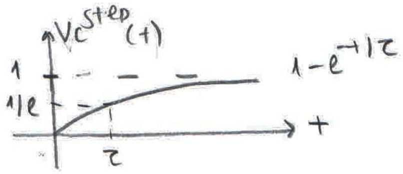
$$h(t) \rightarrow \left(\frac{1}{RC} e^{-t/RC}\right)$$

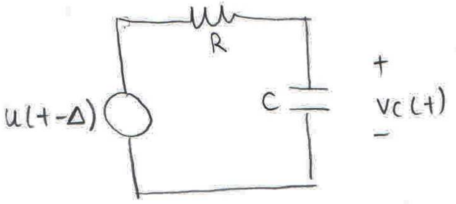
$$= 3 + \left[\frac{1}{RC} - 3\right] e^{-t/RC} \quad t \geq 0$$

Time invariancy

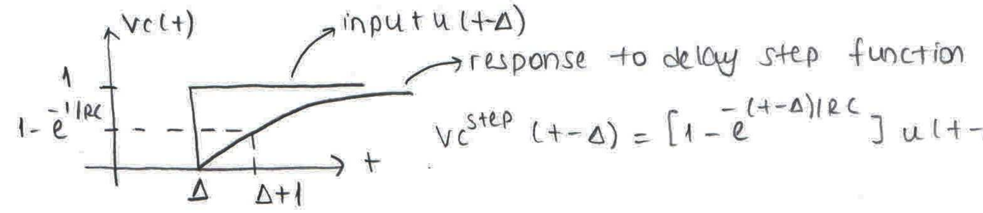


$v_C(0^-) = 0V$





$v_C(\Delta^-) = 0V$



$$v_C^{step}(t-\Delta) = \left[1 - e^{-(t-\Delta)/RC}\right] u(t-\Delta)$$

$$v_C^{step}(t) = (1 - e^{-t/RC})\, u(t)$$

For time invariant systems

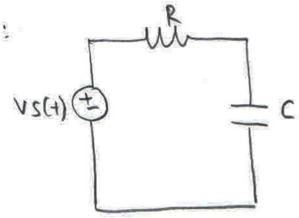
for RC circuit $\left(D + \frac{1}{RC}\right) v_C(t) = \frac{v_s(t)}{RC}$ $\quad v_C(0^-) = 0V \quad \Delta = 0$

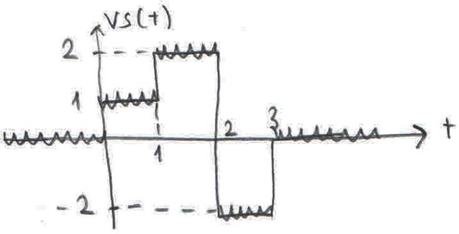
$\left(D + \frac{1}{RC}\right) v_C(t) = \frac{v_s(t-\Delta)}{RC}$ $\quad v_C^{\Delta}(\Delta) = 0V \quad v_C^{\Delta}(t) = v_C^{\Delta=0}(t-\Delta)$

$v_C^{\Delta}(\Delta) = 0$ Initial condition is satisfied

Diff-eqn is also time invariant

Ex:



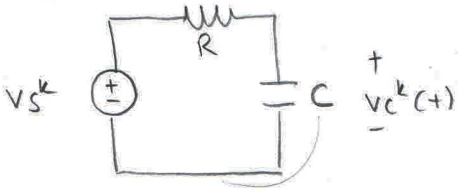


$v_c(0^-)=0V$

Find $v_c(t)$ in terms of step responses of the system

$$v_s(t)=\underbrace{u(t)}_{VS1}+\underbrace{u(t-1)}_{VS2}-\underbrace{4u(t-2)}_{VS3}+\underbrace{2u(t-3)}_{VS4}$$

By linearity of zerostate responses $v_c(t)=v_{c1}(t)+v_{c2}(t)+v_{c3}(t)+v_{c4}(t)$



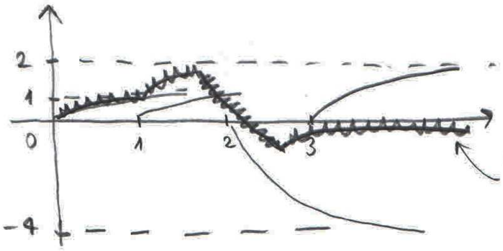
$k\in\{1,2,3,4\}$

$$v_c(t)=v_c^{step}(t)+v_c^{step}(t-1)-4v_c^{step}(t-2)+2v_c^{step}(t-3)$$

↳ response to delayed step input (time invariancy)

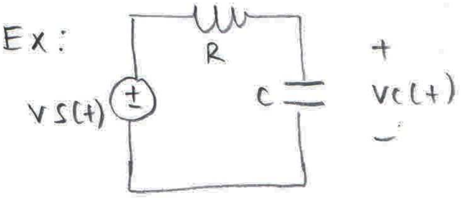
$$v_c^{step}(t)=(1-e^{-t/RC})u(t)$$

$$v_c(t)=(1-e^{-t/RC})u(t)+(1-e^{-(t-1)/RC})u(t-1)-4(1-e^{-(t-2)/RC})u(t-2)+2(1-e^{-(t-3)/RC})u(t-3)$$



$$v_c(t)=\sum_{k=1}^{4}v_c^k(t)$$

Ex:



$v_s(t)=\delta(t)\rightarrow v_c(t)=\frac{1}{RC}e^{-t/RC}u(t)$ impulse resp.

$v_s(t)=u(t)\rightarrow v_c(t)=(1-e^{-t/RC})u(t)$ step response

$v_s(t)=r(t)\rightarrow v_c(t)=(t-RC+RCe^{-t/RC})u(t)$ ramp res

$$\begin{tikzcd}
\delta(t) \arrow[r, shift left, "D^{-1}"] \arrow[d] & u(t) \arrow[l, shift left, "D"] \arrow[r, shift left, "D^{-1}"] \arrow[d] & r(t) \arrow[l, shift left, "D"] \arrow[d, "\Big\} LTI"] \\
h(t) \arrow[r, shift left] & y(t) \arrow[l, shift left] \arrow[r, shift left] & g^{ramp}(t) \arrow[l, shift left]
\end{tikzcd}$$

$$(d/dt)\,y^{step}(t)=((d/dt)(1-e^{-t/RC}))u(t)+(1-e^{-t/RC})\frac{d}{dt}u(t)$$

$$=\frac{1}{RC}e^{-t/RC}u(t)+\frac{1-e^{-t/RC}}{1-e^{-t/RC}}\underset{\substack{\downarrow\\ t=0}}{}\delta(t)$$

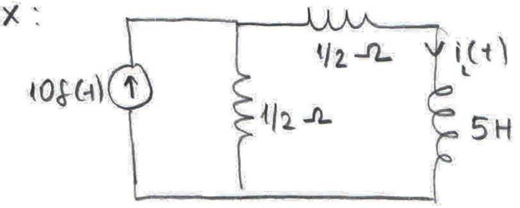
$$h(t)=\frac{1}{RC}e^{-t/RC}u(t)$$

So impulse response can be calculated by

1- Calculating step response

2- $h(t) = \frac{d}{dt}$ (unit-step) response

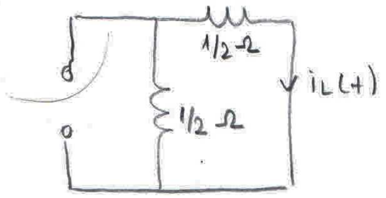
EX:



Find $i_L(t)$ $t \geq 0$

$i_L(0^-) = -2A$

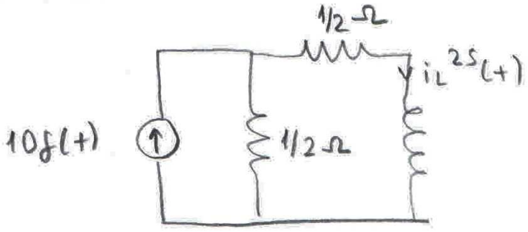
zero input



$\tau = L/R = 5$ sec's

$i_L^{zi}(t) = -2e^{-t/5}$ A

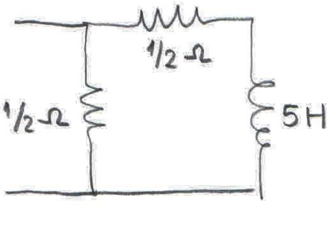
zero state



$i_L^{zs}(t) = i_L^{zs}(0^-) = 0A$

$i_L^{zs}(0^+) = \frac{1}{L}\int_{0^-}^{0^+} v_L(\tau)d\tau = 1A$

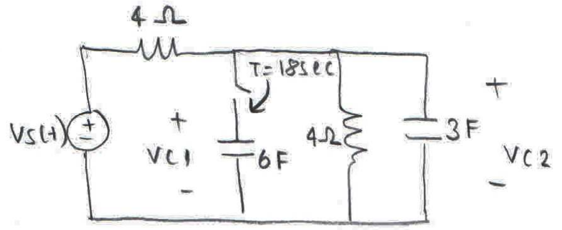
for $t > 0^+$ $i_L(0^+) = 1A$



$i_L^{zs}(t) = e^{-t/5}$ A

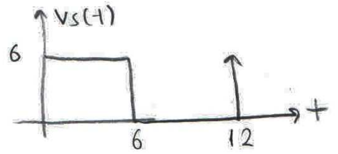
complete soln $= e^{-t/5}$ A

Ex:

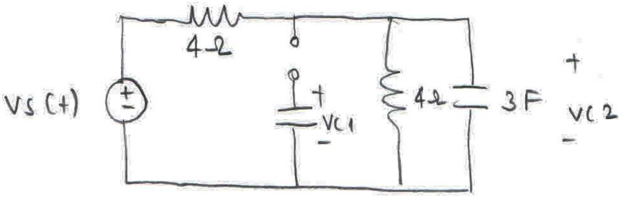


$v_{c1}(0^-) = -2V$ $v_{c2}(0^-) = 7V$

Input:



Before switch closes $t < 18$ sec

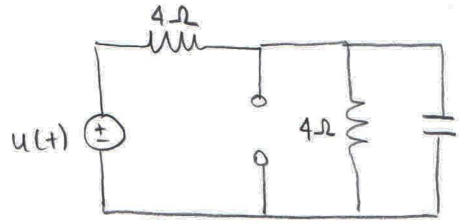


$v_{c1}(t) = v_{c1}(0^-) = -2$ $0^+ < t < 18^-$

$v_{c2}(0^-) = 7V$

$v_{c2}(t) = \underbrace{v_{c2}^{zi}(t)}_{7e^{-t/\tau}u(t)} + v_{c2}^{zs}(t)$ $\tau = 2\Omega \cdot 3F = 6sec$

Idea: Find the unit step response for $v_{c2}(t)$, and find the solution for the given input in terms of $v_{c2}^{unit\ step}(t)$



$V_{C2}^{\text{unitstep}}(t) = \left[\frac{1}{2} - \frac{1}{2}e^{-t/6}\right]u(t)$ zerostate response

$V_{C2}^{\text{unitstep}}(0^+) = 0\text{ V}$ $\quad V_{C2}^{\text{unit}}(\infty) = 1/2\text{ V}$

$h(t) = \frac{1}{12}e^{-t/6}u(t) \longrightarrow$ impulse response

$$V_{C2}^{zs}(t) = V_{C2}^{\text{step}}(t)\times 6 - 6V_{C2}^{\text{step}}(t-6) + h(t-12)\times 6 = 3(1-e^{-t/6})u(t) - 3(1-e^{-(t-6)/6})u(t-6)$$
$$+\frac{1}{2}e^{-(t-12)/6}u(t-12 \qquad 0^+ < t < 18^-$$

$V_{C2}(t) = V_{C2}^{zi}(t) + V_{C2}^{zs}(t)$ complete solutions for $0^+ < t < 18^-$

$V_{C2}(18^-) = 2.14\text{ V}$

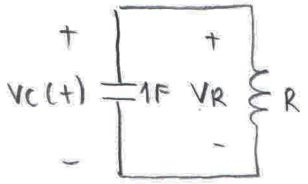
$V_{C1}(18^-) = -2\text{ V}$

After switch closes $t > 18$

$$V_{com}(18^+) = \frac{2.14\times 3 + (-2)\cdot 6}{9} = -0.62\text{ V}$$

$V_{C2}(t) = (-0.62)e^{-(t-18)/6}u(t-18)$ for $t > 18^+$

Time varying and/or Nonlinear First Order Circuits



a.) R ; $R = 1\Omega$ linear time invariant

b.) R ; $R(t) = \frac{1}{1+0.5\cos t}$ linear time varying

c.) R ; Nonlinear resistor with $i_R = V_R^2$ nonlinear time inv

$V_C(0^-) = 1\text{V}$

a.) $C\dot{V_C} = -i_R \qquad \dot{V_C} = -\frac{V_C}{1} \qquad V_C(0^-) = 1\text{V} \qquad V_C(t) = e^{-t/1}\text{ V}$

b.) $C\dot{V_C} = \frac{-V_C(t)}{R(t)} = -(1+0.5\cos t)V_C(t)$

$\frac{d}{dt}V_C(t) = -(1+0.5\cos t)V_C(t)$

$\frac{dV_C(t)}{V_C(t)} = -(1+0.5\cos t)dt \qquad \ln(V_C(t)) = -(t+0.5\sin t) + K$

$V_C(t) = Ae^{-(t+0.5\sin t)}$

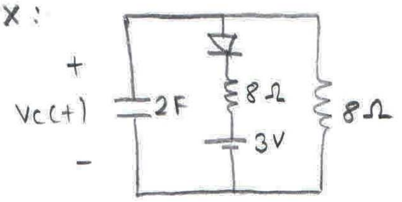
$V_c(0)=1 \qquad V_c(t) = e^{-(t+0.5\sin t)} \quad V$

c-) $C\dot{V_c}(t) = -i_R = -V_c^2(t)$

$\frac{dV_c(t)}{dt} = -V_c^2(t) \quad ; \qquad \frac{dV_c(t)}{V_c^2(t)} = -dt \qquad ; -[V_c(t)]^{-1} = -t+k$

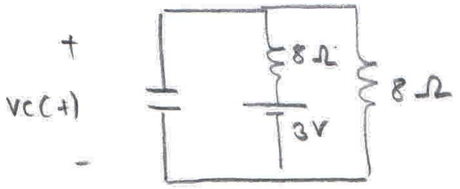
$V_c(t) = \frac{1}{t-k} \qquad V_c(0)=1 \Rightarrow k=-1 \quad \text{then} \quad V_c(t) = \frac{1}{t+1}$

EX:



$V_c(0^-) = 10V \quad V_c(t) = ?$

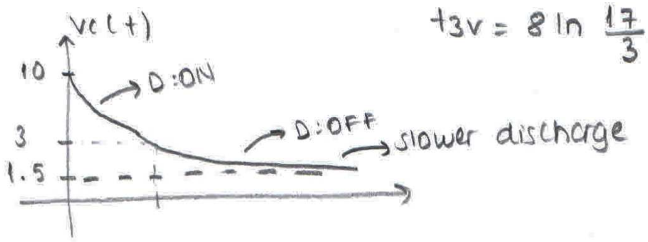
If $V_c(t) > 3V$ D:ON



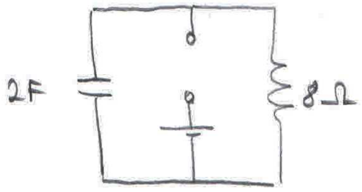
$V(\infty) = 1.5\,V$

$V_c(t) = 1.5 + 8.5\,e^{-t/8} \longrightarrow V_c(t_{3V}) = 3V$

$t_{3V} = 8\ln\frac{17}{3}$



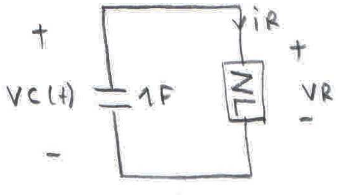
If $V_c(t) < 3$ D:OFF



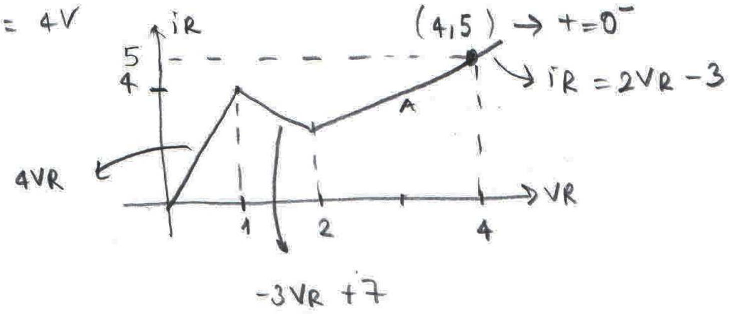
$V_c(t_{3V}^+) = 3V$

$V_c(t) = 3\,e^{-(t-t_{3V})/16} \quad V$

EX:



$V_c(0^-) = 4V$



$C\dot{V_c}(t) = -i_R = -f(V_c)$

$V_c(0^-) = 4V$

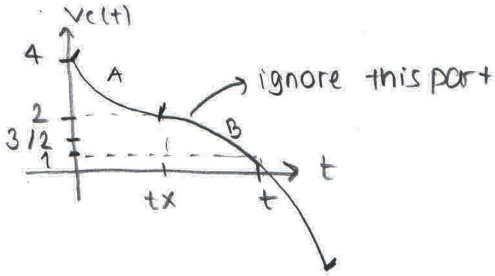
$V_c(0^+) = -f(V_c(0^+))$

a-) $2 < V_c < 4$ In Ⓐ segment

$\dot{V_c}(t) = -i_R = 3 - 2V_c$

$V_c(t)(D+2) = 3 \qquad V_c(0^-) = 4 \; V$ then

$V_c(t) = 2.5\,e^{-2t} + 1.5$



b- $1 < V_c < 2$ $\quad V_c^{\bullet}(t) = -i_R = 3V_R - 7$

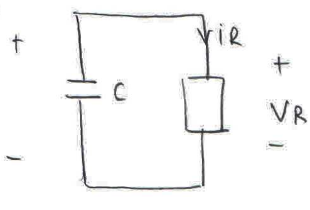
$(D-3)V_c(t) = -7$

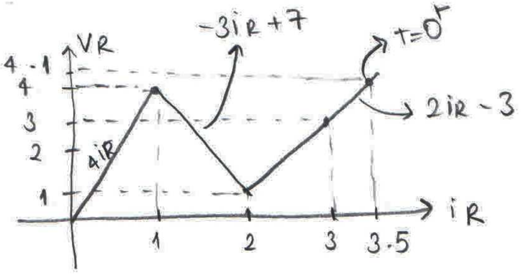
$V_c(t_x) = 2V$

$$V_c(t) = \left[\frac{7}{3} - \frac{1}{3}e^{3(t-t_x)}\right] u(t-t_x)$$

$v_c^{\bullet}(t) = -i_R = -4V_c \quad v_c^{\bullet}(t) = -i_R = -4V_c \quad V_c(t_y) = 1V \quad V_c(t) = e^{-4(t-t_y)} u(t-t_y)$

Ex :





current controlled

$V_c(0^-) = 4.1V$

$Cv_c^{\bullet}(t) = -i_R(t) \quad C = 1F$

at $t = 0^+ \quad v_c^{\bullet}(0^+) = -3.55$

for $4 < V_c < 4.1$

$$V_c^{\bullet}(t) = -i_R(t) = -\frac{V_c(t)+3}{2} \qquad \left(D + \frac{1}{2}\right)V_c(t) = -\frac{3}{2}$$

$V_c(t) = -3 + 7.1e^{-t/2}$ for $4 < V_c(t) < 4.1$

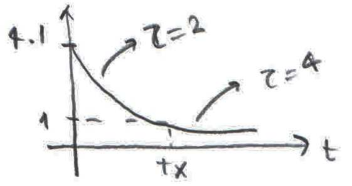
when $1 < v_c(t) < 4$ there are 3 possibilities for $i_R(t)$ that is there are 3 operating points. The solution prefers to be on the same mode that it has travelled upon, so $1 < V_c(t) < 4$ the previous diff. eqn is still valid.

$(D + 1/2)v_c(t) = -3/2$ is still valid since this is the preferred path

$V_c(t) = -3 + 7.1e^{-t/2} \quad 1 < v_c(t) < 4.1$

for $V_c(t) < 1$

$v_c^{\bullet}(t) = -i_R(t) = -\frac{V_c(t)}{4} \qquad V_c(t) = e^{-(t-t_x)/4} u(t-t_x)$ when $t > t_x$

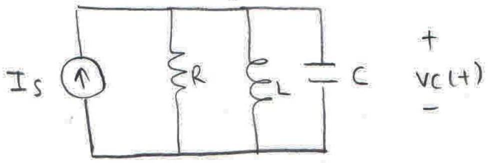


Second Order Systems

parallel RLC ;

$i_L(0^-) = I_o$

$V_c(0^-) = V_o$



$$\frac{v_c(t)}{R} + \underset{\uparrow}{i_L(t)} + Cv_c^{\bullet}(t) = i_s(t)$$

$$i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_L(\tau)d\tau$$

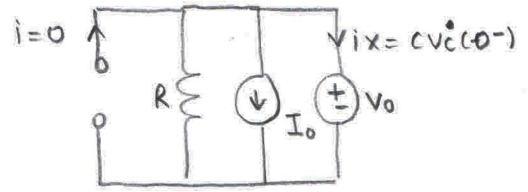
$$\frac{v_C(t)}{R} + i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_C(\tau)\,d\tau + C\dot{v}_C(t) = i_s(t) \qquad t \geq 0$$

↑ integro differential equation

take $\frac{d}{dt}$; $\quad \frac{1}{L}v_C(t) + \frac{1}{R}\dot{v}_C(t) + C\ddot{v}_C(t) = \frac{d}{dt}\,i_s(t)$

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)v_C(t) = \frac{1}{C}\frac{d}{dt}\,i_s(t) \qquad v_C(0^-) = V_0$$

At $t = 0^-$



$i_x = -\frac{V_0}{R} - I_0 \qquad \dot{v}_C(0^+) = \frac{-(V_0 + R I_0)}{RC}$

State equation : 1st Order Matrix differential equations

State equation are first order diff. eqn. set such that right hand side is a linear combination of state variables and input. State variables = $\{v_C(t), I_L(t)\}$

KCL : $\frac{v_C(t)}{R} + I_L(t) + C\dot{v}_C(t) = i_s(t)$

KVL : $L\dot{I}_L(t) = v_C(t)$

$$\begin{bmatrix}\dot{v}_C(t)\\ \dot{I}_L(t)\end{bmatrix} = \begin{bmatrix}-1/RC & -1/C\\ 1/L & 0\end{bmatrix}\begin{bmatrix}v_C(t)\\ I_L(t)\end{bmatrix} + i_s\begin{bmatrix}1/C\\ 0\end{bmatrix}$$

Zero Input Response

Parallel RLC ; $\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)v_C(t) = \cancel{\frac{1}{C}D\,i_s(t)}$ ↗ 0 for zero input

$(D^2 + 2\alpha D + w_0^2)v_C(t) = 0$

For parallel RLC

$2\alpha = 1/RC \qquad w_0^2 = 1/LC$

α : damping factor

w_0 : resonance frequency

$(D^2 + 2\alpha D + w_0^2)\,v_C(t) = 0$

$v_C(0^-) = V_0$

$\dot{v}_C(0^-) = \dot{V}_0$

Constant Coefficient Differential Equation

Guess $v_C(t) = \beta e^{\lambda t}$ by substituting into diff eqn ;

$(\lambda^2 + 2\alpha\lambda + w_0^2)\,\beta\, e^{\lambda t} = 0$

↙ $\beta = 0$ trivial solution

$v_C(t) = 0$ initial condition not satisfy

↘ $\lambda^2 + 2\alpha\lambda + w_0^2 = 0 \qquad \lambda_{1,2} = \frac{-2\alpha \mp \sqrt{4\alpha^2 - 4w_0^2}}{2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2}$

3 possible solutions for λ_1 and λ_2

1. Roots are real and distinct $(\Delta > 0)$, $\alpha > w_0$ and $\lambda_1 \neq \lambda_2$

$Vc(t) = C_1 e^{-2t} + C_2 e^{-3t}$ $\lambda_1 = -2$ $\lambda_2 = -3$

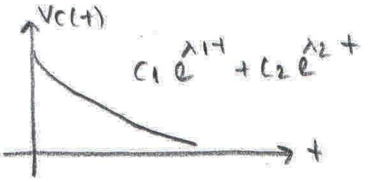
2. Root are real and same $\lambda_1 = \lambda_2$ $(\Delta = 0$ $\alpha = w_0)$

$Vc(t) = C_1 e^{-t} + C_2 t e^{-t}$ $\lambda_1 = \lambda_2 = -1$

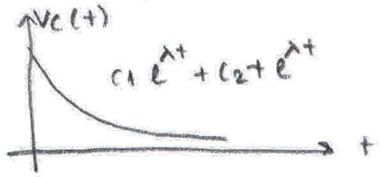
3. Roots are complex, discriminant is negative $\Delta < 0$ $\lambda_1 = \lambda_2^*$ [polynomial has r
coefficients

$Vc(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$ $\lambda_1 = -1 + j3$ $\lambda_2 = -1 - j3$

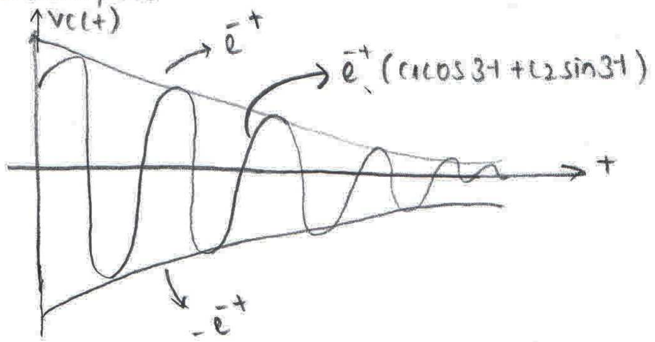
1. Overdamped $(\lambda_1 \neq \lambda_2)$



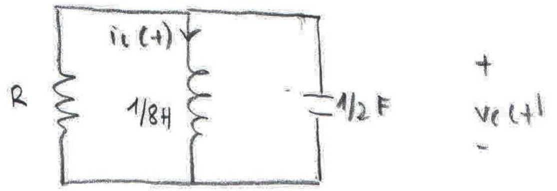
2. Critically damped $(\lambda_1 = \lambda_2)$



3. Underdamped



Ex:



$i_L(0^-) = -4A$

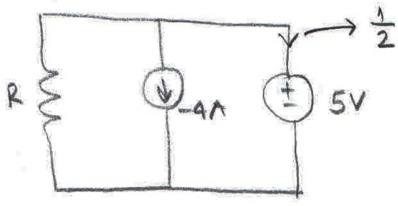
$Vc(0^-) = 5V$

a.) R = 1/5

$(D^2 + 2\alpha D + w_0^2) Vc(t) = 0$

$\dot{Vc}(0^-) = ?$

$\alpha = \frac{1}{2RC} = 5$ $w_0^2 = \frac{1}{LC} = 16$



$\frac{1}{2} \dot{Vc}(0^+) = 4 - \frac{5}{R}$

$\dot{Vc}(0^+) = 8 - \frac{10}{R}$

$\dot{Vc}(0^+) = 42$ V/sec

$\alpha > w_0$ overdamped case

$V_c(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$(D^2+10D+16)\, V_c(t) = 0 \implies \lambda^2 + 10\lambda + 16 = 0 \qquad \lambda_{1,2} = \{-2, -8\}$

$$\left.\begin{array}{l} V_c(t) = C_1 e^{-2t} + C_2 e^{-8t} \\ \dot{V}_c(t) = -2C_1 e^{-2t} - 8C_2 e^{-8t} \\ V_c(0^+) = 5 \qquad \dot{V}_c(0^+) = -42 \end{array}\right] \qquad \begin{bmatrix} 1 & 1 \\ -2 & -8 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -42 \end{bmatrix}$$

$C_1 = -\frac{1}{3} \qquad C_2 = \frac{16}{3}$

$V_c(t) = -\frac{1}{3} e^{-2t} - \frac{16}{3} e^{-8t}$

b- $R = 1/4\ \Omega$;

$\alpha = \frac{1}{2RC} = 4$ & $w_0 = 4 \qquad \alpha = w_0$ critically damped

$(D^2+8D+16)\, V_c(t) = 0$

$(D+4)^2 = 0 \qquad V_c(t) = (5 - 12t)\, e^{-4t} \qquad t \geq 0$

c- $R = 1/3\ \Omega$

$$\left.\begin{array}{l} \alpha = \frac{1}{2RC} = 3 \\ \\ w_0 = 4 \end{array}\right| \ \alpha < w_0 \text{ underdamped}$$

$(D^2+6D+16)\, V_c(t) = 0 \qquad V_c(0^-) = 5$

$\dot{V}_c(0^-) = -\frac{1}{C}\left[\frac{V_0}{R} + I_0\right] = -22$

$\lambda^2 + 6\lambda + 16 = 0$

$\lambda_{1,2} = -3 \mp j\sqrt{7} \qquad V_c(t) = \beta_1 e^{(-3+j\sqrt{7})t} + \beta_2 e^{(-3-j\sqrt{7})t}$

$\dot{V}_c(t) = \beta_1 \lambda_1 e^{\lambda_1 t} + \beta_2 \lambda_2 e^{\lambda_2 t}$

$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -22 \end{bmatrix}$ therefore; $\beta_1 = \frac{5}{2} + j\frac{\sqrt{7}}{2} \qquad \beta_2 = \frac{5}{2} - j\frac{\sqrt{7}}{2}$

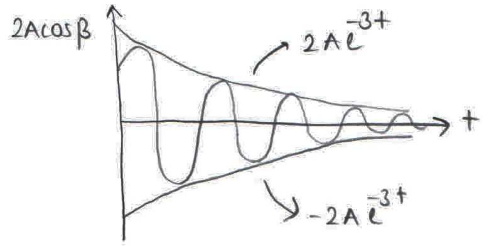
$V_c(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}$

$= 2Re\{\beta_1 e^{\lambda_1 t}\} = 2Re\{\|\beta_1\| e^{j\measuredangle B_1} e^{\lambda_1 t}\} = 2Re\{\|\beta\| e^{j\measuredangle B_1 (\lambda_1^r + j\lambda_1^i)t}\}$

$= 2\|\beta\| e^{\lambda_1^r t} Re\{e^{j(\lambda^i t + \measuredangle B_1)}\} \qquad \lambda_1 = \lambda^r + j\lambda^i$

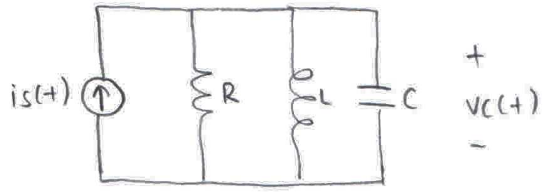
$= 2\|\beta\| e^{\lambda^r t} \cos(\lambda^i t + \measuredangle B_1)$ V

$V_c(t) = 2A e^{-3t} \cos(\sqrt{7}t + \beta)$ select A and B to match initial condition.



$v_C(t) = \gamma_1 e^{-3t}\cos(\sqrt{7}t) + \gamma_2 e^{-3t}\sin(\sqrt{7}t)$

2^{nd} Order Systems



$(D^2 + \frac{1}{RC}D + \frac{1}{LC})v_C(t) = \frac{1}{C}\frac{d}{dt}i_S(t)$

$v_C(t) = L\frac{d}{dt}i_L(t) = LDi_L(t)$

$v_C(0^-) = V_0 \quad I_L(0^-) = I_0$

$(D^2 + \frac{1}{RC}D + \frac{1}{LC})I_L(t) = \frac{1}{LC}i_S(t) \longleftarrow (D^2 + \frac{1}{RC}D + \frac{1}{LC})DI_L(t) = \frac{1}{CL}Di_S(t)$

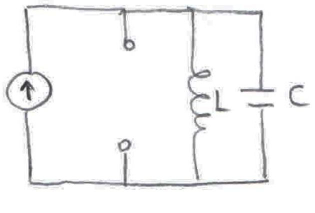
a.) Overdamped $(R = 1/5, L = 1/8, C = 1/2\,F)$ $v_C^{zi}(t) = \alpha e^{-2t} + \beta e^{-8t}$

b.) Critically damped $(R = 1/4, L = 1/8, C = 1/2\,F)$ $v_C^{zi}(t) = \gamma_1 e^{-4t} + \gamma_2 t e^{-4t}$

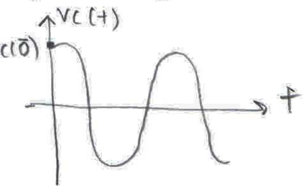
c.) Underdamped system $(R = 1/3, L = 1/8, C = 1/2\,F)$ $v_C^{zi}(t) = Ae^{-3t}\cos\sqrt{7}t + Be^{-3t}\sin\sqrt{7}t$

$= \sqrt{A^2+B^2}\,e^{-3t}\cos(\sqrt{7}t - \tan^{-1}(B/A))$

d.) Lossless system $(R = \infty, L = 1/8, C = 1/2\,F)$



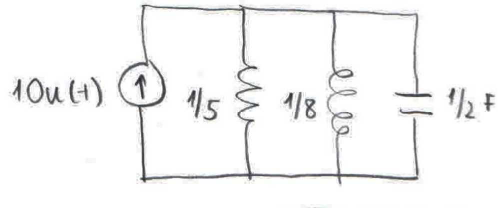
$(D^2+16)v_C^{zi}(t) = 0 \qquad v_C^{zi}(t) = A\cos 4t + B\sin 4t$



Zero State Responses

Ramp, unitstep, impulse response : Remember for the zerostate response, initia condition all zero. Hence the same zero state.

DC input case

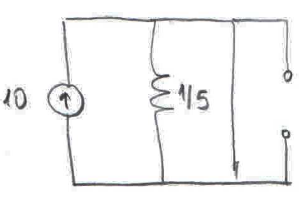


$\longrightarrow$ Overdamped case

$v_C(0^-) = V_0 \qquad I_L(0^-) = I_0$

$v_C(t) = \underbrace{Ae^{-2t} + Be^{-8t}}_{\text{zero input}} + (\underline{\qquad})$ Response

as $t \to \infty$



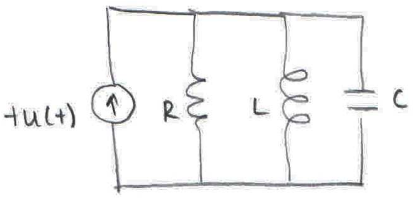
$v_C(\infty) = 0\,V$

$I_L(\infty) = 10\,A$

$I_L(t) = Ce^{-2t} + De^{-8t} + 10$

[A, B, C, D are found from initial conditions]

Ramp Response:



$$(D^2+2\alpha D+w_0^2)\, v_C^{2S}(t) = \frac{1}{C}\frac{d}{dt}(t\,u(t))$$

$$v_C^{2S}(0^-) = I_L^{2S}(0^-) = 0$$

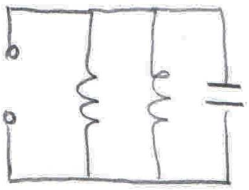
$$(D^2+2\alpha D+w_0^2)\, v_C^{2S}(t) = \frac{1}{C}u(t) \qquad t>0$$

$$(D^2+2\alpha D+w_0^2)\, v_C^{2S}(t) = \frac{1}{C} \qquad t>0$$

$$v_C^{2S}(t) = A + Be^{\lambda_1 t} + Ce^{\lambda_2 t} \qquad t>0$$

$A = \frac{1}{Cw_0}$ Select B and C such that $v_C(0^+)$ and $v_C'(0^+)$ values are satisfied.

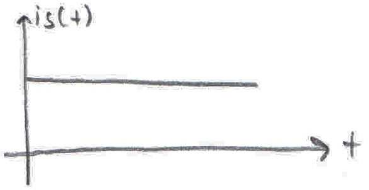
At $t=0^+$



$v_C(0^+) = v_C(0^-) = 0$ | $v_C^{\bullet}(t) = 0$

$I_L(0^+) = I_L(0^-) = 0$ | $I_L^{\bullet}(t) = 0$

Unit Step Response:



$$(D^2+2\alpha D+w_0^2)\, v_C^{2S}(t) = \frac{1}{C}\delta(t)$$

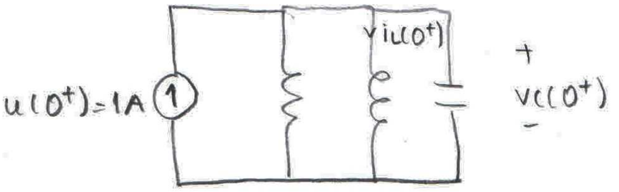
1. Circuit Theory based Approach:

$$(D^2+2\alpha D+w_0^2)\, v_C^{2S}(t) = 0 \qquad t>0$$

$$v_C^{2S}(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \qquad t>0$$

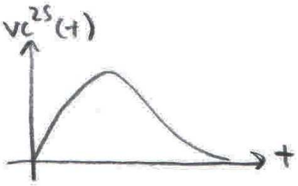
Using $v_C(0^+)=0$ and $v_C(0^+) = 1/C$ find A, B and finalize the step response

At $t=0^+$



$v_C(0^-) = v_C(0^+) = 0$

$I_L(0^-) = I_L(0^+) = 0$



$i_C(0^+) = C v_C^{\bullet}(0^+) = 1 \qquad v_C^{\bullet}(0) = 1/C$

2. Mathematical Approach

$$(D^2+2\alpha D+w_0^2)\, v_C(t) = \frac{1}{C}\overset{g_\epsilon(t)}{\delta(t)}$$

$v_C(t)$ contains $g_\epsilon(t)$

$v_C^{\bullet}(t)$ should contain $g_\epsilon(t)$

Define $\lim_{\epsilon \to 0} g_\epsilon(t) = \delta(t)$

But right hand side diff equation does not contain any $\dot{g}_\epsilon(t)$

So $v_c(t)$ does not contain $g_\epsilon(t)$

$\dot{v}_c(t)$ contain $g_\epsilon(t)$

$\ddot{v}_c(t)$ should contain $\dot{g}_\epsilon(t)$. So this is not also possible

RHS has only $g_\epsilon(t)$, $\ddot{v}_c(t)$ is impulsive

$\dot{v}_c(t)$ does not contain an impulse (step discontinuity)

$v(t) \rightarrow$ does not contain an impulse

Then integrate both sides of diff eqn between 0^- and 0^+

$$\int_{0^-}^{0^+} \ddot{v}_c(t)\,dt' + 2\alpha \int_{0^-}^{0^+} \underset{\substack{\downarrow \\ \text{does not} \\ \text{contain an impulse}}}{\dot{v}_c(t')}\,dt' + \omega_0^2 \int_{0^-}^{0^+} v_c(t')\,dt' = \frac{1}{c} \int_{0}^{0^+} \delta(t)\,dt' = \frac{1}{c}$$

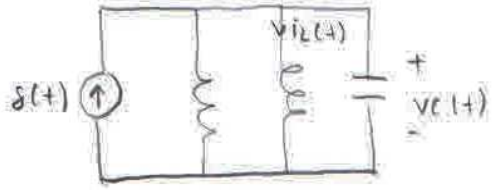
$\dot{v}_c(0^+) - \dot{v}_c(0^-) + 2\alpha \cdot 0 + \omega_0^2 \cdot 0 = 1/c$

since $\dot{v}_c(0^-) = \dot{I}_L(0^-) = v_c(0^-) = I_L(0^-) = 0$

$\dot{v}_c(0^+) = 1/c$

$v_c(0^+) = \int_{0^-}^{0^+} \dot{v}_c(t')\,d' + v_c(0^-) = 0$ Same initial conditions found by circuit theoratical inspection.

Impulse Response:



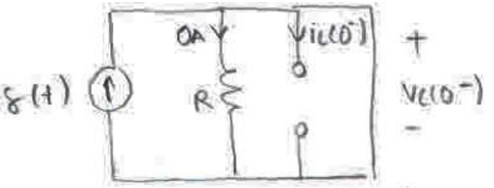
$(D^2 + 2\alpha D + \omega_0^2) v_c(t) = \frac{1}{c} \frac{d}{dt} \delta(t)$

1- Solution by Circuit theoratical Inspection

$(D^2 + 2\alpha D + \omega_0^2) v_c(t) = 0 \quad t > 0$

$v_c(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$

At $0^- < t < 0^+$



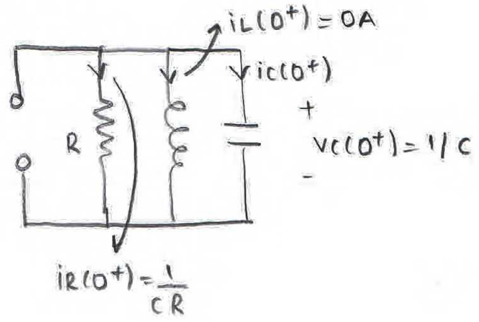
$v_c(0^-) = 0V \quad I_L(0^-) = 0A$

$i_c(t) = \delta(t) \quad 0^- < t < 0^+$

$v_c(0^+) = v_c(0^-) + \frac{1}{c} \int_{0^-}^{0^+} i_c(\tau)\,d\tau = v_c(0^-) + \frac{1}{c} = \frac{1}{c}$

$I_L(0^+) = I_L(0^-) + \frac{1}{L} \int_{0^-}^{0^+} v_L(\tau)\,d\tau = 0$

At $t=0^+$



$i_C(0^+)=-1/RC$

$C\dot{v}_c(0^+)=-1/RC$

$\dot{v}_c(0^+)=-1/RC^2$

From I.C.'s I and II, the impulse response is found.

2. Mathematical Approach for the solution

$$(D^2+2\alpha D+w_0^2)v_c(t)=\frac{1}{C}\dot{\delta}(t)$$

$\ddot{v}_c(t)$ contains $\delta'(t)$, $\dot{v}_c(t)$ contains $\delta(t)$, $v_c(t)$ does not contan $\delta(t)$

$\delta'(t)$ doublet $\quad \int_{0^-}^{0^+} f(t')\dot{\delta}(t')dt'=f'(0)$

$$\int_{0^-}^{0^+}\ddot{v}_c(t')dt'+2\alpha\int_{0^-}^{0^+}\dot{v}_c(t')dt'+w_0^2\int_{0^-}^{0^+}v_c(t)dt=\frac{1}{C}\int_{0^-}^{0^+}1\cdot\delta'(t')dt'=0$$

$$\dot{v}_c(0^+)-\dot{v}_c(0^-)+2\alpha\left[v_c(0^+)-v_c(0^-)\right]+w_0^2\cdot 0=\frac{1}{C}\int_{0^-}^{0^+}\delta'(t')dt'=0$$

$$\dot{v}_c(0^+)+2\alpha v_c(0^+)=0$$

To find another equation to find $v_c(0^+)$ & $\dot{v}_c(0^+)$

D^{-1} $\quad (D^2+2\alpha D+w_0^2)v_c(t)=\frac{1}{C}D\delta(t)$

$(D+2\alpha+w_0^2D^{-1})v_c(t)=\frac{1}{C}\delta(t)$

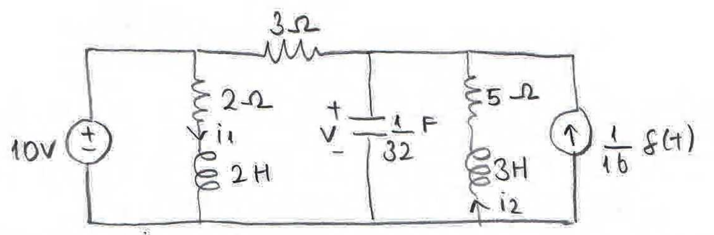
D^{-1} $\quad \int_{0^-}^{0^+}\dot{v}_c(t')dt'+2\alpha\int_{0^-}^{0^+}v_c(t')dt'+w_0^2\int_{0^-}^{0^+}\left[\int_{0^-}^{t}v_c(t'')dt''\right]dt'=\frac{1}{C}$

$v_c(0^+)-v_c(0^-)+2\alpha\cdot 0+w_0^2\cdot 0=\frac{1}{C}$

$v_c(0^+)=\frac{1}{C}$ and using the other equation

$\dot{v}_c(0^+)+\underset{1/RC}{\underset{\downarrow}{2\alpha}}\,\underbrace{v_c(0^+)}_{1/C}=0 \qquad \dot{v}_c(0^+)=-1/RC^2$

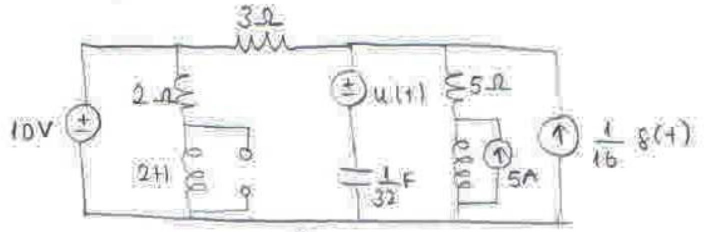
Ex:



$V(0^-)=1V \quad i_1(0^-)=0A$

$i_2(0^-)=5A$

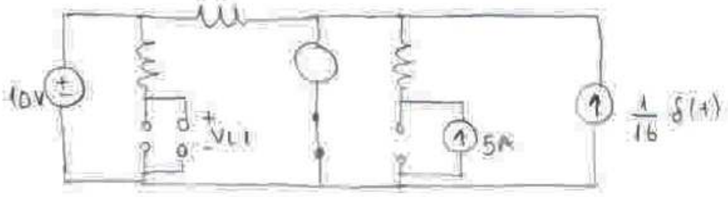
Find $t=0^+$ and $t\to\infty$ solution of the circuit at $0^- < t < 0^+$



initial condition models are implemented
cap, 2H and 3H shown have zero energy
$v_c(0^+)$, $i_1(0^+)$ and $i_2(0^+) = ?$

$$v_c(0^+) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)d\tau$$

$i_c(t) = ?$ $\quad 0^- < t < 0^+$



$$i_c(t) = \frac{1}{16}\delta(t) + 5 + \left(\frac{10-1}{3}\right) \qquad V_c(0^+) = V_c(0^-) + \underbrace{\frac{1}{C}\int_{0^-}^{0^+}\left[\frac{1}{16}\delta(t') + k\right]dt'}_{2}$$

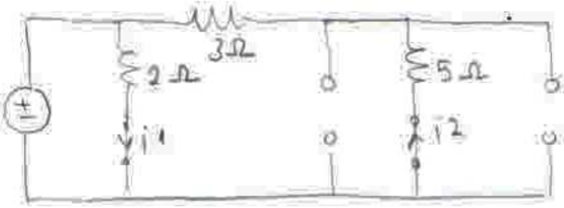
$$V_c(0^+) = 3\ V$$

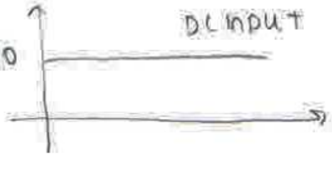
$$i_1(0^+) = i_1(0^-) + \frac{1}{L}\int_{0^-}^{0^+} V_{L1}(t')dt'$$

$V_{L1} = 10V \qquad V_{L2} = 26V$

$i_1(0^+) = i_1(0^-) = 0A \qquad i_2(0^+) = i_2(0^-) = 5A$

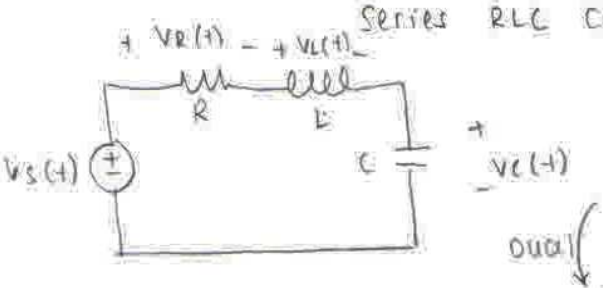
$t \to \infty \quad i_1(t) \to A \quad i_2(t) \to B \quad V_c(t) \to C$





$i_1(\infty) = 5A$
$i_2(\infty) = -\frac{10}{8}A$
$v_c(\infty) = \frac{25}{4}\ V$

Series RLC Circuits



$V_c(0^-) = V_0 \qquad I_L(0^-) = I_0$

= Dual of parallel RLC

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_c(t) = \frac{1}{C}\frac{d}{dt}i_s(t)$$

dual

$V_c \to I_L \quad R \to G \quad L \rightleftarrows C \quad i_s \longleftrightarrow V_s$

$$\boxed{\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)I_L(t) = \frac{1}{L}D\,V_s(t)}$$

by KVL; $-vs(t) + R\, i_L(t) + L \frac{di_L(t)}{dt} + v_C(t) = 0$

$$(R + LD) i_L(t) + v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i_C(\tau) d\tau = vs(t)$$

apply D
divide by L

$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) i_L(t) = \frac{1}{L} D vs(t)$$ series RLC

$$(D^2 + 2\alpha D + w_0^2) D = \frac{1}{L} D vs(t)$$

$\alpha = R/2L \quad w_0 = \sqrt{1/LC}$ series RLC

$\alpha = 1/2RC \quad w_0 = \sqrt{1/LC}$ parellel RLC

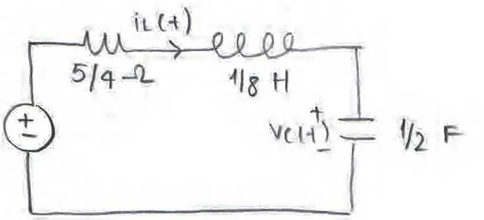
So for both RLC configurations;

$\alpha > w_0 \rightarrow$ overdamped $\quad \gamma_1 e^{-t} + \gamma_2 e^{-2t}$

$\alpha = w_0 \rightarrow$ critically damped $\quad \gamma_1 e^{-t} + \gamma_2 t e^{-t}$

$\alpha < w_0 \rightarrow$ underdamped $\quad \gamma_1 e^{-t} \cos 3t + \gamma_2 e^{-t} \sin 3t$

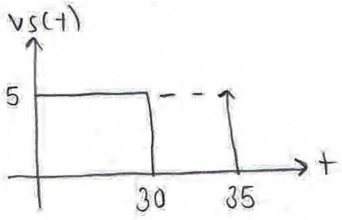
Ex:



$v_C(0^-) = -3V$

$i_L(0^-) = 0A$

Find $v_C(t)$



$\alpha = 5 \quad w_0 = 4$

short and preffered Method

zi zs

$$(D^2 + 2\alpha D + w_0^2)\, i_C^{zi}(t) = 0$$

$\lambda^2 + 10\lambda + 16 = 0 \quad \lambda = \{-2, -8\}$ overdamped

$v_C(t) = c_1 e^{-2t} + c_2 e^{-8t}$

$v_C(0^-) = -3V = c_1 + c_2$

$C\dot{v}_C(0^-) = I_L(0^-) = 0A$;

$c_1 + c_2 = -3$

$-2c_1 - 8c_2 = 0$

$c_1 = -4 \quad c_2 = 1$

$v_C^{zi}(t) = -4e^{-2t} + e^{-8t}$

$v_C^{zs}(t) = ?$

$vs(t) = 15\big(u(t) - u(t-30)\big) + 3\delta(t-35)$

$v_C^{zs} = 15\big(v_C^{step}(t) - v_C^{step}(t-30)\big) + 3v_C^{imp}(t-35)$

$v_C^{step}(t) = 1 + Ae^{-2t} + Be^{-8t}$

$v_C^{step}(0+) = 0V \rightarrow$ Equal to 0^- values

$\dot{v}_C^{step}(0^+) = 0V$

$A + B = -1$

$-2A - 8B = 0$

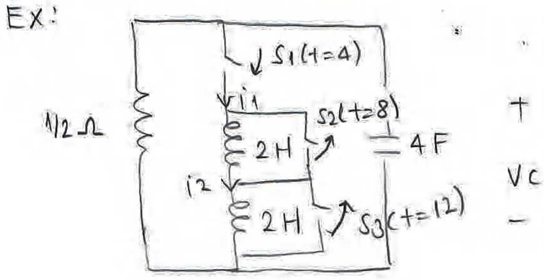
$A = -4/3 \quad B = 1/3$

$$v_C^{step}(t) = \left(1 - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t}\right) u(t)$$

$$h(t) = \frac{d}{dt} v_C^{step}(t) = \underbrace{\frac{v_C^{step}(t)\,\delta(t)}{v_C(0)\,\delta(t)}}_{0} + \left(\frac{8}{3}e^{-2t} - \frac{8}{3}e^{-8t}\right) u(t)$$

$$h(t) = \frac{8}{3}\left(e^{-2t} - e^{-8t}\right) u(t)$$

Ex:

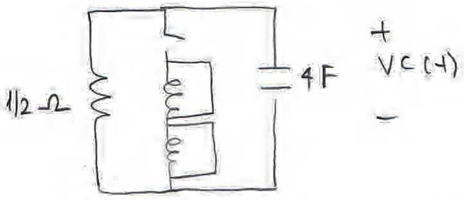


S1 closes at t= 4 sec

S2 opens at t= 8sec

S3 opens at t= 12 sec

0<t<4

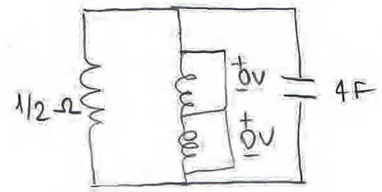


$$i_1(t) = i_1(0^-) + \underbrace{\frac{1}{L}\int_{0^-}^{t} V_{L1}(\tau)d\tau}_{0}$$

$i_1(t) = 1A$ $i_2(t) = -2A$ $Vc(t) = 5e^{-t/2}$ $\tau = RC = 2 sec.$

$Vc(4^-) = 5e^{-2}$ V

4<t<8

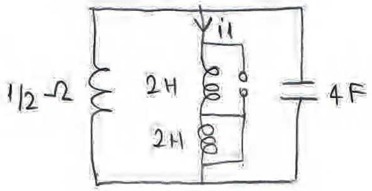


$Vc(4^+) \neq Vc(4^-)$

$Vc(4^+) = 0V$

$Vc(t) = 0$; $4 < t < 8$

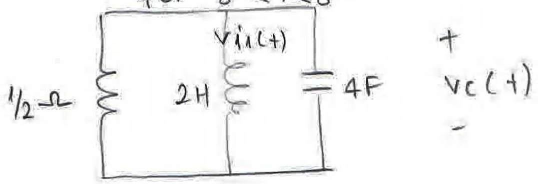
8<t<12



$i_1(8^-) = 1A$ $Vc(8^-) = 0V$ $i_2(8^-) = -2A$

$i_1(8^+) = 1A$ $Vc(8^+) = 0V$ $i_2(8^+) = -2A$

since for $8^- < t < 8^+$



$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)vc(t) = 0$$

$vc(8^+) = 0$ $C\dot{vc}(8^+) = -1$ $\dot{vc}(8^+) = -1/4$

$\alpha = 1/4$ $w_0 = 1/2\sqrt{2}$ $\alpha < w_0$ Underdamped response

8<t<12

$\lambda_{1,2} = \left\{ \frac{1}{4} \mp j\frac{1}{4} \right\}$ Natural frequencies

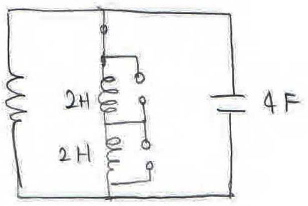
$vc(t) = e^{-(t-8)/4}$

$vc(t) = e^{-(t-8)/4}\left(A\cos\left(\frac{t-8}{4}\right) + B\sin\left(\frac{t-8}{4}\right)\right)u(t-8)$ A,B to be found from I.C

$vc(8^+) = 0 = A$

$vc^{\circ}(8^+) = -\frac{1}{4}$] solution $vc(t) = \left[-e^{-(t-8)/4} \sin\left(\frac{t-8}{4}\right)\right] u(t-8)$ $\quad 8 < t < 12$

$12 < t < \infty$

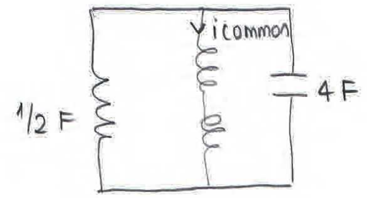


$i_1(12) \neq i_2(12)$ we apply conservation of flux

Total flux $\rightarrow L_1 I_1(12^-) + L_2 I_2(12^-) = (L_1 + L_2)\, I\,common\,(12^+)$

$$-\frac{2}{e} - 4 = 4\,I\,common\,(12^+)$$

$$I\,common\,(12^+) = -1 - \frac{1}{2e}$$



$12 < t < \infty$

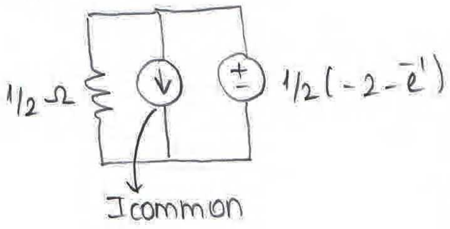
$\alpha = 1/2RC = 1/4$
$w_0 = 1/\sqrt{LC} = 1/4$] critically damped

$(D^2 + 2\alpha D + w_0^2)\, vc(t) = 0 \qquad \lambda_{1,2} = \{-1/4, -1/4\}$

$vc(t) = \left[A\, e^{-(t-12)/4} + B(t-12)\, e^{-(t-12)/4}\right] u(t-12)$

$vc(12) = -0.31$ V

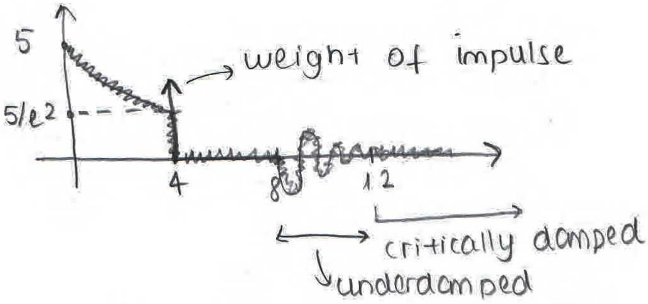
at $t = 12^+$



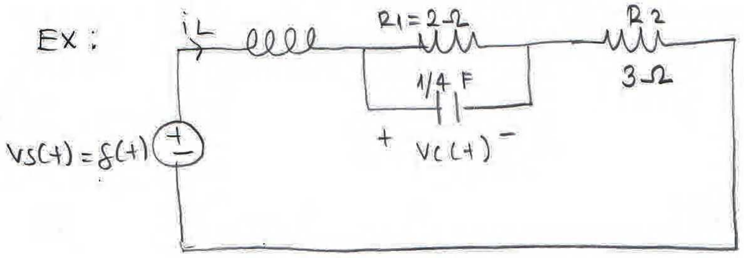
$$CVc^{\circ}(12^+) = \frac{0.31}{1/2} + 1/2\,(2 + e^{-1})$$

Then A and B are found

$vc(t) = e^{-(t-12)/4}\,[-0.31 + 0.37(t-12)] \quad t > 12$



Ex :



$i_L(0^-) = 0 \quad V_c(0^-) = 0$

Find $vc(t) \quad t \geq 0$

by KVL: $-V_s(t) + L\frac{di_L(t)}{dt} + V_C(t) + R_2 i_L(t) = 0$ $\qquad$ $i_L(t) = \frac{V_C(t)}{R_1} + C\frac{dV_C}{dt}$

$$\frac{L}{R_1}\dot{V_C} + CL\ddot{V_C} + V_C + \frac{R_2}{R_1}V_C + R_2 C\dot{V_C} = V_s(t)$$

$$V_C\left[D^2CL + D\left(\frac{L}{R_1} + R_2C\right) + \left(\frac{R_2}{R_1} + 1\right)\right] = V_s(t)$$

$$\left[D^2 + \left(\frac{1}{R_1C} + \frac{R_2}{L}\right)D + \left(\frac{R_2}{R_1}\frac{1}{LC} + \frac{1}{LC}\right)\right]V_C(t) = V_s(t)$$

$(D^2 + 5D + 10)V_C(t) = 4V_s(t)$

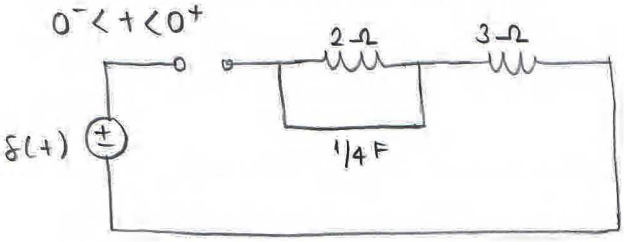
$D^2 + 2\alpha D + \omega_0^2 = 0$ $\qquad$ $\alpha = 2.5$ $\quad$ $\omega_0 = \sqrt{10}$ $\quad$ Underdamped solution

$\lambda^2 + 2\alpha\lambda + \omega_0^2 = 0$

$-\alpha \mp \sqrt{\omega_0^2 - \alpha^2}\,j = \lambda_{1,2}$; $\qquad$ $\lambda_{1,2} = -\frac{5}{2} + j\sqrt{10 - \frac{25}{4}} = \frac{-5 \mp \sqrt{15}\,j}{2}$

$(D^2 + 2\alpha D + \omega_0^2)V_C(t) = 4\,\delta(t)$

$0^- < t < 0^+$



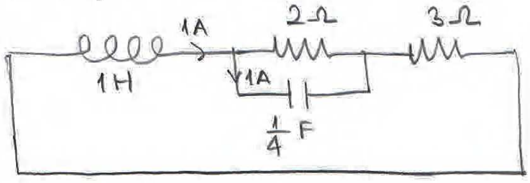
$i_L(t) = 0$ $\quad$ $i_C(t) = 0$

$$V_C(0^+) = V_C(0^-) + \frac{1}{C}\int_{0^-}^{0^+} 0 = 0$$

$$i_L(0^+) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+}\underbrace{\delta(\tau)d\tau}_{V_L(t)} = 1\text{ A}$$

$V_C(0^+) = 0V$ & $\dot{V_C}(0^+) = \frac{i_C(0^+)}{C}$

At $t = 0^+$



$2(1 - i_C(0^+)) = 0$; $\quad$ $i_C(0^+) = 1$ A

$\dot{V_C}(0^+) = \frac{1}{1/4} = 4$ V/sec

$(D^2 + 5D + 10)V_C(t) = 0$ $\qquad$ $V_C(0^+) = 0$ $\quad$ $\dot{V_C}(0^+) = 4$

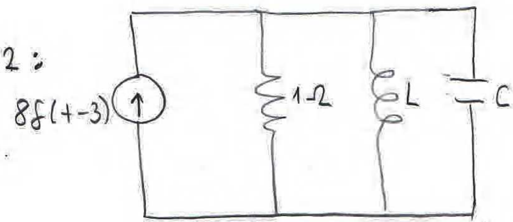
$V_C(t) = e^{5/2t}\left(C_1\cos\left(\frac{\sqrt{15}}{2}t\right) + C_2\sin\left(\frac{\sqrt{15}}{2}t\right)\right)$

$V_C(0^+) = 0 = C_1$

$\dot{V_C}(0^+) = 4$; $\qquad$ $V_C(t) = e^{(5/2)t}\frac{8}{\sqrt{15}}\sin\left(\sqrt{\frac{15}{2}}t\right)$

$C_2 = \frac{8}{\sqrt{15}}$

ZPS VII , 12 :



$V_C(3^+) = 1V$

The response is critically damped $(\alpha = \omega_0)$

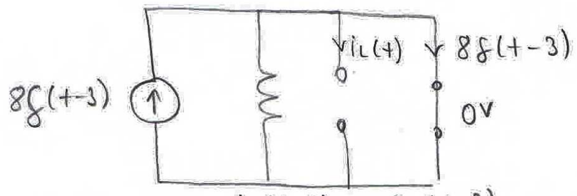
$i_L(0^-) = 0$ $\quad$ $V_C(0^-) = 0$

$\alpha = 1/2RC$ $\quad$ $\omega_0 = \sqrt{1/LC}$

$\left(\frac{1}{2C}\right)^2 = \frac{1}{LC}$ $\qquad$ $L = 4C$

$$\left(D^2 + \frac{1}{RC} D + \frac{1}{LC} \right) V_C(t) = \frac{1}{C} \frac{d}{dt} i_S(t)$$

$3^- < t < 3^+$



$i_L(3^-) = 0 \quad V_L(3^-) = 0V$

$$V_C(3^+) = V_C(3^-) + \frac{1}{C} \int_{3^-}^{3^+} 8\delta(t'-3)\, dt' = \frac{8}{C} = 1\ V$$

$C = 8F \quad L = 32H$

$$V_C(t) = C_1 e^{\lambda(t-3)} + C_2 e^{\lambda(t-3)} (t-3)$$

$V_C(3^+) = C_1 + 0 = 1 \quad C_1 = 1$

$$\dot{V_C}(t) = C_1 \lambda e^{\lambda(t-3)} + C_2 (t-3) \lambda e^{\lambda(t-3)}$$

$$\left(D^2 + \frac{1}{8} D + \frac{1}{256} \right) = \left(D + \frac{1}{6} \right)^2 \quad \lambda = -1/16$$

$$-\frac{1}{16} + 0 + C_2 = -\frac{1}{8} \qquad C_2 = -\frac{1}{16} \quad C_1 = 1$$

$$\boxed{V_C(t) = e^{-\frac{1}{16}(t-3)} - \frac{1}{16} e^{-\frac{1}{16}(t-3)}} \quad \text{for } t > 3$$