EE 201 Lecture Notes

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Chapter 1

Second Order Circuits

1.1 Introducing Second Order Circuits

An understanding of the second order circuits and their modes of operation is fundamental for the analysis of linear time-invariant systems. The phrase "linear time-invariant system" appears frequently in the notes. We should note that this concept is not only central to the electrical engineering, but also important for many other engineering disciplines. In our curriculum, a sequence of courses introduces topics related to linear system theory, such as the second order circuits. The first course in this list is EE201. After EE201, there are some other mandatory courses (EE202, EE301, EE302) and a number of 4th year courses which are linked with the linear system theory.

The response of a linear time-invariant system, which can be circuit or a mechanical system or any other dynamical system, can be interpreted as a joint action of its "modes". (The word mode will become clearer in this chapter.) The superposition of the modes comprise the overall reaction of the system. Specifically, the reaction of an unforced system (a system excited only by the initial conditions, i.e. no external input) is called the natural response. As the name implies this response shows how the system behaves once it is energized at $t = 0$ and then left on its own (no external input or forcing term for $t > 0$). In many applications, you may not be pleased with the way the system behaves and you inject an input to steer the system to a desirable direction. Then the next question is what should be the input to steer the system towards the desired direction. To do that we need to understand the how the system reacts to the external inputs. This chapter presents some answers to such questions. In this chapter, the second order circuits are used to illustrate the answers of such questions.

If we go back to the first order circuits, the first order circuits have a single mode, that is a single way of reacting to a given initial condition. This mode is characterized by an exponential decay and we know that the decay rate is related to the time-constant of the system.

The second order circuits have two modes and the analysis is a little bit more intricate. As a side note, we would like to mention that the first order and second order systems are fundamental for the analysis of N'th order systems, since a higher order system can be decomposed as a cascade of first and second order systems. Hence, a through understanding of second order systems is important for the perception of the whole theory.

1.2 Parallel RLC Circuit

We present the main concepts through the parallel RLC circuit. The parallel and series RLC circuits are classical circuits which are the duals of each other. Hence, their mathematical treatment is identical. We focus on the parallel RLC circuit in this section and briefly examine the series RLC after the completion of this section.

Figure 1.1 shows the parallel RLC circuit configuration. We assume that the circuit is energized with the initial conditions of $V_C(0^-) = V_0$ and $I_L(0^-) = I_0$. The goal is to analyze this circuit for $t \geq 0$ under the external input of $i_s(t)$ and also the initial conditions.

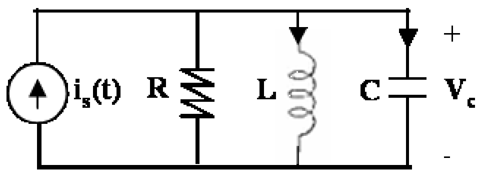


Figure 1.1: Parallel RLC Circuit

We embark on the analysis by writing the KCL equation at the top node of the circuit:

$$-i_s(t) + I_R(t) + I_L(t) + i_C(t) = 0$$

Using the component relations for R, L and C, this equation can be written as follows:

$$\underbrace{\frac{V_C(t)}{R}}_{I_R(t)} + \underbrace{I_L(0^-) + \frac{1}{L}\int_{0^-}^{t} V_C(\tau)d\tau}_{I_L(t)} + \underbrace{C\frac{d}{dt}V_C(t)}_{i_c(t)} = i_s(t), \qquad t > 0 \tag{1.1}$$

In the equation above, the initial condition for the inductor, $I_L(0^-)$, is explicitly present; but the initial condition for the capacitor, $V_C(0^-)$, is absent. The equation (1.1) and the initial condition $V_C(0^-) = V_0$ (which is the initial condition absent in the equation) forms the complete description of the system via an *integro-differential* equation.

By taking the time-derivative of (1.1) for $t > 0$, the integro-differential equation is converted into a differential equation:

$$\frac{1}{R}\frac{d}{dt}V_C(t) + \frac{1}{L}V_C(t) + C\frac{d^2}{dt^2}V_C(t) = \frac{d}{dt}i_s(t) \tag{1.2}$$

The same equation can be written using the operator notation $D \triangleq \frac{d}{dt}$ as follows:

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) = \frac{1}{C}D\left\{i_s(t)\right\} \tag{1.3}$$

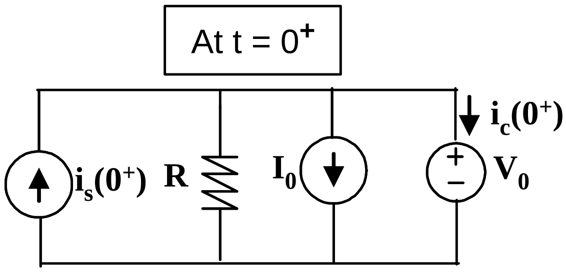
Note that after taking the derivative the initial condition for the inductor vanishes. Hence, we need two initial conditions to find the solution from the differential equation.

We know that a differential equation is not complete unless its initial conditions are specified. Since the differential equation above is with respect to $V_C(t)$, the initial conditions for $V_C(0^-)$ and $\dot{V}_C(0^-)$ should be provided. (The dot on top refers to the derivative of the function.) We are already given the initial condition for $V_C(0^-)$ which is $V_C(0^-) = V_0$; and we need to find the initial condition for $\dot{V}_C(0^-)$. To find that, we can examine, the $t = 0^-$ circuit.

The parallel RLC circuit at $t = 0^+$ is shown in Figure 1.2. Note that, we have replaced the inductor and capacitor in the circuit with the current and voltage sources, respectively. The equivalency of $t = 0^-$ and $t = 0^+$ values is due to the continuity of these circuit variables for the bounded inputs and the replacement of the components for $t = 0^+$ with the sources is possible by the substitution theorem. By inspection of Figure 1.2, the capacitor current at $t = 0^+$ can be written as follows:

$$i_c(0^+) = C\frac{d}{dt}V_C(0^+) = -\frac{V_0}{R} - I_0 - i_s(0^+) \tag{1.4}$$

The required initial condition is then $\frac{d}{dt}V_C(0^+) = -\left(\frac{V_0}{RC} + \frac{I_0 + i_s(0^+)}{C}\right)$. Now, we can write the differential equation along with required initial conditions as follows:



$$V_C(0^-) = V_C(0^+) = V_0$$
$$I_L(0^-) = I_L(0^+) = I_0$$

Figure 1.2: Parallel RLC Circuit at t=0^-

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) = \frac{1}{C} D\left\{i_s(t)\right\}, \quad \begin{array}{l} V_C(0^+) = V_0 \\ \dot{V}_C(0^+) = -\left(\frac{V_0}{RC} + \frac{I_0 + i_s(0^+)}{C}\right) \end{array} \tag{1.5}$$

The last equation completely characterizes the system for $t \geq 0^+$. This means that it is possible to forget that the problem is a circuit analysis problem and work with the given differential equation in a purely mathematical manner and find the solution $V_C(t)$.

The next task is the solution of the differential equation; but before that we present another description for the same circuit. From Figure (1.1), it is possible to write the following two relations:

$$\begin{aligned} \frac{V_C(t)}{R} + I_L(t) + C\frac{d}{dt}V_C(t) &= i_s(t) \\ \underbrace{L\frac{d}{dt}I_L(t)}_{v_L(t)} &= V_C(t) \end{aligned}$$

We retain the differentiated variables on the left hand side and move all others to the right hand side to get the following:

$$\begin{aligned} \frac{d}{dt}V_C(t) &= -\frac{V_C(t)}{RC} - \frac{I_L(t)}{C} + \frac{i_s(t)}{C} \\ \frac{d}{dt}I_L(t) &= \frac{1}{L}V_C(t) \end{aligned}$$

This equation can also be written using matrices. (The matrices are the essential tools for the analysis of linear systems and can significantly ease the mathematical treatment of the topic. Don't get intimidated by matrices, everyone gets, eventually, along very well with matrices!)

$$\begin{bmatrix} \dot{V}_C(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t) \tag{1.6}$$

The equation system shown in (1.6) is a first order matrix differential equation. (Your differential equations teacher would be very happy if you can immediately say that the solution of such equations can be written in terms of $e^{\mathbf{A}t}$ matrix!) The initial conditions to make this equation complete is $V_C(0^-) = V_0$ and $I_L(0^-) = I_0$. Note that, for bounded inputs, i.e. $i_s(t) < M$ for some M, $V_C(0^+) = V_C(0^-) = V_0$ and $I_L(0^-) = I_L(0^+) = I_0$. Note that, there is no need to do a $t = 0^+$ analysis, as we did in Figure 1.2, to find the initial conditions.

Note that it is possible to retrieve the 2nd order scalar differential equation for $V_C(t)$ given in (1.5) from the matrix differential equation given in (1.6). To do that take the derivative of the first equation in (1.6) and substitute $\dot{I}_L(t)$ with the second equation of (1.6). Once this is done, we get the scalar 2nd order differential equation for $V_C(t)$. Hence, both representations are equivalent to each other, it is possible to go back and forth between different representations.

We would like to note that it is possible to write the differential equation for any other circuit variable of the parallel RLC circuit. For example, to write an equation for $I_L(t)$, we can utilize (1.6). First write $V_C(t)$ and $\dot{V}_C(t)$ in terms of $I_L(t)$ and $\dot{I}_L(t)$. Then take the derivative of the second equation to get $\ddot{I}_L(t)$. Finally, replace all $V_C(t)$ and $\dot{V}_C(t)$ appearing in $\ddot{I}_L(t)$ with the inductor current functions. This gives us an equation for $I_L(t)$. We can show these steps as follows:

$$\begin{aligned} \frac{d^2}{dt^2} I_L(t) &= \frac{1}{L}\frac{d}{dt} V_c(t) \\ &\overset{(1.6)}{=} \frac{1}{L}\left(-\frac{V_C(t)}{RC} - \frac{I_L(t)}{C} \right) \\ &\overset{(1.6)}{=} \frac{1}{L}\left(-\frac{L\dot{I}_L(t)}{RC} - \frac{I_L(t)}{C} \right) \end{aligned}$$

It is also possible write a second order differential equation for the resistor current by noting that $I_R(t) = V_C(t)/R$ or $V_C(t) = RI_R(t)$. To write the describing equation for $I_R(t)$, you can replace $V_C(t)$ and its derivatives in (1.5) with $I_R(t)$ and $\dot{I}_R(t)$.

Furthermore you can invent new circuit variables such as $x(t) = V_C(t) + I_L(t)$ $y(t) = V_C(t) + 2I_L(t)$ and write the matrix differential equation satisfied by $x(t)$ and $y(t)$, or you may write the scalar 2nd order differential equation satisfied by $x(t)$. (It may not be clear why you may want to do such a thing at this point.)

Hence there are many ways of describing the same circuit. All of these descriptions are inter-related and we can go back and forth between these equivalent descriptions. (For experienced readers, we note that the natural frequencies of the circuit, or the system poles or the eigenvalues of the **A** matrix is identically the same for every representation. If you do not understand this parenthesis, don't worry!)

1.2.1 Zero-Input Solution

The zero-input solution characterizes the natural response of the circuit. In other words, the zero-input solution is the solution when the circuit is energized at $t = 0^-$ and then left on this own (no external forcing). The question is to find the variation of the circuit variables or the evolution of the circuit variables under the KVL and KCL constraints, given the initial conditions. Stated differently the question is what is happening in the circuit due to the initial conditions in the absence of external input.

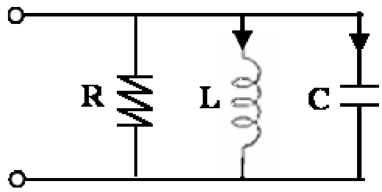


Figure 1.3: Zero-Input Parallel RLC Circuit

The defining equation: In resistive LTI circuits, the equation defining the solution of the circuit is a linear equation system. As we have noted before, the number of unknowns for such a description can be the number of nodes or the number of meshes. For resistive circuits with non-linear elements, the defining equation system is a non-linear equation. The non-linear relation is a quadratic equation if we have component with the relation $V = i^2 R$.

The defining equation for dynamic circuits can be an integro-differential equation as we have noted in (1.1). The integral equations have certain advantages in terms of computation and there is a strong mathematical theory for the integral equations. In this course we prefer the differential equations descriptions for which there is an equally strong theory. (The differential equations can be more fundamental in equation writing since they describe the behaviour of the system via local differences.) The differential equation describing the circuit given in Figure 1.3 is given in (1.5) and reproduced below:

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) \quad = \quad \frac{1}{C} D\left\{i_s(t)\right\}, \text{ with } \begin{array}{l} V_C(0^+) = V_0 \\ \dot{V}_C(0^+) = -\left(\frac{V_0}{RC} + \frac{I_0 + i_s(0^+)}{C}\right) \end{array}$$

When the input $i_s(t) = 0$ (zero-input), the equation characterizing the zero-input solution is:

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) \;=\; 0, \text{ with } \begin{array}{l} V_C(0^+) = V_0 \\ \dot{V}_C(0^+) = -\left(\frac{V_0}{RC} + \frac{I_0}{C}\right) \end{array} \tag{1.7}$$

We present the general solution for the following differential equation:

$$\left(D^2 + 2\alpha D + \omega_o^2\right) V_C(t) = 0 \tag{1.8}$$

We make the following guess for the solution:

$$V_c(t) = ce^{\lambda t}$$

If this function is indeed the solution, then the differential equation (1.7) should be satisfied for all $t > 0$. To check the guess, we insert the function into the differential equation and get the following equation:

$$c(\lambda^2 + 2\alpha\lambda + \omega_o^2) = 0 \tag{1.9}$$

The last equation shows that the guess is correct if $c = 0$ or $\lambda^2 + 2\alpha\lambda + \omega_o^2 = 0$. The solution $c = 0$ becomes $V_C(t) = 0$ which can not be the right solution the unless all initial conditions are zero. Thus, the non-trivial solutions should have λ which satisfies:

$$(\lambda^2 + 2\alpha\lambda + \omega_o^2) = 0 \tag{1.10}$$

This equation is called the *characteristic equation* of the system. The roots of the characteristic equation are called the *natural frequencies*. The roots can be written as follows:

$$\text{Natural Frequencies: } \lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} \tag{1.11}$$

The solution for the zero-input can then be written as follows:

$$V_C(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad t \geq 0 \tag{1.12}$$

where λ_1 and λ_2 are the natural frequencies and c_1 and c_2 are the arbitrary constants. (The constants c_1 and c_2 are set to meet the initial conditions at $t = 0^+$.)

Types of Zero-Input Responses: The zero-input responses are categorized by the natural frequencies, that is the roots of the characteristics equation given in (1.10) determine the type of response.

The possibilities for the roots are:

i. Distinct real roots ($\Delta > 0$, i.e $\alpha > \omega_o$)

ii. Repeated real roots ($\Delta = 0$, i.e. $\alpha = \omega_o$)

iii. Complex conjugate roots ($\Delta < 0$, i.e. $\alpha < \omega_o$)

The parameter Δ is the discrimant value of the quadratic given in (1.10), $\Delta = 4(\alpha^2 - \omega_o^2)$. Each one of these possibilities listed is attributed as a circuit response.

Before the discussion of the responses, we need to introduce some terminology for the parameters α and ω_o appearing in (1.10). The parameter α, which is $1/(RC)$ for the parallel RLC circuit, is called the *damping factor*. The parameter ω_o, which is $1/\sqrt{LC}$ for the parallel RLC, is called the *resonance frequency*.

$$\text{Parallel RLC :} \quad \begin{cases} \alpha = \frac{1}{2RC} \\ \omega_o = \frac{1}{\sqrt{LC}} \end{cases}$$

The terminology will become more apparent with the introduced topics.

Characteristic Equation: $\lambda^2 + 2\alpha\lambda + \omega_o^2 = 0$		
α	:	Damping factor (1/sec, Hz.)
ω_o	:	Natural frequency (1/sec, Hz.)

i. Overdamped Circuits ($\alpha > \omega_o$): The natural frequencies for the overdamped circuits are real valued and distinct.

For the parallel RLC circuit with $R = 1/5\ \Omega$, $L = 1/8$ H and $C = 1/2$ F, we have $\alpha = 5$ and $\omega_o = 4$. Since $\alpha > \omega_o$, the response is *over*damped, i.e. damping is *greater* than than ω_o. The characteristic equation for this circuit is $\lambda^2 + 10\lambda + 16 = 0$ and the natural frequencies are $\lambda = \{-8, -2\}$. The zero-input solution is then

$$V_C^{zi}(t) = c_1 e^{-2t} + c_2 e^{-8t}, \quad t \geq 0.$$

The constants c_1 and c_2 are to be determined according to the initial conditions at $t = 0^+$.

ii. Critically Damped Circuits ($\alpha = \omega_o$): The natural frequencies for critically damped circuits are real valued and repeated.

Let's take $R = 1/4\ \Omega$, $L = 1/8$ H and $C = 1/2$ F, then $\alpha = \omega_o = 4$ and the characteristic equation is $\lambda^2 + 8\lambda + 16 = 0$. The natural frequencies are $\lambda = -4$. The zero-input solution is then

$$V_C^{zi}(t) = c_1 e^{-4t} + c_2 t e^{-4t}, \quad t \geq 0.$$

iii. Underdamped Circuits ($\alpha < \omega_o$): The natural frequencies for critically damped circuits are complex valued and two natural frequencies are the complex conjugates of each other.

Let's take $R = 1/3\ \Omega$, $L = 1/8$ H and $C = 1/2$ F in the parallel RLC circuit, then $\alpha = 3$ and $\omega_o = 4$. It is clear that the damping coefficient *smaller* than the natural frequency, hence the name *under*damped. The characteristic equation for this system is $\lambda^2 + 6\lambda + 16 = 0$. The natural frequencies are $\lambda = -3 + j\sqrt{7}, -3 - j\sqrt{7}$. The zero-input solution is then:

$$V_C^{zi}(t) = c_1 e^{(-3-j\sqrt{7})t} + c_2 t e^{(-3+j\sqrt{7})t}, \quad t \geq 0$$

The solution for this case is complex valued. It may not be clear from this representation that the solution we get is a reasonable solution representing the physics of the problem, that is if the solution is complex valued, it would be very hard for us to interpret as the capacitor voltage! Luckily, as it is shown belown, there are no such interpretation problems and the theory extends smoothly towards complex valued natural frequencies.

Let's assume that $V_C(0^+)$ and $\dot{V}_C(0^+)$ values are provided. Then to find $V_C^{zi}(t)$, we need to set c_1 and c_2 appearing in (1.13) to meet the initial conditions at $t = 0^+$.

$$\begin{aligned} V_C(0^+) &= c_1 + c_2 \\ \dot{V}_C(0^+) &= \lambda_1 c_1 + \lambda_2 c_2 \end{aligned} \tag{1.13}$$

Here λ_1 and λ_2 are complex valued natural frequencies, which is $\lambda_1 = (-3 - j\sqrt{7})$ and $\lambda_2 = (-3 + j\sqrt{7})$ for the presented example.

The claim is that if $V_C(0^+)$ and $\dot{V}_C(0^+)$ are real valued, then $c_1 = c_2^*$ (c_1 and c_2 are complex conjugates of each other). To show this claim, we take the complex conjugate of the equations in (1.13), the resultant equation is as follows:

$$\begin{aligned} V_C(0^+) &= c_1^* + c_2^* \\ \dot{V}_C(0^+) &= \lambda_1^* c_1^* + \lambda_2^* c_2^* \end{aligned}$$

But since λ_1 and λ_2 are complex conjugates of each other, the equation system reduces to:

$$\begin{aligned} V_C(0^+) &= c_2^* + c_1^* \\ \dot{V}_C(0^+) &= \lambda_1 c_2^* + \lambda_2 c_1^* \end{aligned} \tag{1.14}$$

Now we compare (1.13) and (1.14). If the equation system given in (1.13) has a unique solution (which is the case, think about why?), then that solution for (1.13) should be solution of (1.14). Due to the uniqueness of the solution, we have $c_1 = c_2^*$ and $c_2 = c_1^*$. This shows that the claim is indeed correct.

Going back to (1.13), we can now write the zero-input solution for $t > 0$ as

$$\begin{aligned}
V_C^{zi}(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
&\overset{(a)}{=} c_1 e^{\lambda_1 t} + c_1^* e^{\lambda_1^* t} \\
&\overset{(b)}{=} 2\mathrm{Re}\left\{c_1 e^{\lambda_1 t}\right\} \\
&\overset{(c)}{=} 2\mathrm{Re}\left\{||c_1|| e^{j\measuredangle c_1} e^{\lambda_1 t}\right\} \\
&= 2||c_1||\mathrm{Re}\left\{e^{j\measuredangle c_1} e^{\lambda_1 t}\right\} \\
&\overset{(d)}{=} 2||c_1||\mathrm{Re}\left\{e^{j\measuredangle c_1} e^{(\lambda_1^r + j\lambda_1^i)t}\right\} \\
&= 2||c_1||\mathrm{Re}\left\{e^{\lambda_1^r t} e^{j(\lambda_1^i t + \measuredangle c_1)}\right\} \\
&\overset{(e)}{=} 2||c_1|| e^{\lambda_1^r t}\mathrm{Re}\left\{e^{j(\lambda_1^i t + \measuredangle c_1)}\right\} \\
&\overset{(f)}{=} 2||c_1|| e^{\lambda_1^r t}\cos(\lambda_1^i t + \measuredangle c_1) \\
&\overset{(g)}{=} d_1 e^{\lambda_1^r t}\cos(\lambda_1^i t + d_2)
\end{aligned}$$

In line (a): The complex conjugacy relation between the parameters is used.
In line (b): Summation of a function and its complex conjugate yields two times the real part of the function.
In line (c): The constant c_1 is expressed in polar coordinates.
In line (d): The constant λ_1 is expressed as $\lambda_1 = \lambda_1^r + j\lambda_1^i$.
In line (e): The real valued function $e^{\lambda_1^r t}$ is pulled out of the real operator.
In line (f): Euler's formula.
In line (g): Undetermined constants are replaced with a new set of undetermined constants d_1 and d_2.

If we go back to the parallel RLC circuit example, the zero-input is then

$$V_C^{zi}(t) = d_1 e^{-3t}\cos(\sqrt{7}t + d_2), \quad t \geq 0$$

Again note that by selecting d_1 and d_2, it is possible to meet any given initial condition at $t = 0^+$.

Comparison of Under/Critical and Overdamped Responses:

(to be written...)

1.2.2 Zero-State Solution

The zero-state solution assumes the circuit is not energized initially, that is the initial voltage of all capacitors and the initial current of all inductors are zero. The zero-stage solution is the response due to the applied input in the absence of any initial conditions, that is when the system is at rest initially.

The general approach is as follows:

i. Find the initial conditions at $t = 0^+$ by inspecting the $t = 0^+$ circuit.

ii. Solve the differential equation for $t > 0$ with the $t = 0^+$ initial conditions.

For the zero-state solution, the circuit is initially at rest; therefore $t = 0^-$ values for the capacitor voltages and inductor currents are zero. From our earlier discussions, we know that the capacitor voltages and inductor currents is a continuous function of time if the input is bounded. As discussed before, this is a consequence of the terminal equations for the capacitor ($V_C(t) = V_C(0)+1/C\int_0^t I_C(t')dt'$) and the inductor ($I_L(t) = I_L(0)+1/L\int_0^t V_L(t')dt'$). Therefore, if there is no impulsive source in the circuit, $V_C(0^+) = V_C(0^-)$ and $I_L(0^+) = I_L(0^-)$ is granted. The continuity of the $V_C(t)$ and $I_L(t)$ is the golden rule that should always be remembered.

Below we find the zero-state response to ramp, unit step and impulse input. The method shown below can also be used to directly write the complete solution, i.e. solving a circuit without zero-input and zero-state decomposition. As we have discussed before, this decomposition enables us to apply the superposition for the zero-stage responses. Therefore it is very useful if we have a superposition of elementary functions at the input. In this case, the zero-state response to the superposed input is the superposition of zero-state responses to each input. This is the reason (the possibility of superposition) that makes the zero-state responses important to us.

i. Ramp Response: Figure 1.4 shows the circuit at $t = 0^+$ when the input is the ramp function.

An inspection of this circuit, reveals that all circuit variables in this current has the value of zero $t = 0^+$, that is $I_C(0^+) = 0$ and therefore $\dot{V}_C(0^+) = 0$. Hence the $t = 0^+$ initial conditions for the ramp input is $V_C(0^+) = \dot{V}_C(0^+) = 0$ and the first step of the procedure given above is completed.

The differential equation for the ramp input, i.e. $i_s(t) = r(t)$ is as follows:

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) \quad = \quad \frac{1}{C}D\left\{i_s(t)\right\} = \frac{1}{C}u(t), \qquad \begin{array}{l} V_C(0^+) = 0 \\ \dot{V}_C(0^+) = 0 \end{array}$$

For $t > 0$, the differential equation is

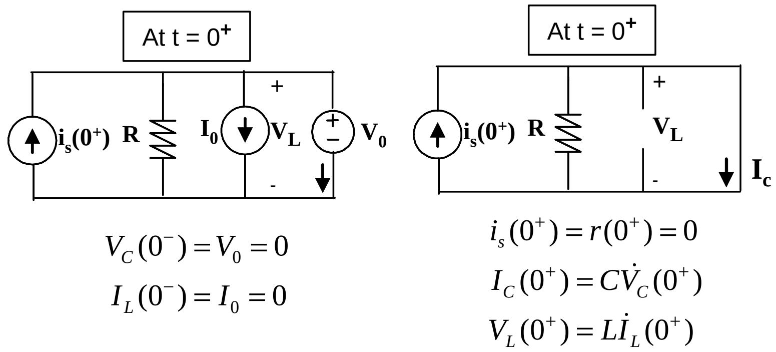


Figure 1.4: At $t = 0^+$ circuit for the ramp input

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) = \frac{1}{C}, \qquad t > 0$$

The zero-state solution is then

$$V_C^{zs}(t) = L + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad t > 0$$

Here λ_1, λ_2 are the natural frequencies and the constants c_1, c_2 should be selected the meet the initial conditions at $t = 0^+$.

ii. Step Response: Figure 1.5 shows the circuit at $t = 0^+$ when the input is the step function.

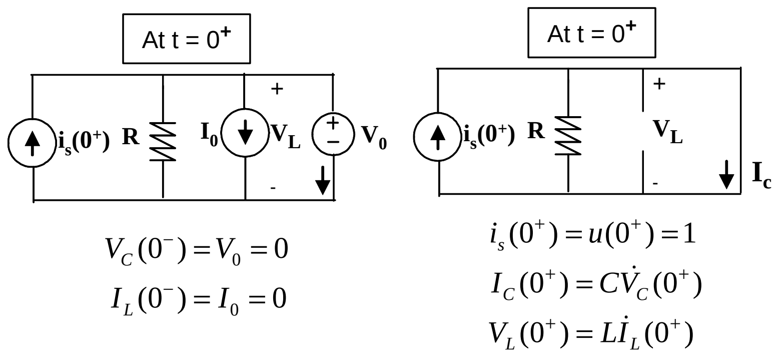


Figure 1.5: At $t = 0^+$ circuit for the unit step input

A simple analysis of the $t = 0^+$ which does not contain any dynamic elements, but only sources and resistors reveals that $V_C(0^+) = 0$ and $V_C(0^+) = 1/C$. It should be noted that $V_C(0^+)$ is due to the continuity of the capacitor voltage for the bounded inputs (the golden rule!) and does not follow from any analysis.

Then the differential equation for the unit step input, i.e. $i_s(t) = u(t)$ is as follows:

$$(1.15)\ \left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) \;=\; \frac{1}{C}D\{i_s(t)\} = \frac{1}{C}\delta(t), \qquad \begin{array}{l} V_C(0^+) = 0 \\ \dot{V}_C(0^+) = 1/C \end{array}$$

On the right hand side of the differential equation we have a $\delta(t)$ function which is quite a problem if we were presented with such a differential equation at a differential equations course; but note that for $t > 0$, the differential equation is

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_C(t) \;=\; 0, \qquad t > 0$$

The zero-state solution for $t > 0$, is then

$$V_C^{zs}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad t > 0$$

Here λ_1, λ_2 are the natural frequencies and the constants c_1, c_2 should be selected the meet the initial conditions at $t = 0^+$.

iii. Impulse Response: Figure 1.6 shows the circuit that is be analyzed for $V_C(0^+)$ and $\dot{V}_C(0^+)$ values.

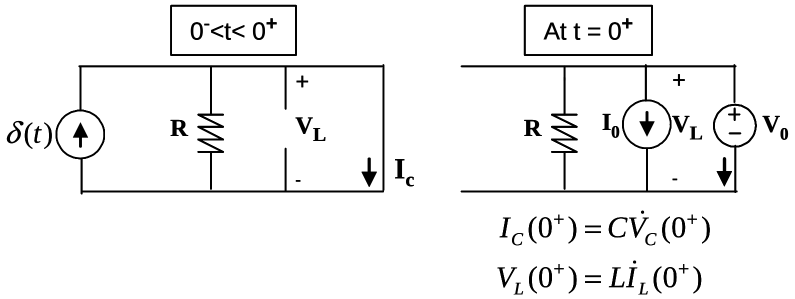


Figure 1.6: At $t = 0^+$ circuit for the impulse input

The input is not a bounded one, therefore we can not apply the rule stating that the capacitor voltage and inductor current is continuous. The circuit on the left side of Figure 1.6 shows that $I_C(t) = \delta(t)$ "during the application of the impulse". Hence $V_C(0^+) = V_C(0^-) + 1/C \int_{0^-}^{0^+} i_c(t')dt' = 1/C$. How about $I_L(0^+)$? To find $I_L(0^+)$, we need to find $V_L(t)$ during the application of the impulse. It can be noted from Figure 1.6 that $V_L(t) = 0$ in between 0^- and 0^+ and hence $I_L(0^+) = 0$.

To solve the differential equation written for $V_C(t)$, we need the initial conditions for V_C at $t = 0^+$. To find $V_C(0^+)$, we examine $t = 0^+$ circuit, which is the circuit just after the application of the impulse. This circuit is given on the right side of Figure 1.6. In this circuit $V_0 = 1/C$ and $I_0 = 0$. Then a resistive circuit analysis reveals that $I_C(0^+) = -V_C(0+)/R = -1/(RC)$, then $\dot{V}_C(0^+) = -1/(RC^2)$. We are done with the finding of the initial conditions.

The differential equation for the unit impulse input, i.e. $i_s(t) = \delta(t)$ is as follows:

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) \;=\; \frac{1}{C}D\left\{i_s(t)\right\} = \frac{1}{C}\dot{\delta}(t), \qquad \begin{array}{l} V_C(0^+) = 1/C \\ \dot{V}_C(0^+) = -1/RC^2 \end{array}$$

On the right hand side of the differential equation, we have the derivative of the $\dot{\delta}(t)$ function which is called the doublet function. (A rigorous solution of a differential equation with the impulse input and its derivatives requires familiarity with the generalized functions. A semi-rigorous but convincing method of solution is presented as an appendix to this chapter.) The function $\dot{\delta}(t)$ on the right hand side of the differential equations can cause nervousness and nausea to some readers, but readers should be assured that for $t > 0$, the differential equation reduces to

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_C(t) \;=\; 0, \qquad t > 0$$

The zero-state solution for $t > 0$, is then

$$V_C^{zs}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad t > 0$$

Here λ_1, λ_2 are the natural frequencies as before and the constants c_1, c_2 should be selected the meet the initial conditions at $t = 0^+$.

Remark #1: We would like to note that the solution of the differential equations with a forcing term of $\delta(t)$ (and its derivatives) is not a straightforward problem. The technique presented in this chapter passes the difficulties associated with the impulse by analyzing $t = 0^+$ circuit. A time-domain verification of the $t = 0^+$ circuit analysis result using differential equations knowledge (no circuit theory) requires fairly sophisticated mathematical analysis knowledge. Another approach for the solution of such differential equations is the transform domain methods, i.e. the Laplace domain, which is throughly presented in EE202 within the context of s-domain circuit analysis.

Remark #2: As noted before instead of having a 2nd order scalar differential equation for $V_C(t)$, we may write a 1st order matrix differential equation for $V_C(t)$ and $I_L(t)$. Both equations characterize the same circuit and their solution is identical. It should always be remembered that alternative characterizations can be always useful i.) to justify your solution and ii.) to find alternative methods to reach the same solution. In many problems the method to the solution can be more critical and lead to a better conceptual understanding and generalizations of the problem in hand.

The 1st order matrix differential equations present an alternative method for circuit characterization. In EE202, this topic is investigated under the heading of state equations. The state equations have a number of advantages in comparison to scalar differential equations.

For parallel RLC, the state equation defining the circuit is given in (1.6) and reproduced below for convenience.

$$\begin{array}{c}\text{State Equation for}\\ \text{Parallel RLC Circuit}\end{array} \Rightarrow \begin{bmatrix} \dot{V}_C(t) \\ \dot{I}_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t)$$

Some of the advantages of the equation above in comparison to the scalar differential equation are:

- The state equation description does not contain the derivative of the input on the right hand side. Therefore, if we use this alternative characterization for step-response calculation, we do not have an $\delta(t)$ popping up as a forcing term as in (1.15).

- For any bounded input the initial conditions for $V_C(t)$ and $I_L(t)$ at $t = 0^-$ are transferred as it is to $t = 0^+$.

- It is fairly easy to show that solution for bounded inputs can be written in the following form:

$$(1.16)\quad \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = e^{\mathbf{A}t} \begin{bmatrix} V_C(0^-) \\ I_L(0^-) \end{bmatrix} + \int_{0^-}^{t} e^{\mathbf{A}(t-t')} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s(t')dt', \quad t > 0$$

  Here $e^{\mathbf{A}t}$ is a function of a matrix for a specific t, and it is defined as $e^{\mathbf{A}t} = \sum_{k=0}^{\infty}(\mathbf{A}t)^k/(k!)$. An advantage of the solution given in (1.16) is that the effect of the input appears under an integral sign. Therefore, the response to the input of $\delta(t)$ (the impulse response) can be easily found as the limit of the responses to the bounded inputs of $\delta_\epsilon(t)$, as in (**??**). So it is possible to find $t = 0^+$ value for the impulse response as follows:

$$\begin{aligned}(1.17)\quad \begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} &= e^{\mathbf{A}t} \underbrace{\begin{bmatrix} V_C(0^-) \\ I_L(0^-) \end{bmatrix}}_{\mathbf{0}} + \int_{0^-}^{0^+} e^{\mathbf{A}(0^+-t')} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \delta(t')dt' \\ &= \begin{bmatrix} 1/C \\ 0 \end{bmatrix}\end{aligned}$$

  This result matches the initial conditions found from $t = 0^+$ circuit. Note that the circuit theory based method to find $t = 0^+$ values requires the understanding of "during of application of impulse" which is not required for state-equation based solution.

  Furthermore, the impulse response for $t > 0$ can be written as:

$$
\begin{aligned}
\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} &= e^{\mathbf{A}t} \begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} + \int_{0^+}^{t} e^{\mathbf{A}(t-t')} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \underbrace{\delta(t')}_{0} dt' \\
&= e^{\mathbf{A}t} \begin{bmatrix} V_C(0^+) \\ I_L(0^+) \end{bmatrix} \\
&= e^{\mathbf{A}t} \begin{bmatrix} 1/C \\ 0 \end{bmatrix}, \quad t \geq 0
\end{aligned}
$$

This shows that the impulse response for $V_C(t)$ and $I_L(t)$ can be found by multiplying $e^{\mathbf{A}t}$ matrix and $\mathbf{b} = [1/C,\ 0]^T$ vector. This result is valid for any circuit having state variables listed in $\mathbf{x}(t)$ vector and state equation of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\, i_s(t)$.

Remark #3: (A Mechanical Analogy) To illustrate the differences in between the overdamped and underdamped responses, we present a mechanical analog for the parallel RLC circuit. The mechanical analogy carries our everyday intuition of dynamical systems to the RLC circuits.

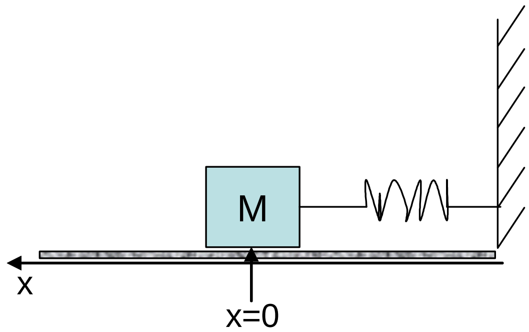


Figure 1.7: Mechanical Analog of 2nd Order RLC circuit

Figure 1.7 shows a mass M attached to a linear spring with the spring constant k on a rough surface with friction. We assume that the system is energized at $t = 0^-$ and released at $t = 0$. There is no external force on the system for $t > 0$. Our goal is to analyze what happens after we release.

We assume that the origin of the selected coordinate system coincides with the equilibrium position of the mechanical system. In other words when the mass is at $x = 0$, the spring is not compressed, i.e. the force exerted by the spring on the mass is zero. The position of the mass is denoted by the function $x(t)$. Our goal is to find $x(t)$ for $t > 0$.

The initial energy can be deposited to the mechanical system in two ways. The mass can have a non-zero initial speed or the spring can be initially compressed.

Mass with a non-zero speed at $t = 0^-$ leads to the storage of kinetic energy, $\frac{1}{2}M\dot{x}^2(t)$. Initially compressed spring leads to the storage of the potential energy, $\frac{1}{2}kx^2(t)$. The initial rest conditions (no initial energy) for the mechanical system is then $x(0^-) = 0$, $\dot{x}(0^-) = 0$.

The analogy with the RLC circuit can be established by noting that the stored electrical energy in the circuit at time t is $\frac{1}{2}CV_C(t)^2$ and the stored magnetic energy is $\frac{1}{2}LI_L(t)^2$. For the parallel RLC circuit, if we choose to denote $I_L(t)$ as $x(t)$, then $\dot{x}(t) = V_C(t)/L$. Hence the electrical and magnetic energy at time t can be written as $\frac{1}{2}\frac{C}{L^2}x(t)^2$ and $\frac{1}{2}Lx(t)^2$ respectively. It should be noted that the energy relations for the electrical and mechanical systems are closely related. (Note that by setting the mass as $M = C/L^2$ and spring constant k as L, we can reproduce the same functional relations for mechanical and electrical systems. It should be noted our goal in making this analogy is to understand the differences between underdamped and overdamped responses, not the construction of an exact mechanical analog of an electrical circuit. Therefore attention should not be focused on the arbitrary system constants such as M, k or R, L, C; but we should focus on the functional relations.)

By Newton's axioms, we can write that $F_{\text{net}} = m\frac{d^2}{dt^2}x(t)$ and for the particular mechanical system $F_{\text{net}} = -kx(t) - \mu\dot{x}(t)$. In the last equation $\mu\dot{x}(t)$ is the force due to friction. (In Physics 105, the friction force is, in general, denoted as μ. Here we assume that the friction force is proportional to velocity of the object. The friction used in this model is the viscous friction which is the friction faced by a moving object in a liquid or gas. Figure 1.7 does not explicit show the type of friction. If you want, you may consider that the object M is floating on the surface of a pool filled with some fluid.)

The governing equation for the position of the mass is as follows:

$$M\frac{d^2}{dt^2}x(t) + \mu\frac{d}{dt}x(t) + kx(t) = 0, \quad \text{with} \quad \begin{aligned} x(0^-) &= X_0 \\ \dot{x}(0^-) &= \dot{X}_0 \end{aligned}$$

The same equation can be written in the standard form as shown below:

$$\frac{d^2}{dt^2}x(t) + \underbrace{\frac{\mu}{M}}_{2\alpha}\frac{d}{dt}x(t) + \underbrace{\frac{k}{M}}_{\omega_0^2}x(t) = 0, \quad \text{with} \quad \begin{aligned} x(0^-) &= X_0 \\ \dot{x}(0^-) &= \dot{X}_0 \end{aligned}$$

From the last equation, we can note the damping factor and resonant frequency constants of the system, $\alpha = \mu/2M$ and $\omega_o^2 = k/M$. (Note that if $M = C$,$k = 1/L$,$\mu = 1/R$, $\alpha = R/2C$ and $\omega_o^2 = 1/LC$ exactly matching the differential equation of the parallel RLC circuit given in (1.7).) Note that the friction only effects damping coefficient and it is analogous to R in the RLC circuit in this sense.

Underdamped Case (μ is small): When μ is sufficiently small $\alpha < \omega_0$ and the mechanical system will be underdamped. The word "underdamped" is specifically chosen to imply that the damping (or the friction force) is small or below a certain threshold for this case. The response for this case is in the form:

$$x(t) = d_1 e^{\lambda_1^r t} \cos(\lambda_1^i t + d_2)$$

where d_1 and d_2 are chosen to satisfy the initial conditions and λ_1^r and λ_1^i are the real and imaginary parts of the natural frequency λ_1.

The most important observation about this solution is that $x(t)$ changes its sign infinitely many times until it comes to a final rest In other words, the object crosses the equilibrium point ($x = 0$) and proceeds towards, say, the positive x-axis. After some time, the object slows down ($\dot{x}(t)$ is negative valued) and then stops temporarily (when $\dot{x}(t) = 0$, note that when it stops it stretches/compresses the spring to the maximum level and at this instant kinetic energy is zero) and then speeds up in the opposite direction that is towards negative x-axis direction. After some time, it crosses the equilibrium point and slows down due to the compression of the spring (storage of potential energy). Then stops and repeats the some motion. At every oscillation cycle, the system losses some part of its energy due to friction. Because of this, it can never attain the level of spring compression/streching that it has achieved in the earlier cycles. The object is said have decaying oscillations.

Now, read the previous paragraph with the following replacements: Mass $\rightarrow$ Capacitor; Inductor $\rightarrow$ Spring; Friction $\rightarrow$ Conductance ($1/R$, i.e. no friction $R \rightarrow \infty$); Kinetic Energy $\rightarrow$ Electrical Energy; Potential Energy $\rightarrow$ Magnetic Energy.

The important conclusion is, one more time, that the underdamped systems due to small friction (or small ohmic losses) oscillates around the equilibrium point and eventually comes to a stop.

Overdamped Case (μ is large): When μ is sufficiently large $\alpha > \omega_0$ and the mechanical system will be overdamped. The response for this case is in the form:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Here λ_1, λ_2 are the distinct and negative valued real numbers which are the natural frequencies of the system. The parameters c_1, c_2 are, as before, chosen to satisfy the initial conditions.

In this case the response does not have oscillations across the equilibrium point. The response is the summation of two monotonic functions. You can show that depending on the initial conditions, the response either have a single equilibrium crossing (only one sign change) or no equilibrium crossings ($x(t)$ keeps its sign, that is remains positive or negative and monotonically approaches the equilibrium state of $x(t) = 0$). In both cases, system reaches the equilibrium point via an exponential decay.

Note that the presence or absence of oscillations is determined by the damping factor which is directly influenced by the amount of friction (or resistance) in the system. Again note that, the type of the response is the property of the system, not its initial conditions or external input.

Lossless Case ($\mu = 0$)**:** When μ is equal to zero, there is no friction in the system and the mechanical system has sustained oscillations. In other words the object moves back and forth across the equilibrium point. The solution in this case is

$$x(t) = d_1 \cos(\lambda_1^i t + d_2)$$

where $\lambda_{1,2} = \pm j\,\omega_o$. For the lossless case, the oscillation frequency is the resonance frequency ω_o. The object moves back and forth across $x = 0$ point, ω_o times at every 2π seconds.

Remark #4: The topic of second order circuits is also examined under the topic of frequency response in EE202. Especially the underdamped case will be brought under scrutiny for the description of RLC resonators.

1.3 Series RLC Circuit

The series RLC circuit is the dual of the parallel RLC circuit. The circuit is shown in Figure 1.8.

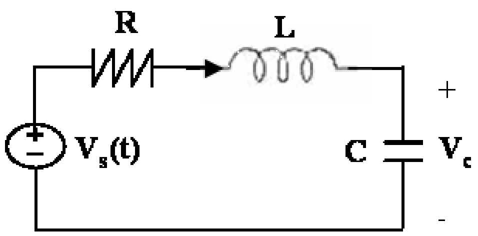


Figure 1.8: Series RLC Circuit

We can write the differential equation characterizing the circuit as follows:

$$\begin{aligned} V_R(t) + V_L(t) + V_C(t) &= v_s(t) \\ RI_L(t) + L\frac{d}{dt}I_L(t) + V_C(0^-) + \frac{1}{C}\int_{0^-}^{t} I_L(t')dt' &= v_s(t), \quad t \geq 0 \end{aligned}$$

By taking the derivative of the last relation, we can get the following differential equation:

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) I_L(t) = \frac{1}{L}D\{v_s(t)\}, \quad t \geq 0 \tag{1.18}$$

Note that, the differential equation given above can be written by noting the duality of the parallel RLC and series RLC circuit. To do that one needs to replace $R \to 1/R$, $L \to C$, $C \to L$ and $V_C(t) \to I_L(t)$, $i_s(t) \to V_s(t)$ in (1.3).

The differential equation for $V_C(t)$ can be written by noting that $I_L(t) = CD\{V_C(t)\}$ for the series RLC circuit. By substituting $CD\{V_C(t)\}$ for $I_L(t)$ in (1.18), we get:

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) CD\{V_C(t)\} \;=\; \frac{1}{L}D\{v_s(t)\}$$

After the cancellation of D operator, we get the differential equation for $V_C(t)$:

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) V_C(t) \;=\; \frac{1}{LC}v_s(t) \tag{1.19}$$

The only difference between the parallel and series RLC circuits as the definition of damping factor and resonant frequency. All results and conclusions given for the parallel RLC is equally applicable to the series RLC circuit.

1.4 Examples

Example 1.1 *Find $V_C(t)$ for $t \geq 0$. The initial conditions are $V_C(0^-) = -3$ V and $I_L(0^-) = 0$ A.*

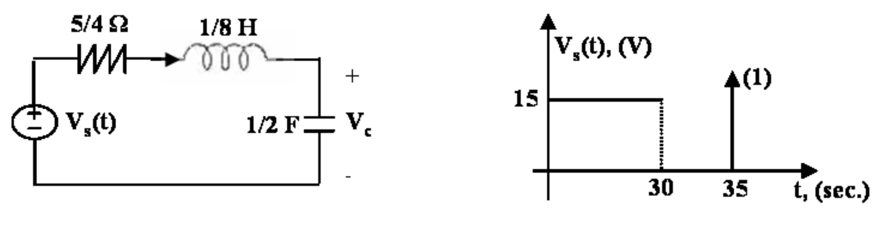


Figure 1.9: Example 1.1

Solution:

The differential equation for $V_C(t)$ has been previously given in (1.19). Once the given R, L and C values are substituted, we get:

$$\left(D^2 + 10D + 16\right) V_C(t) \quad = \quad 16v_s(t)$$

We use zero-input and zero-state decomposition of the circuit to find the complete solution for $V_C(t)$.

Zero-Input Solution:

The initial conditions for $V_C(t)$ at $t = 0^+$ is needed. Since the input is bounded (The input is zero!), $V_C(0^+) = V_C(0^-)$. We also need $\dot{V}_C(0^+)$. The value for $\dot{V}_C(0^+)$ can be found by analyzing $t = 0^+$ circuit. The circuit is given in Figure 1.10.

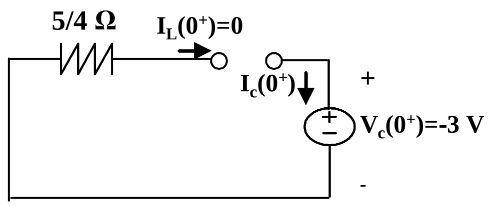


Figure 1.10: Example 1.10

It can be noted from the zero-input circuit at $t = 0^+$ that $\dot{V}_C(0^+) = 0$. Then the differential equation

$$\left(D^2 + 10D + 16\right) V_C^{zi}(t) \quad = \quad 0, \quad \text{with} \quad \begin{array}{l} V_C(0^+) = -3 \\ \dot{V}_C(0^+) = 0 \end{array}$$

should be solved. It can be noted that the natural frequencies of the circuit is $\lambda = \{-2, -8\}$. The system is overdamped. The zero-input solution can be written in the form:

$$V_C^{zi}(t) = c_1 e^{-2t} + c_2 e^{-8t}, \quad t \geq 0 \tag{1.20}$$

The constant c_1 and c_2 should be determined from the initial conditions at $t = 0^+$. We find the constants by solving the following equation system.

$$\begin{aligned} V_C^{zi}(0^+) &= c_1 + c_2 = -3 \\ \dot{V}_C^{zi}(0^+) &= -2c_1 - 8c_2 = 0 \end{aligned}$$

The solution of the equation system is $c_1 = -4$ and $c_2 = 1$. Finally the zero-input solution is then:

$$V_C^{zi}(t) = -4e^{-2t} + e^{-8t}, \quad t \geq 0 \tag{1.21}$$

Zero-State Solution:

The external input $v_s(t)$ can be written as follows:

$$v_s(t) = 15\,(u(t) - u(t-30)) + \delta(t-35)$$

The zero-state response to this input is then

$$V_C^{zs}(t) = 15\left(V_C^{\text{step}}(t) - V_C^{\text{step}}(t-30)\right) + V_C^{\text{impulse}}(t-35)$$

The superposition of the zero-state responses is the main reason that we decompose the solution into zero-input and zero-state parts. By calculating the step-response and impulse response of series RLC circuit (which are the standard responses), it is possible to present a solution to a fairly complicated input given in this example.

Step-Response: Like all zero-state responses, the initial conditions at $t = 0^-$ are all zero. The circuit is not energized at $t = 0^-$. Since the input is bounded, (the input is unit step function); $V_C(0^-)$ and $I_L(0^-)$ values are transferred as it is to $t = 0^+$ values. Then $V_C(0^+) = 0$ and $I_L(0^+) = 0$ and we do a $t = 0^+$ to find $\dot{V}_C(0^+)$.

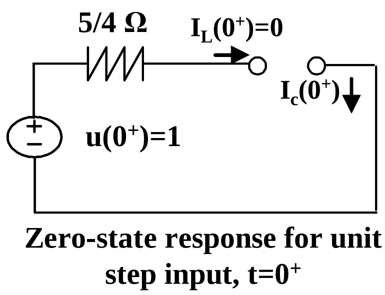


Figure 1.11: Step response calculation, $t = 0^+$

Figure 1.11 shows that $\dot{V}_C(0^+) = 0$. Then the differential equation for the step input is then

$$\left(D^2 + 10D + 16\right) V_C^{zs}(t) \quad = \quad 16u(t), \quad \text{with} \quad \begin{array}{l} V_C(0^+) = 0 \\ \dot{V}_C(0^+) = 0 \end{array}$$

The solution for the step input is then:

$$V_C^{zs}(t) = 1 + d_1 e^{-2t} + d_2 e^{-8t}, \quad t \geq 0$$

Again d_1 and d_2 are constants to be determined from $t = 0^+$ initial conditions. These constants can be found as $d_1 = -4/3$ and $d_2 = 1/3$. Then the step response can be written as follows:

$$V_C^{\text{step}}(t) = 1 - \frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t}, \quad t \geq 0$$

Impulse-Response: The impulse response can be calculated by finding the initial conditions at $t = 0^+$ and then solving the differential equations for $t > 0$. Even though, this is possible; we use the knowledge that the impulse response is the derivative of the step response and immediately write the response as:

$$\begin{aligned} V_C^{\text{impulse}}(t) &= \frac{d}{dt}V_C^{\text{step}}(t) \\ &= \frac{8}{3}\left(e^{-2t} - e^{-8t}\right), \quad t \geq 0 \end{aligned}$$

Complete Solution: The solution can be written as the summation of the zero-input and the zero-state solutions which is

$$\begin{aligned} V_C(t) &= V_C^{zi}(t) + V_C^{zs}(t) \\ &= \left(-4e^{-2t} + e^{-8t}\right) + 15(V_C^{\text{step}}(t) - V_C^{\text{step}}(t-30)) + 35V_C^{\text{impulse}}(t-35), \quad t \geq 0 \end{aligned}$$

This concludes the example.

...
(to be continued, January 13, 2013)