EE201, (Inst: Çağatay Candan)

28. 9. 2009

Lecture Notes of

Kamil Nar, Fall 2009-10

$I_c = C \frac{d}{dt} V_c(t)$

Lumped circuit assumption

frequency is small $\Rightarrow$ $\lambda >>$ size of the circuit.

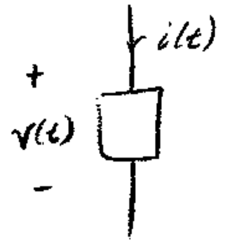
charge coulomb

current $\frac{d}{dt} q(t)$

energy / work : $w = q \cdot \Delta V$

power: $p(t) = \frac{d}{dt} w(t) = \frac{d}{dt} q \Delta V = i \cdot \Delta V$

Passive Sign Convention



Note: ① Polarity of the voltage v(t)

② Direction of the current

$\Downarrow$

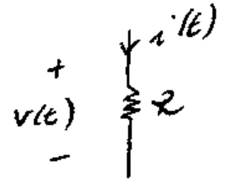
current entering positive terminal

power $\rightarrow$ $p(t) = v(t) \cdot i(t)$

Then for a component

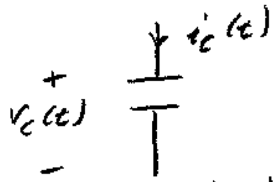
when $p(t) \geq 0 \quad \forall t \rightarrow$ passive component

$p(t) < 0 \quad \exists t \rightarrow$ active component



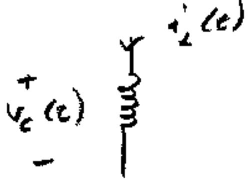
$v(t) = i(t) \cdot R$

$p(t) = v(t) \cdot i(t) = [i(t)]^2 \cdot R \geq 0 \rightarrow$ passive comp.



$i_c(t) = c \cdot \frac{d}{dt} V_c(t)$

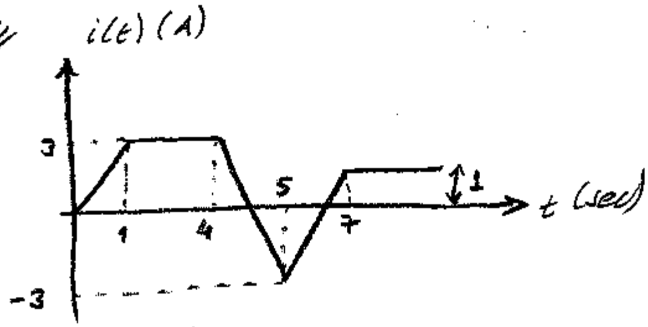
↳ Farad's (capacitance)

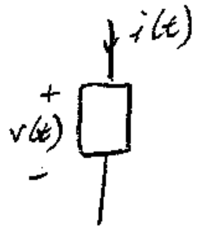


$v_L(t) = L \frac{d}{dt} i(t)$

↳ Henry's (inductance)

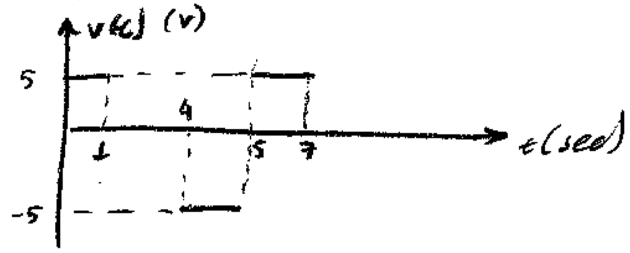
Ex



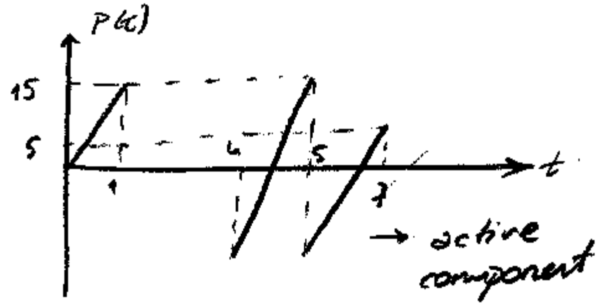


a) Find $p(t)$

b) Evaluate total energy dissipated during seconds [0,8]



a)



b) $p(t) = \frac{d}{dt} w(t)$

$$w(t) = \int_a^t p(\tau)\, d\tau + C$$

$$\Delta w = w(8) - w(0)$$

$$= \left(\int_a^8 p(\tau) d\tau + c \right) - \left(\int_a^0 p(\tau) d\tau + c \right)$$

$$= \underbrace{\int_0^8 p(t)\, dt}_{\text{Area}} = -2,5 \text{ Joules}$$

2,5 Joules is generated & delivered to other components in first 8 seconds.

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$p(t) = v(t)\, i(t)$ → $p(t) \geq 0$ → comp. absorbs energy at time t

→ $p(t) < 0$ → comp delivers energy

Passive - Active Components

Passive → $\int_{-\infty}^{t} p(\tau) d\tau \geq 0 \quad \forall t$

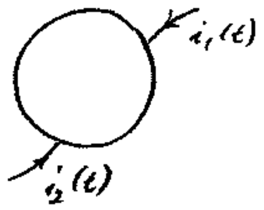
(Resistor, capacitor, inductor)

Active → $\int_{-\infty}^{t} p(\tau) d\tau < 0 \quad \exists t$

(Generator, Battery)

Kirchoff's Current Law (KCL)

→ Conservation of charge
Total charge in a closed sphere is constant



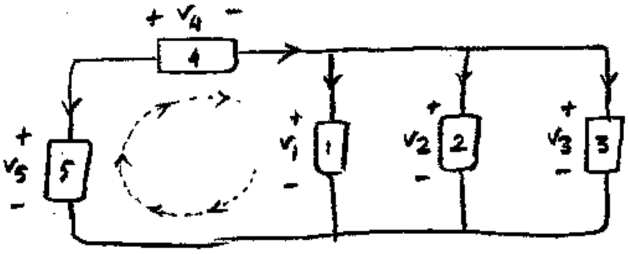
$$\left.\begin{aligned} Q(t) &= c \\ \frac{dQ(t)}{dt} &= 0 \end{aligned}\right\} \quad i_1(t) + i_2(t) = 0$$

$$\Updownarrow \quad \text{Since } Q(t) = \int_{-\infty}^{t} \left(i_1(t) + i_2(t) \right) dt$$

Branch : where components lie
Node : intersection of branches
Cut-set : any partition of the circuit dividing it into two. (inside & outside)

Mesh : Individual cells on a planar circuit
Loop : arbitrary closed path of branches

Kirchoff's Voltage Law (KVL)



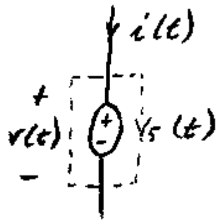
Conservation of Energy

$$q V_5 + q(-V_4) + q(-V_1) = 0$$

$$V_5 - V_4 - V_1 = 0 \longrightarrow \text{KVL}$$

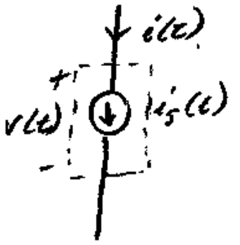
Components
R, L, C

Ideal Voltage Source

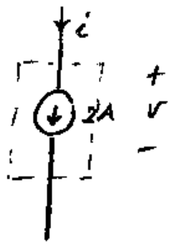


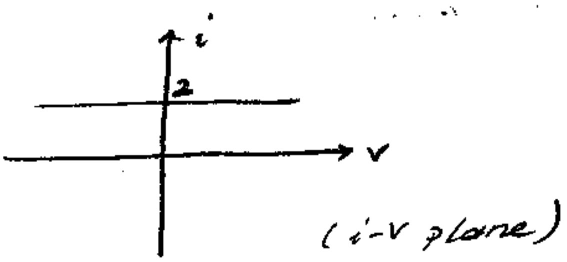
$v(t) = v_s(t) \quad \forall t$
$i(t)$ can be anything
$i(t) \in \mathbb{R}$

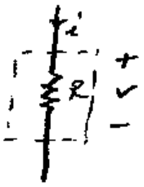
Ideal Current Source

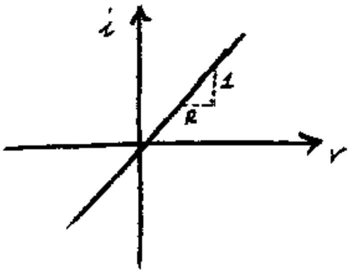


$v(t) \in \mathbb{R}$
$i(t) = i_s(t) \quad \forall t$
(Dual of voltage source)







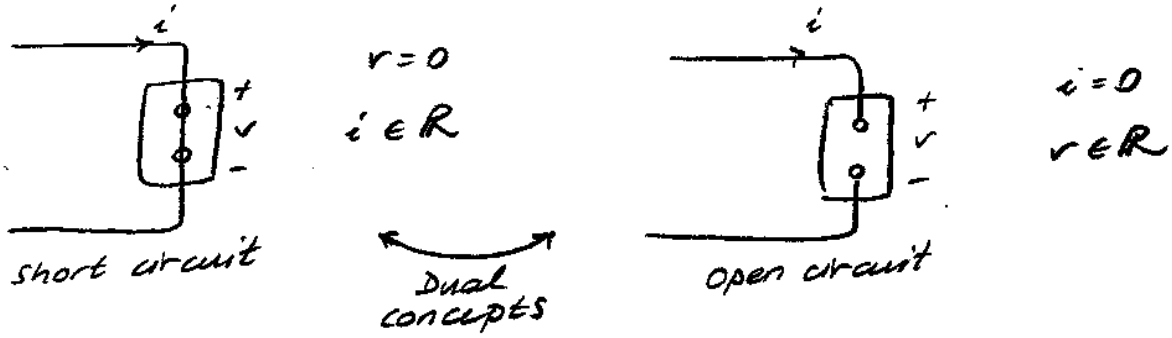


Resistors

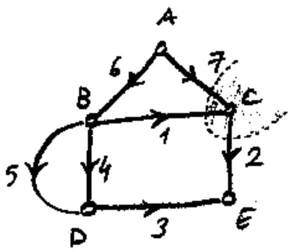
1) Time invariant → $v(t) = i(t) \cdot R$

2) Time-varying → $v(t) = i(t) \cdot R(t)$

Ohm's Law is valid since $v(t^*) = R(t^*) \cdot i(t^*)$ should be satisfied at time $t = t^*$



Graph Theory



1) collection of nodes & branches : graph

2) with assigned directions : directed graph

3) mesh , loop : closed paths of branches

$\{1, 2, 3, 4\}$ $\{1, 2, 3, 5\}$

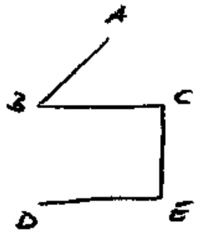
4) node , cut-sets

$\{C\}$ & $\{A, B, D, E\}$

$\{C, E\}$ & $\{A, B, D\}$

Finding minimum number of equations to be satisfied for all KVL and KCL equations to hold

5) trees : → do not contain a loop
→ reaches all nodes
→ it is connected

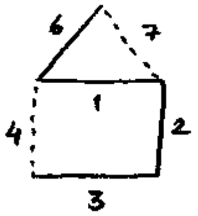


valid tree

6) co-tree : branches not included in tree

{6, 1, 2, 3} & {4, 7}
tree co-tree

7) Fundemental Loop :
Given a tree, any loop that can be formed by taking of the tree and a single branch from co-tree

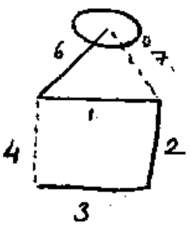


{6, 7, 1} → fund. Loop

{1, 2, 3, 4} → fund. Loop


→ Depending on tree, fundamental loops change.

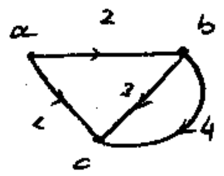
8) Fundamental cut-set :
Given a tree, a cut-set with only one branch from tree and the rest of the branches from co-tree



{6, 7}
{1, 4, 7}
{2, 4}
{3, 4}

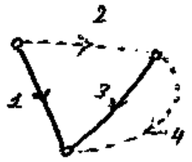
it can have only one of 1, 2, 3 and 6 ; and the rest can be 4 and/or 7.

Ex//



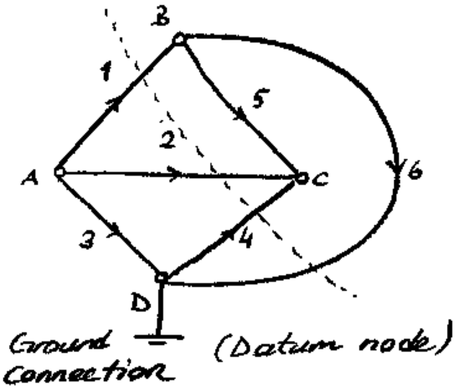
write KCL eqn. at nodes a, b & c :

@a : $i_1 + i_2 = 0$
@b : $-i_2 + i_3 + i_4 = 0$
@c : $i_1 + i_3 + i_4 = 0$



{1, 2}
{3, 2, 4}
} Fund. Cut-sets

The Incidence Matrix



The incidence matrix shows the connection between $\{A, B, C, D\}$

$$A = [A_{ik}] = \begin{cases} 1, & k^{th} \text{ branch leaves node } i \\ -1, & k^{th} \text{ branch enters node } i \\ 0 \end{cases}$$

$$\begin{array}{ll|cccccc} & & 1 & 2 & 3 & 4 & 5 & 6 \\ \text{row1} & A & 1 & 1 & 1 & 0 & 0 & 0 \\ \text{row2} & B & -1 & 0 & 0 & 0 & 1 & 1 \\ \text{row3} & C & 0 & -1 & 0 & -1 & -1 & 0 \\ \text{row4} & D & 0 & 0 & -1 & 1 & 0 & -1 \end{array} = A \longrightarrow$$

linearly dependent
no need for one of the rows

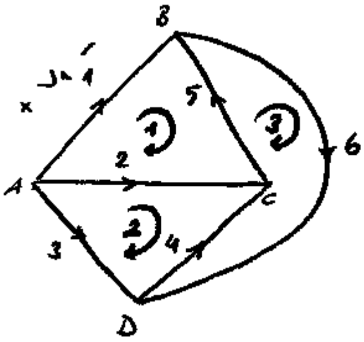
$$A \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_6 \end{bmatrix} = \begin{bmatrix} i_1 + i_2 + i_3 \\ -i_1 + i_5 + i_6 \\ -i_2 - i_4 - i_5 \\ -i_3 + i_4 - i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from KCL's

$$[1 \ 1 \ 0 \ 1 \ 0 \ -1] \cdot \begin{bmatrix} i_1 \\ \vdots \\ i_6 \end{bmatrix} = [0] \longrightarrow$$ for cut-set ;
linear combination of row 2 & 3

The Mesh Matrix

$$\mathcal{M} = [\mathcal{M}_{ik}] = \begin{cases} 1, & \text{branch } k \text{ is in mesh } i \text{ and reference directions agree} \\ -1, & \text{branch } k \ldots \text{ do not agree} \\ 0, & \text{otherwise} \end{cases}$$



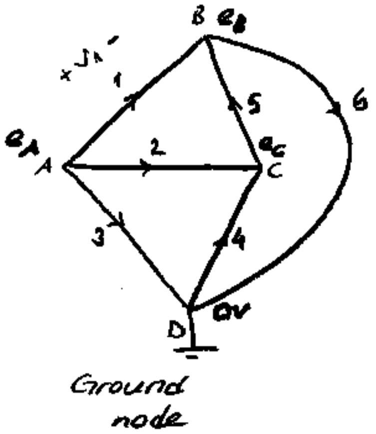
$$\mathcal{M} = \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_6 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 - V_5 \\ \vdots \\ \end{bmatrix}$$

$= [0]$ from KVL's

For mesh $\{1, 6, 3\}$, $[1 \ 0 \ -1 \ 0 \ 0 \ 1]$ is a linear combination of row 1, 2 & 3.

$\Rightarrow$ no need to include in outer mesh's KVL's

Analysis with Incidence Matrix (Node Analysis)



reduced incidence matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\underline{\underline{A}} \cdot \underline{i} = \underline{0} \quad \text{where} \quad \underline{i} = \begin{bmatrix} i_1 \\ \vdots \\ i_6 \end{bmatrix} \rightarrow \text{branch currents}$$

e_A, e_B & e_C are three unknowns from which all circuit variables can be solved.

e_A, e_B, e_C : node voltages

D: Ground node is assumed to be <u>zero</u> voltage level

then; $V_1 = e_A - e_B$

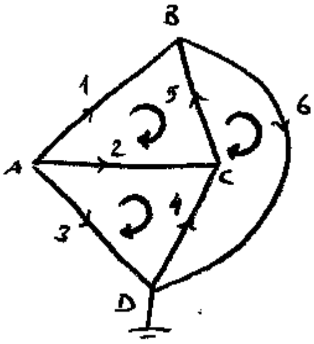
$V_2 = e_A - e_C$

...

$$\begin{bmatrix} V_1 \\ \vdots \\ V_6 \end{bmatrix} = \underline{V} = \underline{\underline{A}}^T \cdot \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix}$$

unknowns to solve for

Analysis with Mesh Matrix (Mesh Analysis)



reduced mesh matrix

$$\underline{\underline{M}} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \begin{bmatrix} V_1 \\ \vdots \\ V_6 \end{bmatrix} = \underline{0} \quad (\text{by KVL})$$

$i_1 = I_A$

$i_2 = I_B - I_A$

$i_3 = -I_B$

$i_4 = I_C - I_B$

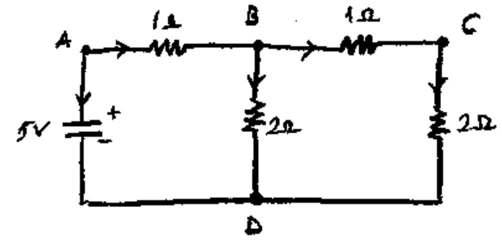
$i_5 = -I_A + I_C$

$i_6 = I_C$

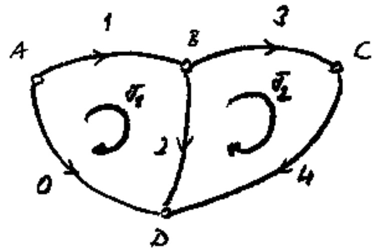
I_A, I_B & I_C : Mesh currents

$$\underline{i} = \underline{\underline{M}}^T \cdot \underbrace{\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix}}_{\text{mesh currents}}$$

Ex



mesh analysis using graph theory



KVL's $\Rightarrow$ $\underline{\underline{M}} \cdot \underline{v} = \underline{0}$
$\hookrightarrow$ branch voltages

Mesh 1 : $-V_0 + V_1 + V_2 = 0$

mesh 2 : $-V_2 + V_3 + V_4 = 0$

Note : $V_0 = 5V$ from the circuit given

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underline{\underline{M}} \cdot \underline{V} = \underline{0} \ldots ①$$

② .... $i = \underline{\underline{M}}^T \cdot J \rightarrow$ mesh currents

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{V} = \underline{\underline{R}} \cdot \underline{i} + \underline{V_S} \ldots ③$$

Solution 1) Multiply ③ by $\underline{\underline{M}}$

$$\underline{\underline{M}} \cdot \underline{v} = \underline{\underline{M}} \cdot \underline{\underline{R}} \cdot \underline{i} + \underline{\underline{M}} \cdot \underline{V_s}$$

2) $\underline{\underline{M}} \cdot \underline{v} = \underline{0}$ and $\underline{i} = \underline{\underline{M}}^T J \longrightarrow 0 = (M R \cdot M^T) J + M \cdot V_s$

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Tellegen's Theorem

Given a graph, branch voltages (v_k) and branch currents satisfy

$$\underset{k=1}{\overset{N \leftarrow \text{\# of branches}}{\sum}} v_k i_k = 0$$

Proof: $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$; $\underline{i} = \begin{bmatrix} i_1 \\ \vdots \\ i_N \end{bmatrix}$

$$\sum_{k=1}^{N} v_k i_k = \underline{v}^T \cdot \underline{i} = (\underline{\underline{A}}^T \cdot \underline{e})^T \cdot \underline{i} = \underline{e}^T \underline{\underline{A}}\, \underline{i} = \underline{e}^T \cdot \underbrace{(\underline{\underline{A}} \cdot \underline{i})}_{\underline{0}\ (\text{KCL's})} = \underline{e}^T \underline{0} = \underline{0}$$

A: reduced incidence matrix

e: node voltages vector

Applications:

1) Power conservation

$$\sum_{k=1}^{N} v_k \cdot i_k = 0 \Rightarrow \sum_{k=1}^{N} P_k = 0$$

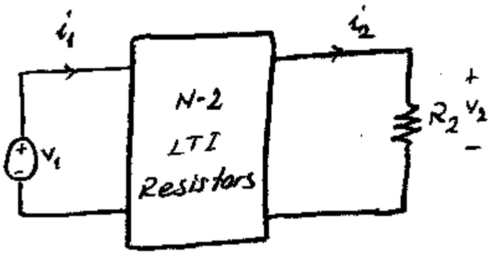
2) Two different circuits with same graph, say c and $\hat{c}$ satisfy

$$\sum_{k=1}^{N} v_k \cdot \hat{i}_k = \sum_{k=1}^{N} \hat{v}_k \cdot i_k = 0$$

As they have the same A matrix, which gives the proof

Network Theorems.

Ex



(LTI Resistors obey Ohm's Law $v_k = R_k \cdot i_k$)

c

$R = 1$

$i_2 = i_1 = 1A$

$v_2 = 1V$

$v_1 = 4V$

$\hat{c}$

$\hat{R} = 2$

$\hat{v}_1 = 6V$

$\hat{i}_1 = 1.2A$

$R_k = \hat{R}_k \quad (k \neq 2)$

$\hat{v}_2 = ?$

(1) $\sum_{k=1}^{N} v_k \cdot \hat{i}_k = 0$

$$\underset{4}{v_1}\,(-\underset{1,2}{\hat{i}_1}) + \underset{1}{v_2}\,\underset{?}{\hat{i}_2} + \underbrace{\sum_{k=3}^{N} v_k\, \hat{i}_k}_{x} = 0$$

(2) $\sum_{k=1}^{N} \hat{v}_k \cdot i_k = 0$

$$\underset{6}{\hat{v}_1} \cdot (-\underset{1}{i_1}) + \underset{?}{\hat{v}_2}\,\underset{1}{i_2} + \underbrace{\sum_{k=3}^{N} \hat{v}_k\, i_k}_{x} = 0$$

Note $v_k = R_k \cdot i_k$ & $\hat{v}_k = R_k \cdot \hat{i}_k$

then $v_k \cdot \hat{i}_k = \hat{v}_k \cdot i_k$

using $\hat{i}_2 = \frac{\hat{v}_2}{\hat{R}_2}$

(1) $-4,8 + \frac{1}{2}\hat{v}_2 + x = 0$

$-6 + \hat{v}_2 + x = 0$

$$\boxed{\hat{v}_2 = 2.4\ V}$$

Duality

① Dual Graph

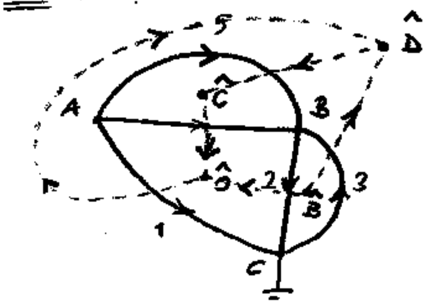
Remember that node analysis has an associated $\underline{\underline{A}}$ matrix; and similarly mesh analysis has an associated $\underline{\underline{M}}$ matrix.

Dual graphs have the same $\underline{\underline{A}}$ (for Node Analysis) and $\underline{\underline{M}}$ (for Mesh analysis) matrices.

How to construct Dual Graphs:

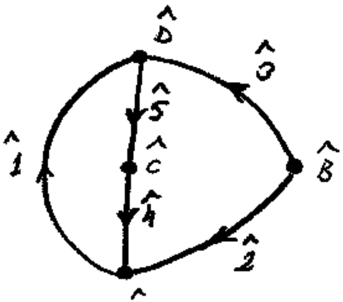
Given a graph
1) Pick centers of meshes
2) Direction of branches are generated by 90° of rotation in clock-wise direction

Ex



Graph

$\underline{\underline{A}}$ $\quad \underline{\underline{A}}\, \underline{i} = 0$



It's Dual

$\underline{\underline{M}} \cdot \underline{v} = 0$

$\underline{\underline{A}}$ and $\underline{\underline{M}}$ are row equivalent. or equal (?)

Important Note:

There can be more than one dual graph; but if you enforce KCL at datum node (Reference node) to be equivalent to the KVL corresponding to the outer mesh (as in the example) dual graph is unique.

Dual Circuits

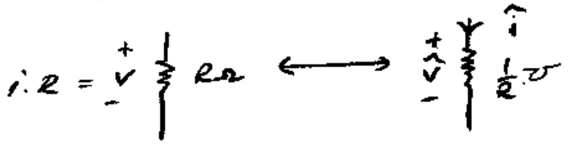
Dual components

1) Resistance

$$R \xleftrightarrow{\text{dual}} \hat{i} = v \cdot \hat{x}$$

$v = i \cdot R$

Another resistor with conductance $\frac{1}{R}$



$$\hat{i} = \frac{\hat{v}}{1/R} = \hat{v} \cdot R$$

2) Capacitor

Inductor

$$Q = C \cdot V \xleftrightarrow{\text{dual}} \phi(t) = L \cdot i(t)$$

charge

flux

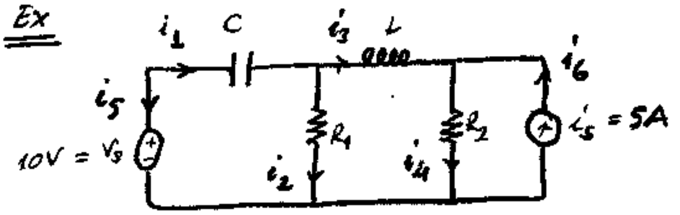
x Farad capacitor ⟷ X Henry inductor

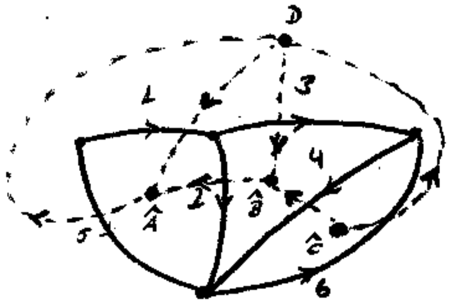
$$i_c(t) = C \cdot \frac{d}{dt} V_c(t) \qquad V_L(t) = L \cdot \frac{d}{dt} i(t)$$

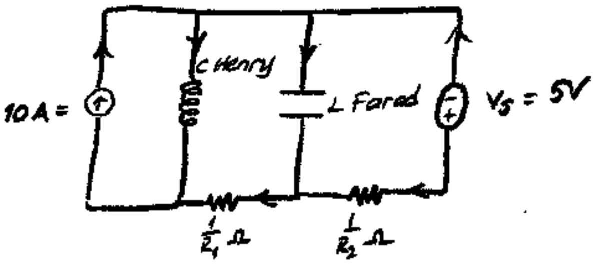
3) Current source ⟷ Voltage source

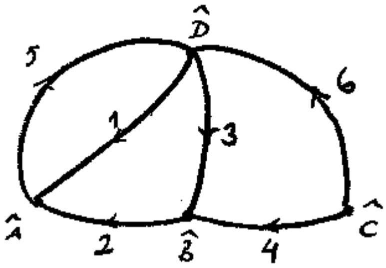
4) Short circuit ⟷ Open circuit

Ex





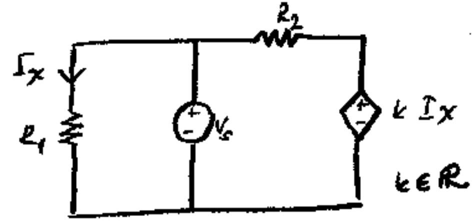




Components

R, L, C, V_s, I_s

Dependent Sources

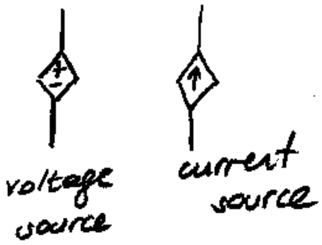


I_x : Reference current

x branch : reference branch

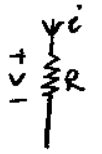
Current controlled Voltage source

1) CCVS
2) CCCS
3) VCCS
4) VCVS

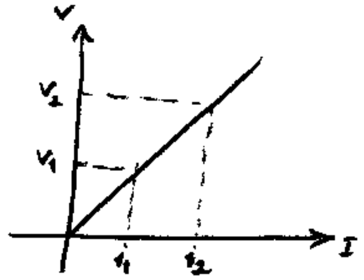


Resistors

→ Linear time invariant resistors: (LTI)



$v = i \cdot R$



$V_1 = i_1 \cdot R$

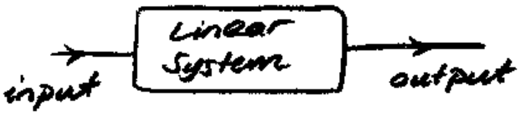
$V_2 = i_2 \cdot R$

If I have $i_1 + i_2$ as the current through the resistor

$V_{1+2} = (i_1 + i_2) \cdot R$

$= V_1 + V_2$

Linear System:

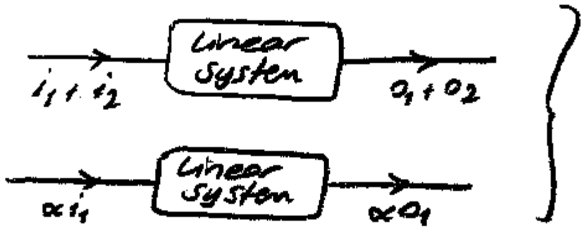


i_1, i_2 as inputs

$\downarrow \quad \downarrow$

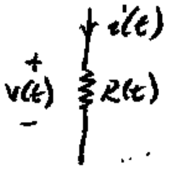
o_1, o_2 as the corresponding outputs

then

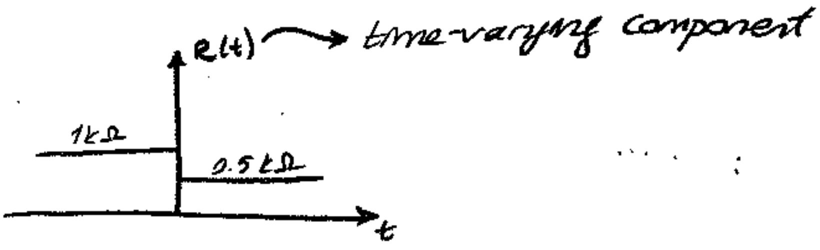


System is linear

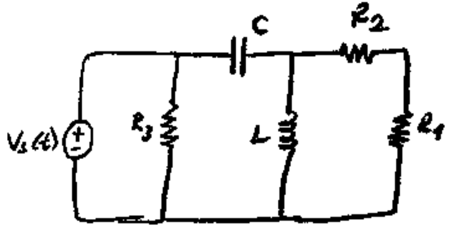
→ Time-invariant Systems



$v(t) = R(t) \cdot i(t)$

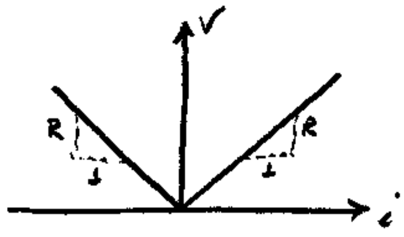


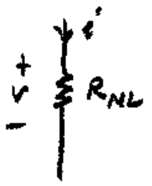
LTI Systems



<u>Note</u>: Resistor is memoriless component as it does not relate V & I with their derivatives.

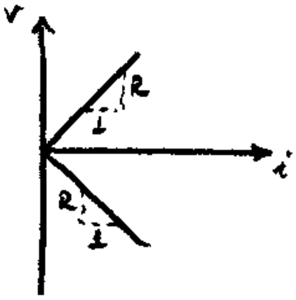
Non-Linear Resistor

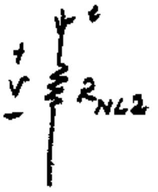




$v = f_{R_{NL}}(i)$

$v = |i \cdot R|$

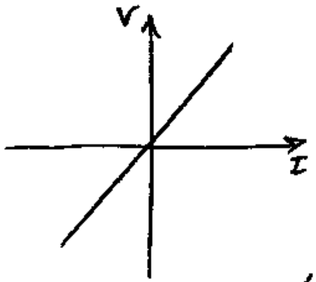




$$i = f_{R_{NL2}}(v)$$

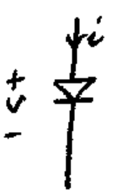
$$i = \frac{|v|}{R}$$

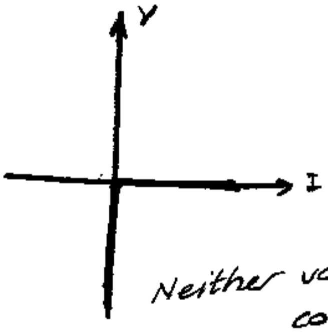
A component is called <u>current-controlled</u> (voltage controlled) if its branch voltage (current) is a function of its current (voltage)



Both voltage and current controlled

Diode





Neither voltage-current controlled

Classification of Circuits

↳ Dynamic (R, L, C combinations)
(requires derivatives and/or integrations)

↳ Memoryless (Resistances and sources)

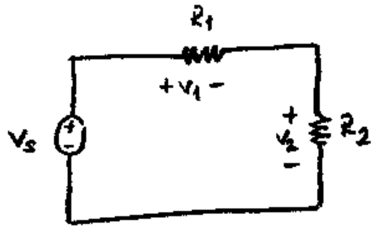
Passive & Active Components

$$\int_{-\infty}^{t} p(\tau)\,d\tau < 0 \quad \exists t \longrightarrow$$ component is active (generating energy)

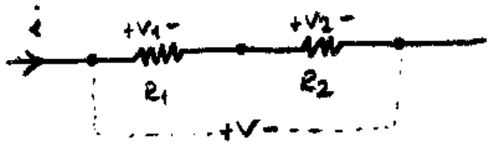
Circuits with active components ⟶ active circuits

Any circuit with physically realizable components are passive (R, L, C)

Voltage / Current Division



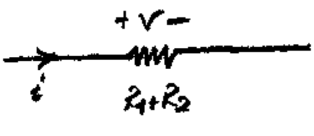
Series Combination of LTI Resistors:



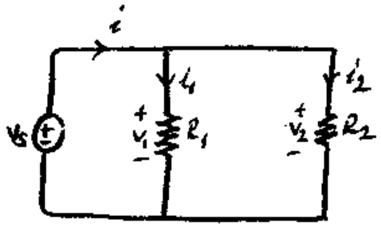
$V = V_1 + V_2 \ldots$ KVL

$$V = i_1 R_1 + i_2 R_2$$

$$V = i(R_1 + R_2)$$



Parallel combination of LTI Resistors



$$v_1 = v_2 = V_s$$

$$i = i_1 + i_2$$

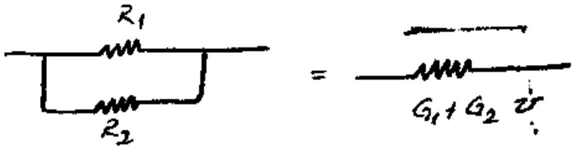
$$i = \frac{V_1}{R_1} + \frac{V_2}{R_2}$$

$$i = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_s \longrightarrow V_s = \left(\frac{R_1 R_2}{R_1 + R_2}\right) i$$

R → ohm Ω

G → mho ℧, Siemens

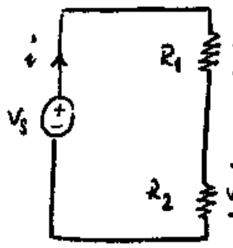
G : conductance $G = 1/R$



N resistors in series: $R_{eq} = \sum_{k=1}^{N} R_k$

N resistors in parallel $R_{eq} = \left(\sum_{k=1}^{N} \frac{1}{R_k}\right)^{-1}$

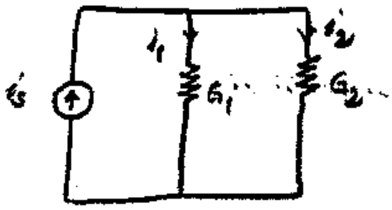
Voltage Division



$$v_1 = V_s \cdot \frac{R_1}{R_1 + R_2}$$

$$v_2 = V_s \cdot \frac{R_2}{R_1 + R_2}$$

current Division

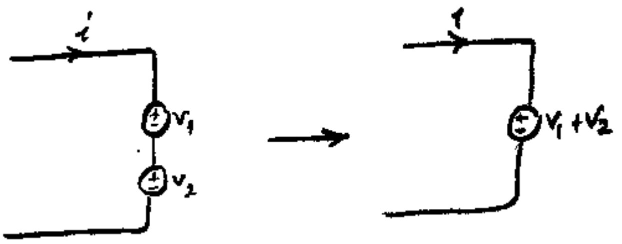


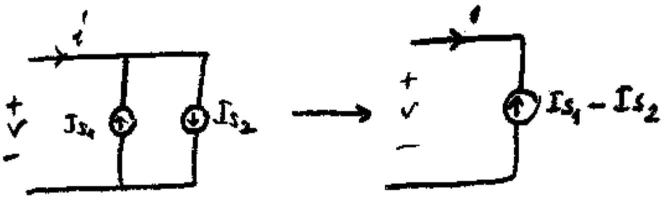
$$i_1 = \frac{R_2}{R_1 + R_2} i_s$$

$$i_1 = \frac{1/R_1}{1/R_2 + 1/R_1} i_s$$

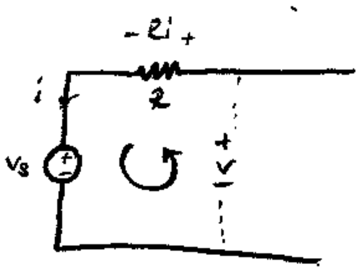
$$i_1 = \frac{G_1}{G_1 + G_2} i_s$$

Source Addition

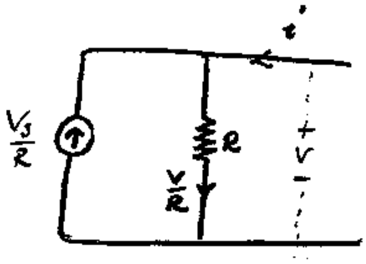




Source Transformation



Voltage Source In Series

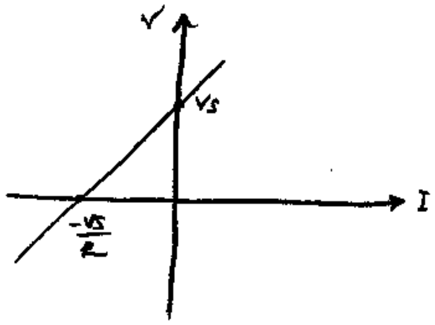


Current Source In Parallel

I-V characteristics

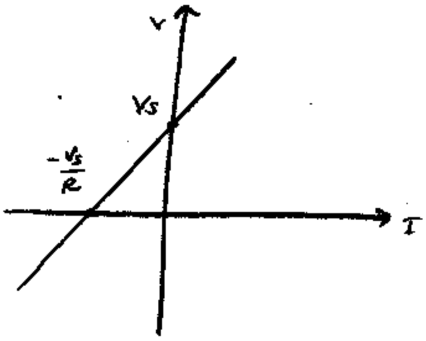
$$+V_s - V + Ri' = 0$$

$$v = V_s + Ri'$$

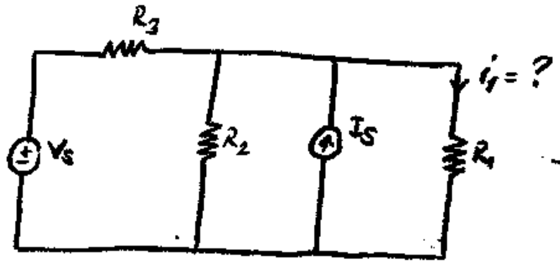


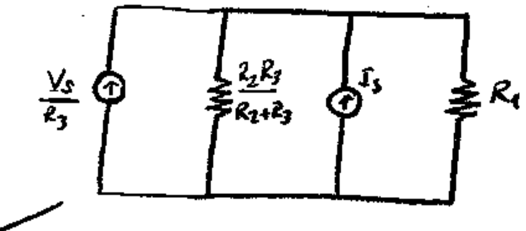
$$\frac{V_s}{R} + i' - \frac{V}{R} = 0$$

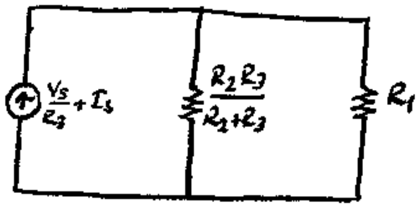
$$V = V_s + iR$$



Ex

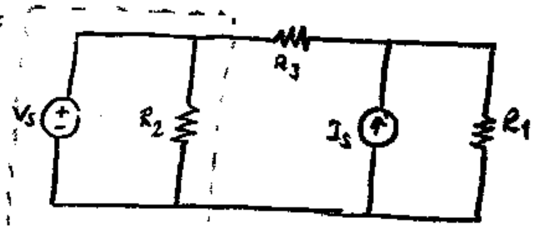


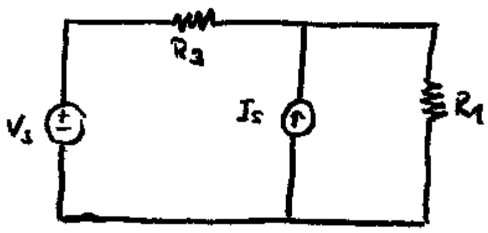


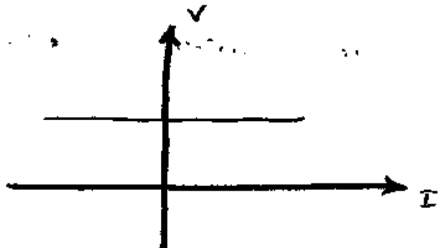


Keep R_1 when you are interested in voltage or current on R_1.

Ex

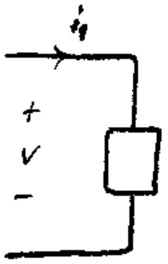




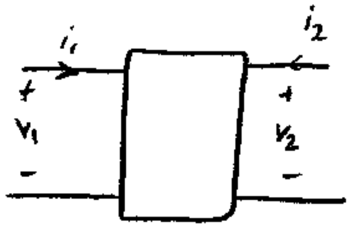


I-V characteristic is the same as a voltage source alone

One Port Circuits



one-part

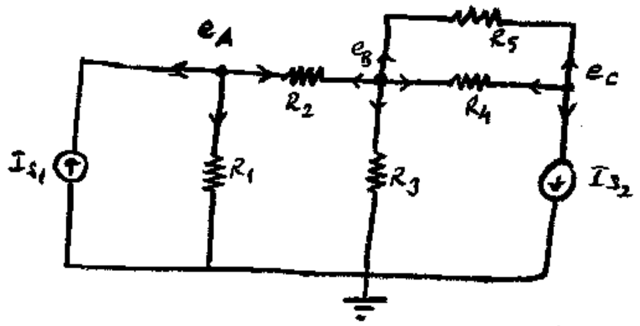


Two-port

Multi-port

Node-Mesh Analysis Techniques

Ex Method: 1) Introduce datum node
2) Introduce node voltages ($e_A, e_B \dots$)
3) Write KCL at each node except datum node



KCL at e_A:

$$-I_{s_1} + \frac{e_A - 0}{R_1} + \frac{e_A - e_B}{R_2} = 0$$

KCL at e_B:

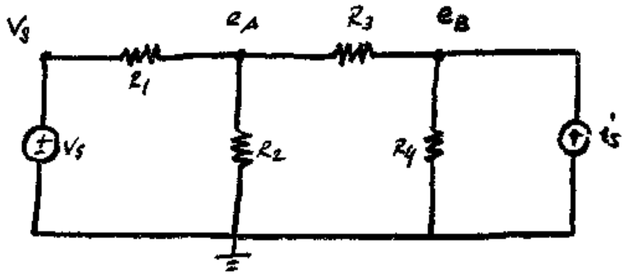
$$\frac{e_B - 0}{R_3} + \frac{e_B - e_C}{R_4} + \frac{e_B - e_C}{R_5} + \frac{e_B - e_A}{R_2} = 0$$

KCL at e_C:

$$I_{s_2} + \frac{e_C - e_B}{R_4} + \frac{e_C - e_B}{R_5} = 0$$

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} & -\frac{1}{R_4} - \frac{1}{R_5} \\ 0 & -\frac{1}{R_4} - \frac{1}{R_5} & \frac{1}{R_4} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} e_A \\ e_B \\ e_C \end{bmatrix} = \begin{bmatrix} I_{s_1} \\ 0 \\ -I_{s_2} \end{bmatrix}$$

Node Analysis with Voltage Sources

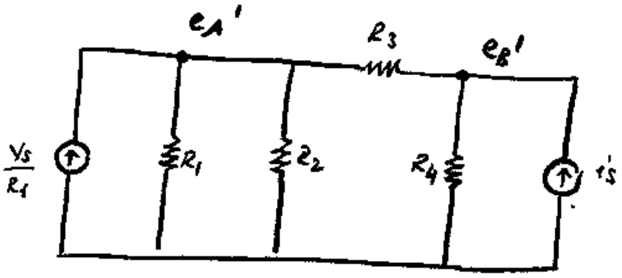


KCL at e_A:

$$\frac{e_A - V_S}{R_1} + \frac{e_A - 0}{R_2} + \frac{e_A - e_B}{R_3} = 0$$

KCL at e_B:

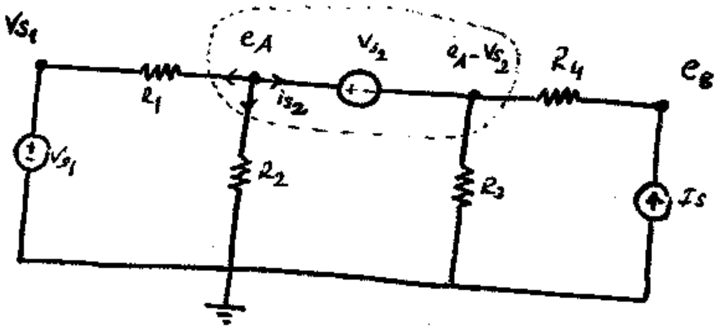
$$-i_s + \frac{e_B - e_A}{R_3} + \frac{e_B - 0}{R_4} = 0$$



$$\frac{e_A'}{R_1} + \frac{e_A'}{R_2} + \frac{e_A'-e_B'}{R_3} - \frac{V_s}{R_1} = 0$$

$$\frac{e_B'}{R_4} + \frac{e_B'-e_A'}{R_3} - i_s = 0$$

Supernode:



KCL at e_A: $\frac{e_A - V_{s_1}}{R_1} + \frac{e_A}{R_2} + i_{s_2} = 0$ - - - - - - ①

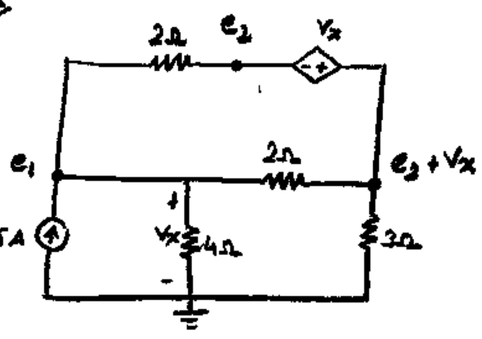
KCL at e_B: $\frac{e_B - e_A + V_{s_2}}{R_4} - I_s = 0$

KCL at "$e_A - V_{s_2}$" node: $\frac{e_A - V_{s_2}}{R_3} + \frac{e_A - V_{s_2} - e_B}{R_4} - i_{s_2} = 0$ - - - - ②

Instead of ① and ②, KCL at supernode:

$$\frac{e_A - V_{s_1}}{R_1} + \frac{e_A}{R_2} + \frac{e_A - V_{s_2}}{R_3} + \frac{e_A - V_{s_2} - e_B}{R_4} = 0$$

ex



KCL at e_1

$$-5 + \frac{e_1 - 0}{4} + \frac{e_1 - e_2 - \overset{e_1}{V_x}}{2} + \frac{e_1 - e_2}{2} = 0$$

KCL at supernode:

$$\frac{e_2 + V_x}{3} + \frac{e_2 + V_x - e_1}{2} + \frac{e_2 - e_1}{2} = 0$$

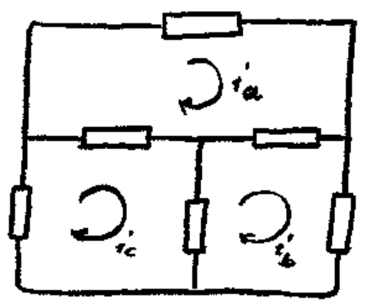
$$\left.\begin{aligned} 3e_1 - 4e_2 &= 20 \\ -e_1 + 8e_2 &= 0 \end{aligned}\right\} \begin{aligned} e_1 &= 8V \\ e_2 &= 1V \end{aligned}$$

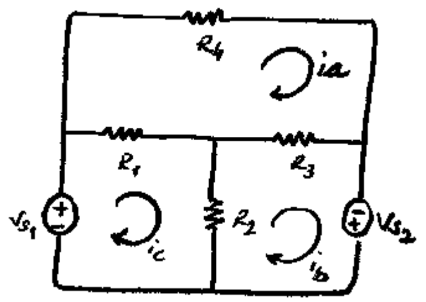
$V_{3\Omega} = e_2 + V_x = e_2 + e_1 = 9V$

$P_{3\Omega} = V_{3\Omega} \cdot i_{3\Omega} = V_{3\Omega} \cdot \frac{V_{3\Omega}}{3} = 27 W$

Mesh Analysis

only applicable to planar circuits





KVL for i_a mesh

$$V_{R_3}+V_{R_1}+V_{R_4}=0$$

$$i_{R_3}R_3+i_{R_1}R_1+i_{R_4}R_4=0$$

$$R_4\cdot i_a+R_3(i_a-i_b)+R_1(i_a-i_c)=0$$

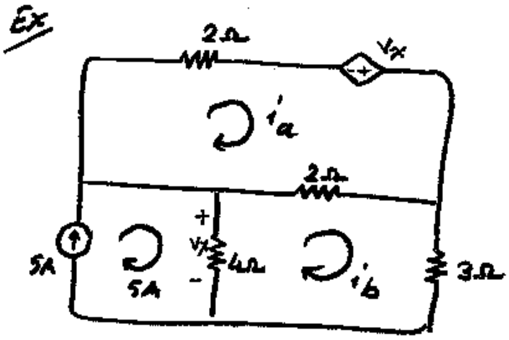
KVL for i_b mesh:

$$-V_{S_2}+R_2(i_b-i_c)+R_3(i_b-i_a)=0$$

KVL for i_c mesh:

$$-V_{S_1}+R_1(i_c-i_a)+R_2(i_c-i_b)=0$$

Ex



KVL for

$$i_a:\ 2(i_a-i_b)+2\,i_a-\overset{\curvearrowright\ 4\cdot(5-i_b)}{V_x}=0$$

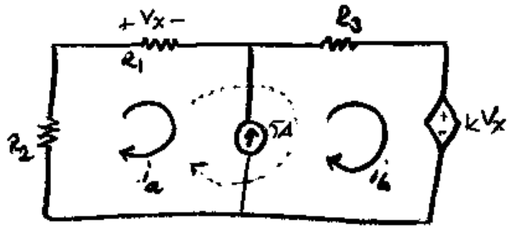
$$i_b:\ 3\,i_b+4(i_b-5)+2(i_b-i_a)=0$$

$$\begin{bmatrix}4 & 2\\ -2 & 9\end{bmatrix}\begin{bmatrix}i_a\\ i_b\end{bmatrix}=\begin{bmatrix}20\\ 20\end{bmatrix}$$

$$\begin{bmatrix}i_a\\ i_b\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}9 & -2\\ 2 & 4\end{bmatrix}\begin{bmatrix}20\\ 20\end{bmatrix}=\begin{bmatrix}7/2\\ 3\end{bmatrix}$$

Supermesh

1. Problem with current sources in the interior of the circuit!



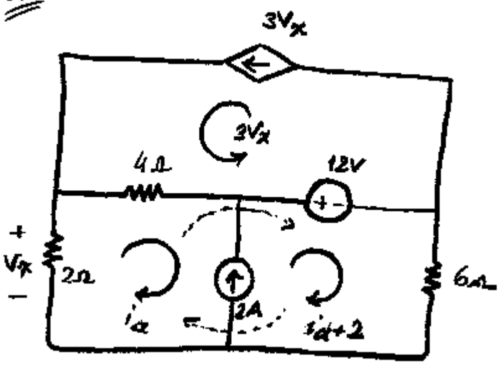
$$R_2\,i_a+R_1\,i_a-V_{cs}=0$$

$$kV_x+V_{cs}+R_3\cdot i_b=0$$

KVL around supermesh:

$$R_2\,i_a+R_1\,i_a+R_3\,i_b+k\underset{\uparrow\ i_a R_1}{V_x}=0$$

Ex



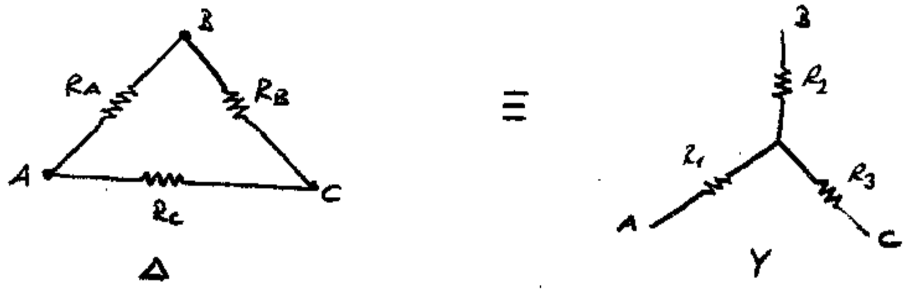
$V_x = -2 i_a$

Around supermesh

$4(i_a - 6 i_a) + 12 + 6(2 + i_a) + 2 i_a = 0$

$i_a = 2A$

Δ-Y Transformation

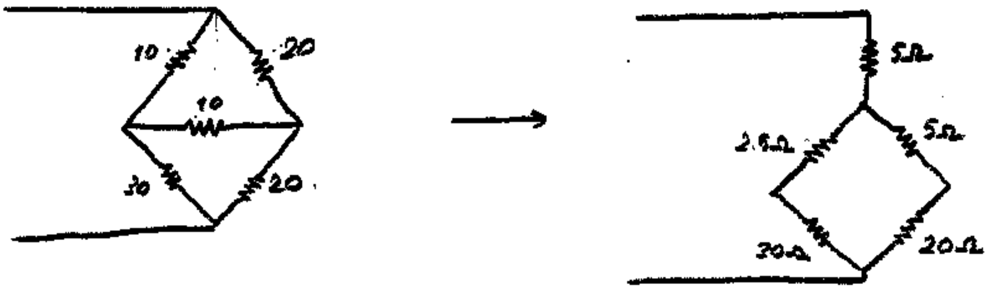


$$R_1 = \frac{R_A R_C}{R_A + R_B + R_C} \qquad R_A = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_3}$$

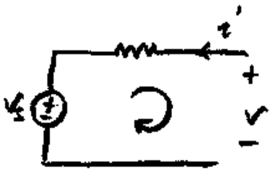
$$R_2 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad R_B = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1}$$

$$R_3 = \frac{R_B R_C}{R_A + R_B + R_C} \qquad R_C = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2}$$

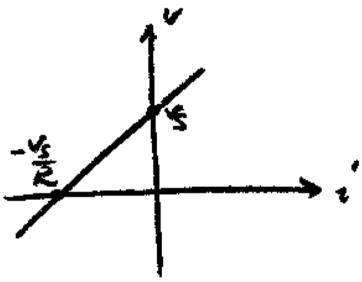
Ex Wheat Stone Bridge

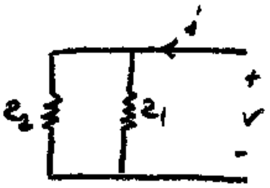


Input Resistance Calculation



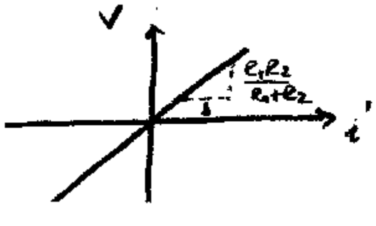
$-V_S + R(-i) + V = 0$



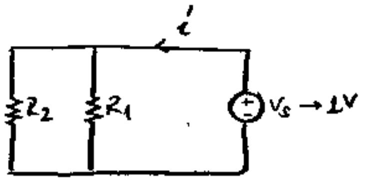


$i' = \frac{V}{R_1} + \frac{V}{R_2}$

$\hookrightarrow V = \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \cdot i'$

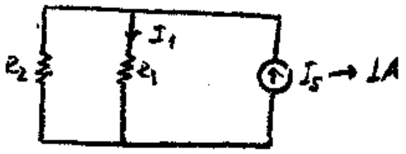


Another method to find i-v char. (when you have only resistors)



$i = \frac{1}{R_1} + \frac{1}{R_2}$ when $V_s = 1V$

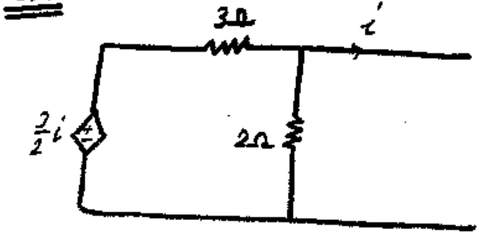
$$\frac{1V}{\frac{1}{R_1} + \frac{1}{R_2}} = R_{in} \longrightarrow R_{in} = R_1 \| R_2$$



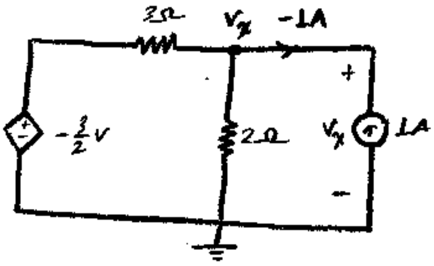
$I_1 = \frac{R_2}{R_1+R_2}$ $\quad I_s \to 1A$

$V = R_1 I_1 = \frac{R_1 R_2}{R_1+R_2}$

Ex



$R_{in} = ?$



Find $V_x \longrightarrow \frac{V_x}{1} = R_{in}$

KCL at V_x : $\frac{V_x - 0}{2} - 1 + \frac{V_x + \frac{3}{2}}{3} = 0$

$V_x = 0.6\,V$

$R_{in} = 0,6$

Linearity

Operators: a mapping between input and output

$$\xrightarrow[f(t)]{} \boxed{\frac{d}{dt}} \xrightarrow[f'(t)]{} \qquad \xrightarrow[f_i(t)]{} \boxed{\text{System}} \xrightarrow[f_o(t)]{}$$

Linear systems:

$f_{out} = L\{f_{in}\}$

$\hookrightarrow$ a linear system

⓪ $f_{out}^{1} = L\{f_{in}^{1}\}$

① $L\{\alpha f_{in}^{1}\} = \alpha f_{out}^{1}$

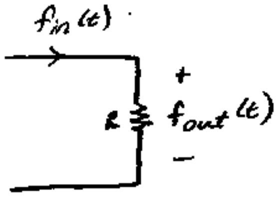
② $L\{f_{in}^{1} + f_{in}^{2}\} = f_{out}^{1} + f_{out}^{2}$

From ① and ②

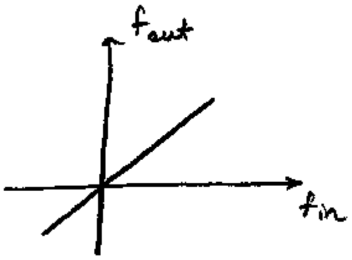
$L\{0\} = 0$

$L\{-f_{in}{}^{2}\} = -f_{out}{}^{1}$

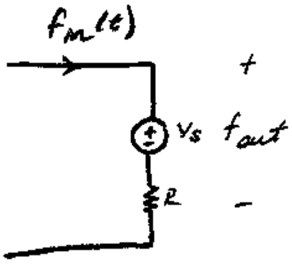
Ex

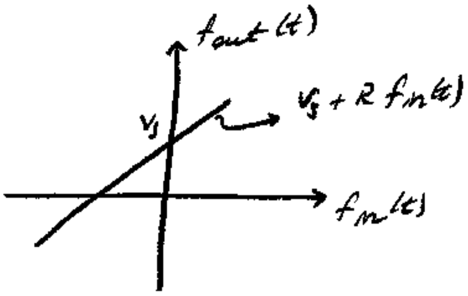


$f_{out}(t) = R \cdot f_{in}(t)$



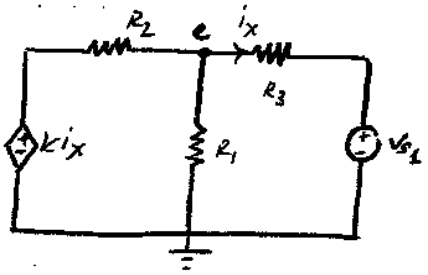
Ex





Not a linear system because of V_s source!

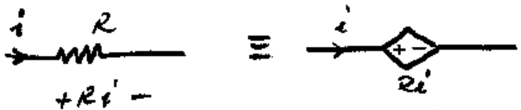
Circuits with a single source:



$$\frac{e}{R_1} + \frac{e - ki_x}{R_2} + \frac{e - V_{s_1}}{R_3} = 0$$

$$(\ldots)\, e = V_{s_1}$$

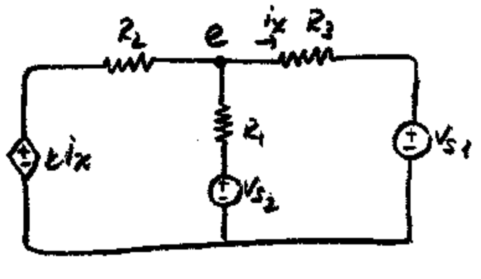
when V_{s_1} is doubled, all circuit variables are also doubled.



Never treat dependent sources as an independent sources!

Superposition Priciple:

A linear system with multiple inputs can be decomposed into several circuits with single input and addition of all output variables in the decomposition results in the solution of multiple input circuit.



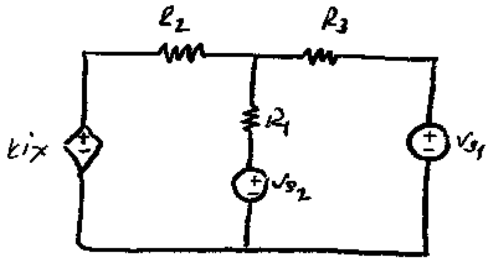
$\alpha e = \beta V_{s_1} + \gamma V_{s_2}$

$\alpha e^1 = \beta V_{s_1}$

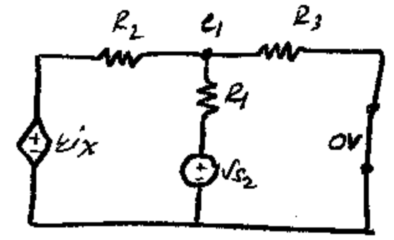
$\alpha e^2 = \qquad \gamma V_{s_2}$

$e = e^1 + e^2$

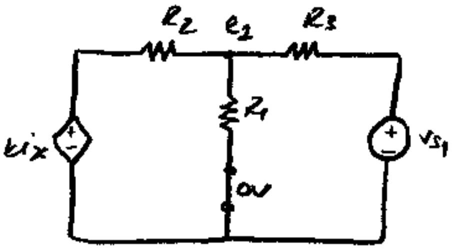
Turning OFF sources (killing the source)



$\xrightarrow{\text{kill } V_{s_1}}$ ($V_{s_1} = 0V$)



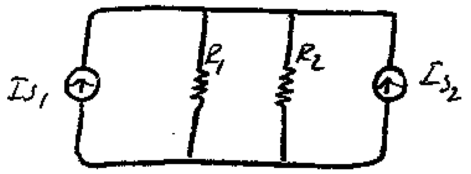
kill $\downarrow V_{s_2}$



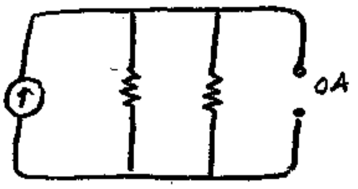
1) solve both circuits for e_1 & e_2

2) $e = e^1 + e^2$

Independent current sources



$\xrightarrow{\text{Turn off } I_{s_2}}$ $I_{s_2} = 0A$



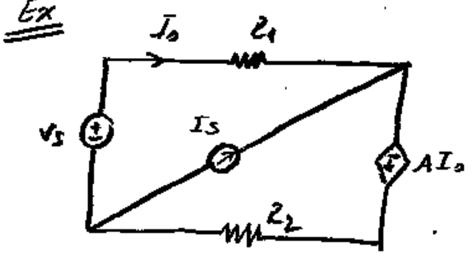
superposition principle applies to all linear systems hence all linear circuits To turn off

① voltage source ⟶ Replace with short circuit ($V_s = 0$)

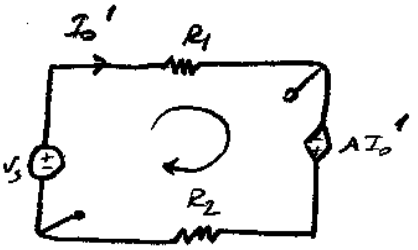
② current source ⟶ Replace with open circuit ($I_s = 0$)

Do <u>not</u> try to turn off dependent sources.

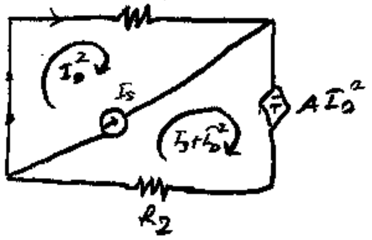
Ex



$\xrightarrow{\text{kill } I_s}$



$\downarrow$ kill Vs



$$I_o^1 (R_1 + R_2) - AI_o^1 - V_S = 0$$

$$I_o^1 = \frac{V_S}{R_1 + R_2 - A}$$

$$+ I_o^2 R_1 - I_o^2 \cdot A + (I_o^2 + I_s) R_2 = 0$$

$$I_o^2 = \frac{-I_s \cdot R_2}{R_1 + R_2 - A}$$

$$I_o = I_o' + I_o^2$$

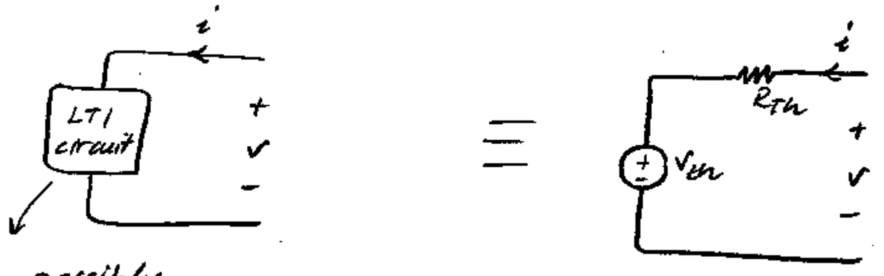
$$I_o = \frac{V_s - R_2 I_s}{R_1 + R_2 - A}$$

Thevenin-Norton Equivalent Circuits

1) circuit simplification methods
    - i - resistance combination
    - ii - adding sources
    - iii - source transformation
    - iv - Δ-Y transformation

2) circuit analysis tools / methods:
    - i - superposition principle
    - ii - mesh analysis
    - iii - node analysis

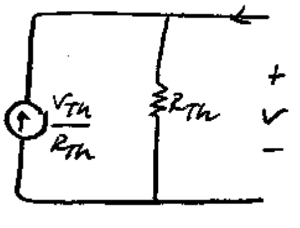
Apply KVL, KCL conditions and terminal equations to form a system of equations for the solution of the circuit.



with possibly independent sources

V_{Th} : Thevenin voltage

R_{Th} : Thevenin resistance



Norton equivalent

v_{oc} : open circuit voltage

i_{sc} : short circuit current

Procedure:

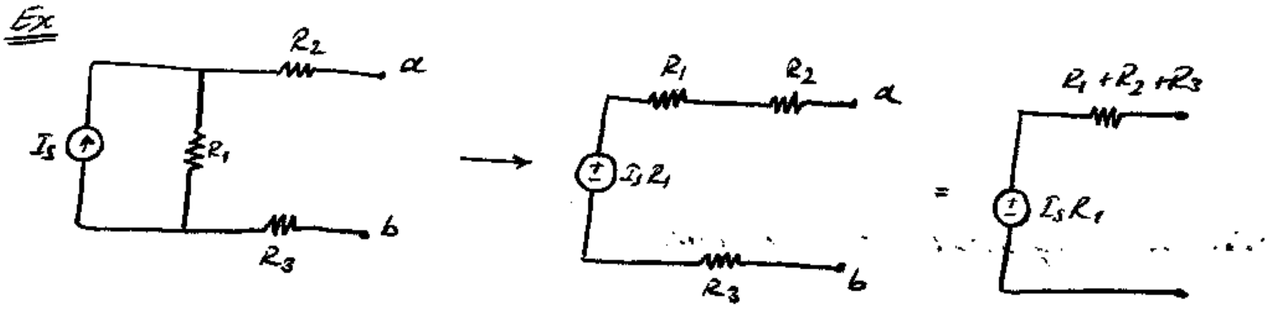
1) Find Voc
2) Find R_{Th} → Kill all independent sources and find Req seen from a-b terminals

→ Kill all independent sources and apply 1V and find i_s of 1V source → $Req = 1V/i_s$

Procedure 2:

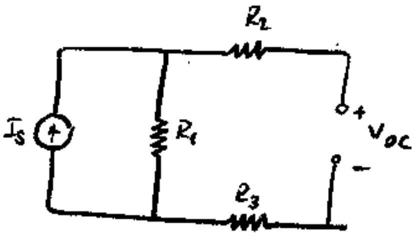
1) Find Voc
2) Find i_{sc}
3) $\frac{V_{oc}}{i_{sc}} = R_{Th}$

Ex



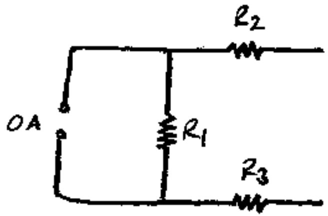
Procedure 1

1) V_{oc} = ?



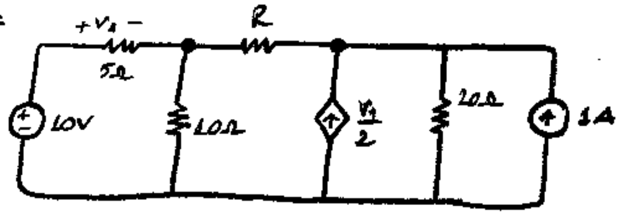
$V_{oc} = I_s \cdot R_1$

2) R_{Th} = ? Kill all independent sources ($V_s = 0V$; $I_s = 0A$)



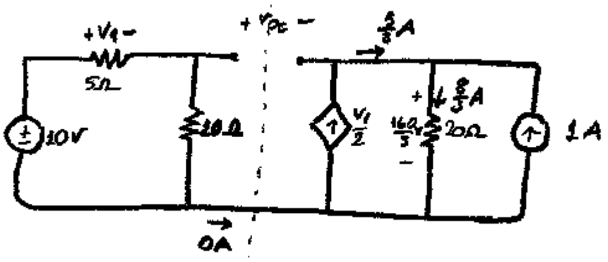
$R_{eq} = R_1 + R_2 + R_3$

Ex



Procedure 2 :

①

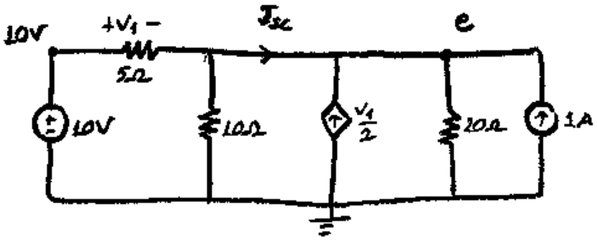


$V_1 = 10V \cdot \frac{5}{5+10} = \frac{10}{3}V$

$V_{oc} = V_a - V_b$

$= \frac{20}{3}V - \frac{160}{3}V = -\frac{140}{3}V$

② I_{sc} = ?



KCL at e :

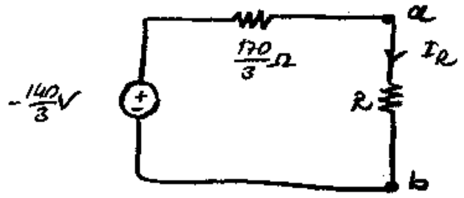
$$\frac{e-10}{5} + \frac{e}{10} - \frac{\overset{10-e}{V_1}}{2} + \frac{e}{20} - 1 = 0$$

$$(4+2+10+1)e = 40+100+20$$

$$e = \frac{160}{17}V$$

$$I_{sc} = \frac{10-e}{5} - \frac{e}{10} = -\frac{14}{17}A$$

$$R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{-140/3}{-14/17} = \frac{170}{3}\Omega$$



a) when $R=\infty$, $V_R = V_{oc} = -\frac{140}{3}$V

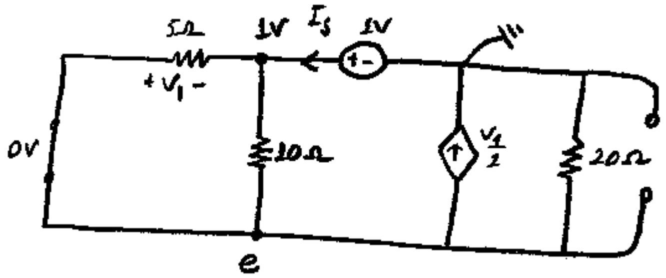
b) when $R=0$, $I_R = I_{sc}$

c) For any R:

$$I_R = \frac{-140/3}{\frac{170}{3} + R}$$

Ex (same question) $R_{Th} = ?$ using external source

Killing independent sources



$V_1 = e - 1$

$$\frac{e-1}{5} + \frac{e-1}{10} + \frac{V_1}{2} + \frac{e}{20} = 0$$

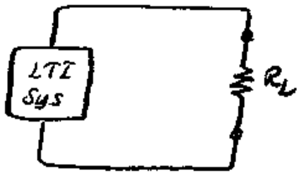
$$e = \frac{16}{17}V$$

$$I_s = \frac{1-e}{5} + \frac{1-e}{10} = \frac{3}{170}A$$

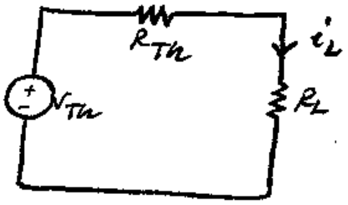
$$R_{Th} = \frac{1V}{I_s} = \frac{170}{3}\Omega$$

Maximum Power Transfer

In many situations, you would like to maximize energy/power delivered to a specific component.



Power consumed by load resistance R_L:



$$i_L' = \frac{V_{Th}}{R_{Th}+R_L} \quad ; \quad V_L = i_L' \cdot R_L$$

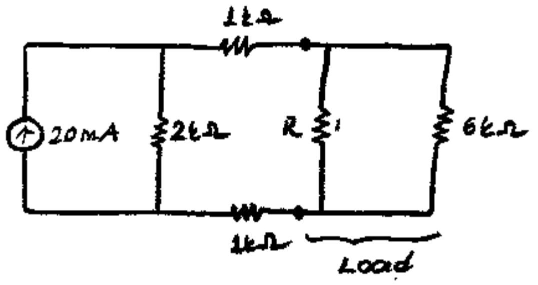
$$P_{R_L} = i_L' \cdot V_L = (i_L')^2 \cdot R_L = V_{Th}^2 \cdot \frac{R_L}{(R_L+R_{Th})^2}$$

to maximize $P(R_L) \longrightarrow \frac{dP}{dR_L} = 0$

$$\frac{d}{dR_L}(P_{R_L}) = V_{Th}^2 \frac{(R_L+R_{Th})^2 - 2R_L(R_L+R_{Th})}{(R_L+R_{Th})^4} = 0 \Rightarrow R_L = \{-R_{Th}, R_{Th}\}$$

to maximize the power transfer: $R_L = R_{Th}$

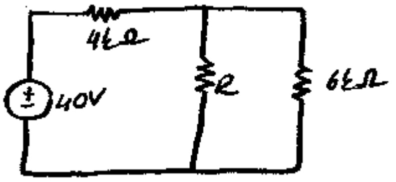
Ex



Find the voltage & power delivered to the load, when power transfer maximized.

$R_{eq} = 1k\Omega + 2k\Omega + 1k\Omega = 4k\Omega$

$V_{oc} = 40V$

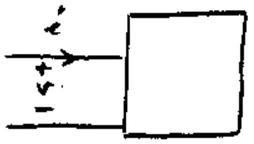


$R \parallel 6k\Omega = 4k\Omega \Rightarrow R = 12k\Omega$

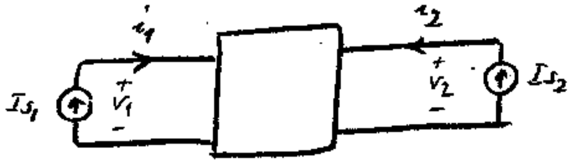
$V_{Load} = 20\,V$

$P_{Load} = 0,1$ watt.

One Port - Two Port



One-port



2-port

R: Resistance Parameters

$$\underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\text{output circuit variables}} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \xrightarrow{} \text{current sources}$$

input sources

G: Conductance Parameters

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (G^{-1} = R)$$

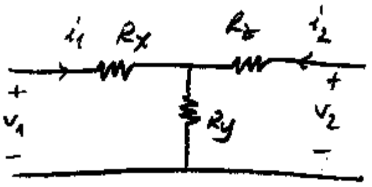
Hyprid Parameters (I)

$$\begin{bmatrix} V_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ V_2 \end{bmatrix}$$

Transfer Parameters (ABCD Parameters)(Chain Parameters)

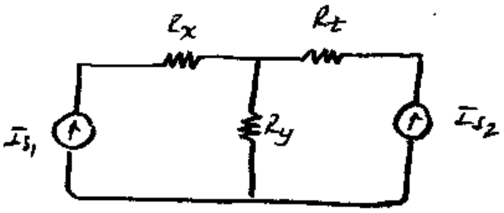
$$\begin{bmatrix} V_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \underbrace{\begin{bmatrix} V_2 \\ -i_2 \end{bmatrix}}_{\text{not } i_2 \text{ but } (-i_2)}$$

Ex



Find R parameters

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{22} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$



$$V_1 = R_x \cdot I_{s_1} + R_y (I_{s_1} + I_{s_2})$$

$$= \begin{bmatrix} R_x + R_y & R_y \end{bmatrix} \cdot \begin{bmatrix} I_{s_1} \\ I_{s_2} \end{bmatrix}$$

$$V_2 = R_z \cdot I_{s_2} + R_y (I_{s_1} + I_{s_2})$$

$$= \begin{bmatrix} R_y & R_y + R_z \end{bmatrix} \begin{bmatrix} I_{s_1} \\ I_{s_2} \end{bmatrix}$$

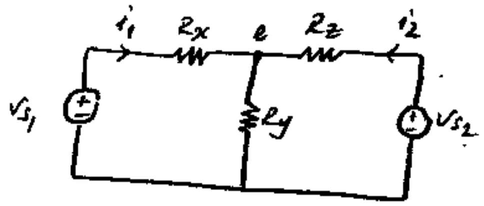
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_x + R_y & R_y \\ R_y & R_y + R_z \end{bmatrix} \begin{bmatrix} I_{s_1} \rightarrow i_1 \\ I_{s_2} \rightarrow i_2 \end{bmatrix}$$

Find G parameters

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = G \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \longrightarrow \underline{\underline{G}} = \underline{\underline{R}}^{-1}$$

$$G = \begin{bmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{bmatrix} \cdot (R_{11} R_{22} - R_{12} R_{21})^{-1}$$

G parameters using analysis.



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\frac{e - V_{s_1}}{R_x} + \frac{e}{R_y} + \frac{e - V_{s_2}}{R_z} = 0$$

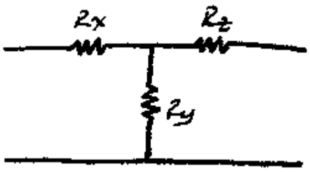
$$i_1 = \frac{V_{s_1}}{R_x} - \frac{e}{R_x}$$

$$\left(\frac{1}{R_x} + \frac{1}{R_y} + \frac{1}{R_z} \right) e = \frac{V_{s_1}}{R_x} + \frac{V_{s_2}}{R_z}$$

$$i_1 = \underbrace{\left(\frac{1}{R_x} - \frac{R_y R_z}{R \cdot R_x} \right)}_{g_{11}} V_{s_1} - \underbrace{\frac{R_y}{R}}_{g_{12}} V_{s_2}$$

$$e = \frac{R_y R_z V_{s_1} + R_x R_y V_{s_2}}{R_x R_y + R_y R_z + R_x R_z \rightarrow R}$$

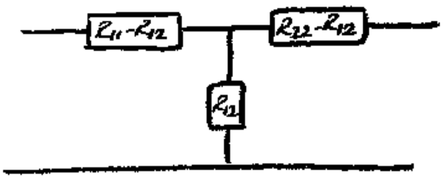
T-network



$R_{11} = R_x + R_y$

$R_{12} = R_{21} = R_y$

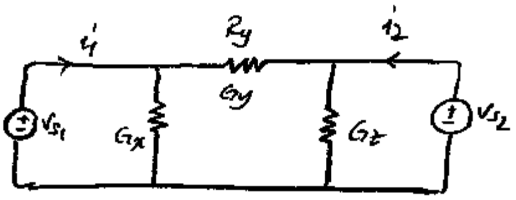
$R_{22} = R_y + R_z$



A given R parameters can be used to form T-network.

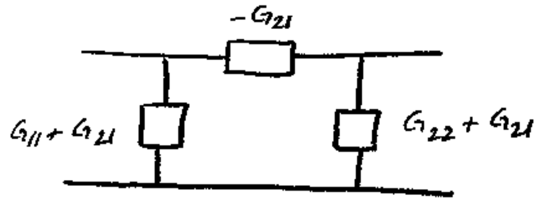
G Parameters

π network



$$i_2 = Vs_2 G_z + (Vs_2 - Vs_1) G_y$$

$$= \underset{g_{21}}{\underset{\downarrow}{-G_y \cdot Vs_1}} + \underset{g_{22}}{\underset{\downarrow}{(G_y + G_z) Vs_2}}$$



More On Two Ports.

① Symmetry:

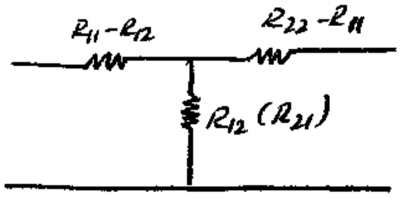
Does R, G have to be symmetric ?

Symmetry occurs if we do not have dependent sources

(only resistances → symmetry)

(inductor, capacitor, RLC → Symmetry)

$R : (R_{11}, R_{12}, R_{21}, R_{22})$; $R_{12} = R_{21}$

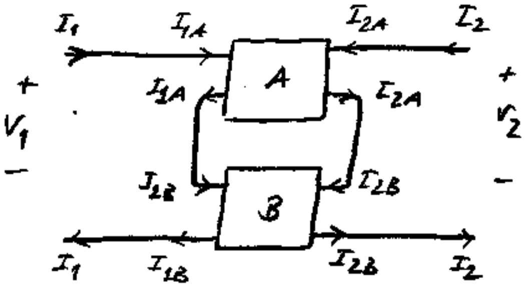


$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

when ① $i_1 = 0$: $i_2 = I_s \longrightarrow (V_1, V_2) = (R_{12} I_s, R_{22} I_s)$

② $i_1 = I_s$; $i_2 = 0 \longrightarrow (V_1, V_2) = (R_{11} I_s, R_{21} I_s)$

② Interconnection of Two Ports



Series form

R parameters: R_A, R_B combined

$$\underline{R} = \underline{R_A} + \underline{R_B}$$

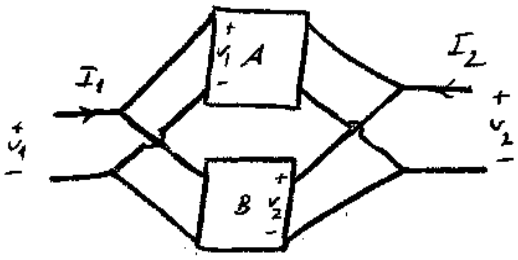
$V_1 = V_{1A} + V_{1B}$

$V_2 = V_{2A} + V_{2B}$

$I_{2A} = I_{2B} = I_2$

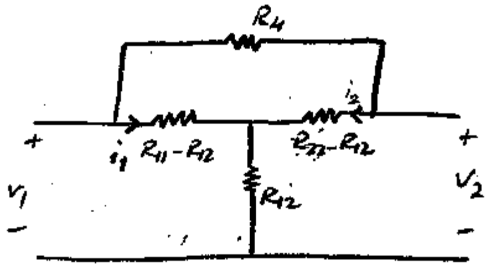
$I_{1A} = I_{2B} = I_1$

Parallel connection of Two-Ports

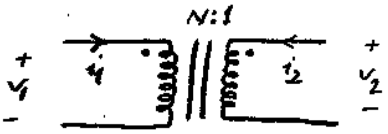


$$G_{combined} = G_1 + G_2$$

③ Two Ports with Feedback Connection



④ Transformers:



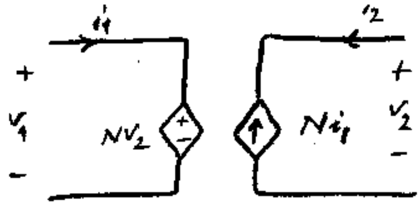
N:1 (turns ratio)

$$\frac{V_1}{V_2} = N$$

• indicates + part.

$$\frac{i_1}{i_2} = -\frac{1}{N}$$

Ideal transformer:



Note: Physically, transformers can only operate with AC input. BUT, in EE201, we use transformers in DC circuit analysis to practice analysis with transformers.

$P_1 = v_1 \cdot i_1$ $\quad P_1 + P_2 = 0$

$P_2 = v_2 \cdot i_2$

Two Ports (continued)

Hybrid I:

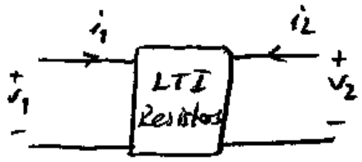
$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = H \cdot \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

Hybrid II:

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = H^{-1} \cdot \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

On parameter calculation

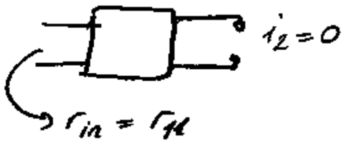
R parameters:



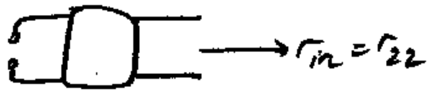
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$V_1 = r_{11} i_1 + r_{12} i_2$$
$$V_2 = r_{21} i_1 + r_{22} i_2$$

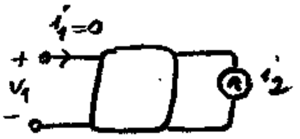
$$r_{11} = \frac{V_1}{i_1}\bigg|_{i_2=0}$$



$$r_{22} = \frac{V_2}{i_2}\bigg|_{i_1=0}$$

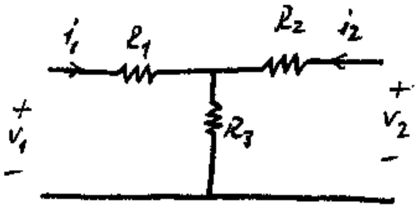


$$r_{12} = \frac{V_1}{i_2}\bigg|_{i_1=0}$$



$$r_{21} = \frac{V_2}{i_1}\bigg|_{i_2=0}$$

EX



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$r_{11} = \frac{V_1}{i_1}\bigg|_{i_2=0} = R_1 + R_3$$

$$r_{22} = \frac{V_2}{i_2}\bigg|_{i_1=0} = R_2 + R_3$$

$$r_{12} = \frac{V_1}{i_2}\bigg|_{i_1=0} = R_3$$

$$r_{21} = \frac{V_2}{i_1}\bigg|_{i_2=0} = R_3$$

Note: $\underline{R}$ is symmetric ($r_{12} = r_{21}$) as expected since the network consists of only resistances.

EX Express $\underline{H}$ parameters in terms of $\underline{R}$ parameters.

$$V_1 = r_{11} i_1 + r_{12} i_2 \quad (1)$$
$$V_2 = r_{21} i_1 + r_{22} i_2 \quad (2)$$

$$V_1 = h_{11} i_1 + h_{12} V_2$$
$$i_2 = h_{21} i_1 + h_{22} V_2$$

$$h_{11} = \frac{v_1}{i_1}\Big\downarrow_{v_2=0}$$

From (2) $\longrightarrow$ $i_2 = -\frac{r_{21}}{r_{22}}\, i_1$

Substitute in (1) $\longrightarrow$ $v_1 = r_{11}\cdot i_1 + r_{12}\cdot\left(-\frac{r_{21}}{r_{22}}\right) i_1$

$$h_{11} = r_{11} - \frac{r_{12}\cdot r_{21}}{r_{22}}$$

$$h_{11} = \frac{r_{11}\cdot r_{22} - r_{12}\cdot r_{21}}{r_{22}} = \frac{|\underline{\underline{R}}|}{r_{22}}$$

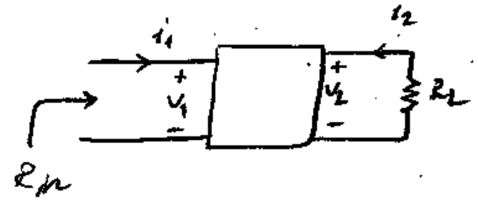
$$h_{21} = \frac{i_2}{i_1}\Big\downarrow_{v_2=0} = -\frac{r_{21}}{r_{22}}$$

$$h_{22} = \frac{i_2}{v_2}\Big\downarrow_{i_1=0} = \frac{1}{r_{22}} \quad [\text{from } (2)]$$

$$h_{12} = \frac{v_1}{v_2}\Big\downarrow_{i_1=0} = \frac{r_{12}}{r_{22}}$$

Note: If $\underline{\underline{R}}$ matrix is symmetric $(r_{12} = r_{21}) \longrightarrow$
$\underline{\underline{H}}$ has the property $h_{12} = -h_{21}$

Terminated Two Ports



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$R_{in} = \frac{v_1}{i_1}$

2 equations from 2-port component
1 equation from load

$$v_2 = -R_L\cdot i_2$$

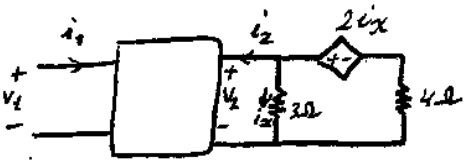
$$(2) \rightarrow \; -R_L\cdot i_2 = r_{21}\, i_1 + r_{22}\, i_2 \;\rightarrow\; i_2 = \frac{-r_{21}}{r_{22}+R_L}\, i_1$$

$$v_1 = r_{11}\cdot i_1 + r_{12}\cdot \underset{\hookrightarrow\ \frac{-r_{21}}{r_{22}+R_L} i_1}{i_2}$$

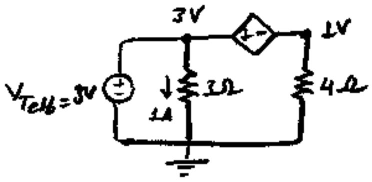
$$\frac{v_1}{i_1} = r_{11} - \frac{r_{12}\cdot r_{21}}{r_{22}+R_L}$$

$$R_{in} = \frac{r_{11}\cdot r_{22} - r_{12}\cdot r_{21} + r_{11}\cdot R_L}{r_{22}+R_L}$$

Ex Find R {ZPS-II}



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$



$$i_t = \frac{3}{3} + \frac{1}{4} = \frac{5}{4} A$$

$$R_{th} = \frac{V_{test}}{i_{test}} = \frac{3V}{5/4\ A} = \frac{12}{5} \Omega = 2{,}4\Omega$$

R = ?

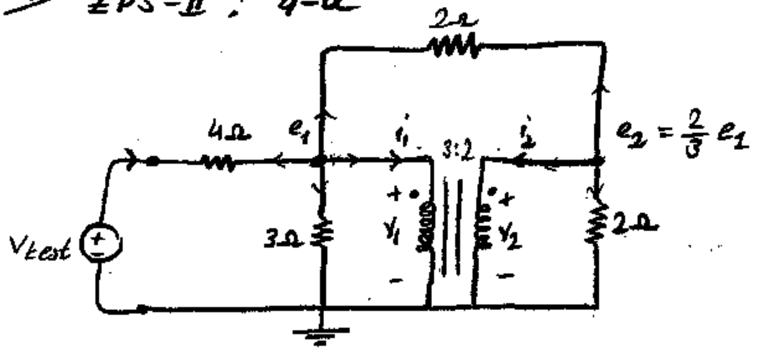
$$\left.\begin{aligned} v_2 &= -2{,}4\, i_2 \quad \text{(Load equation)} \\ v_2 &= 2i_1 + 4i_2 \end{aligned}\right\} i_1 = -3{,}2\, i_2$$

$$(1) \quad v_1 = 3i_1 + 2i_2 = 3i_1 + 2 \cdot \frac{-5}{16} i_1 = \frac{19}{8} i_1$$

$$R = \frac{v_1}{i_1} = \frac{19/8\ i_1}{i_1} .$$

$$\boxed{R = 2.375\Omega}$$

Ex ZPS-II ; 4-a



Reminder

For a n:1 transformer

$$\frac{V_1}{V_2} = \frac{n}{1}$$

$$\frac{i_1}{i_2} = -\frac{1}{n}$$

$$\text{KCL at } e_1 : \quad \frac{e_1}{3} + \frac{e_1 - V_t}{4} + \frac{e_1 - \frac{2}{3}e_1}{2} + i_1 = 0$$

$$\text{KCL at } e_2 : \quad \frac{\frac{2}{3}e_1}{2} + \frac{\frac{2}{3}e_1 - e_1}{2} + i_2 = 0 \quad \rightarrow i_2 = -\frac{3}{2} i_1$$

$$3e_1 + 4i_1 = V_t$$

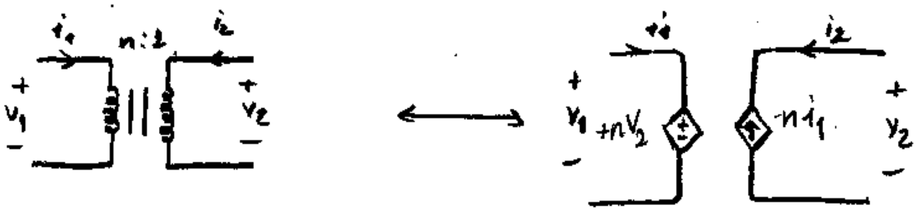
$$e_1 - 9i_1 = 0$$

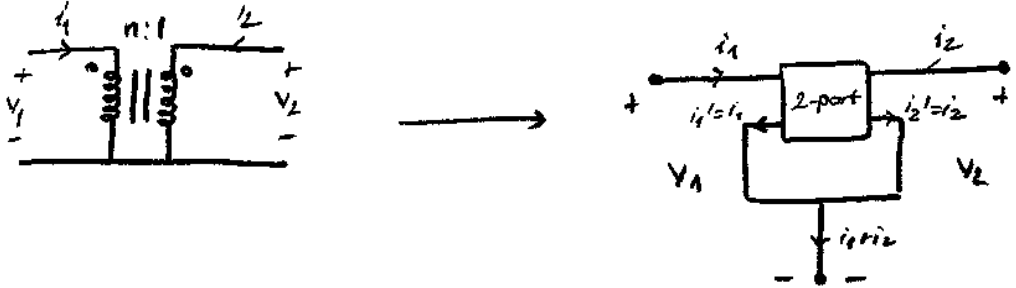
$$\begin{bmatrix} 3 & 4 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} V_t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \frac{-1}{31} \begin{bmatrix} -9 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} V_t \\ 0 \end{bmatrix} = -\frac{1}{31} \begin{bmatrix} -9V_t \\ -V_t \end{bmatrix} = \begin{bmatrix} \frac{9}{31} V_t \\ \frac{1}{31} V_t \end{bmatrix}$$

$$i_{test} = \frac{V_{test} - e_1}{4} = \frac{(1 - 9/31)V_t}{4} = \frac{11}{62} V_{test}$$

$$\boxed{R = \frac{V_{test}}{i_{test}} = \frac{62}{11} \Omega}$$





3-Terminal 2-port!

Then for 3 terminal 2 ports

$i_1' = i_1$ & $i_2' = i_2$

does not necessarily hold. You don't need to check this.

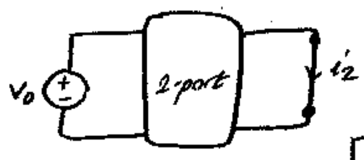
Reciprocity

① $\underline{\underline{R}}$, $\underline{\underline{G}}$, $\underline{\underline{H}}$ parameters have same special properties (symmetry or anti-symmetry (for $\underline{\underline{H}}$)) when 2-port in question does include only resistors, (inductors, capacitors)

Ex $r_{12} = r_{21}$, $g_{12} = g_{21}$, $h_{12} = -h_{21}$

② Reciprocity makes use of observation ① and is also used in measurements in experiments.

①



(g_{21})

$\boxed{i_1 = i_2}$

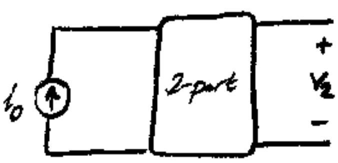
(g_{12})

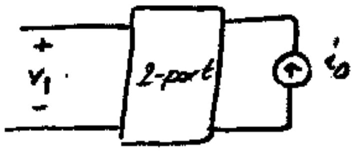
Check!

without dependent sources, symmetry is ensured.

$i_2 = g_{21} v_1 + g_{22} v_2$

②





(r_{21})

(r_{12})

$\boxed{v_1 = v_2}$

$v_2 = r_{21} i_1 + r_{22} i_2$

③

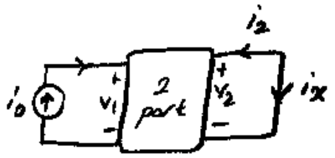
Check 2PS-II
for reciprocity

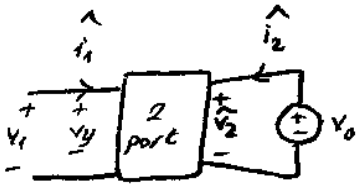
h_{21}

h_{12}

Sources: $i_0(t) = V_0(t) \longrightarrow i_2 = V_1$ (note that i_2 is reversed)

Proof of reciprocity:
③





we would like to prove $i_x = V_y$

$$\sum_{k=1}^{N} V_k \cdot \hat{i}_k = \sum_{k=1}^{N} \hat{v}_k \, i_k = 0 \quad \text{(Tellegen's Theorem)}$$

N: # of branches

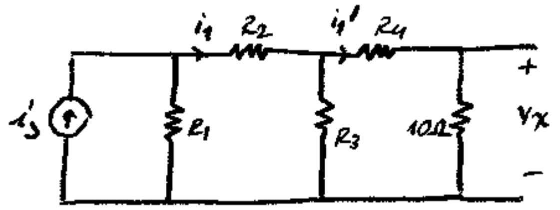
$$\underset{\swarrow \, 0}{V_1 \cdot (-\hat{i}_1)} + \underset{\swarrow \, 0}{V_2(-\hat{i}_2)} + \cancel{\sum_{k=3}^{N} R_k i_k \hat{i}_k} = \hat{V}_1 \underset{\swarrow \, -i_0}{(-i_1)} + \hat{V}_2 \underset{\swarrow \, V_0}{(-i_2)} + \cancel{\sum_{k=3}^{N} R_k \hat{i}_k i_k}$$

$$0 = \hat{V}_1 \cdot (-i_0) + V_0(-i_2)$$

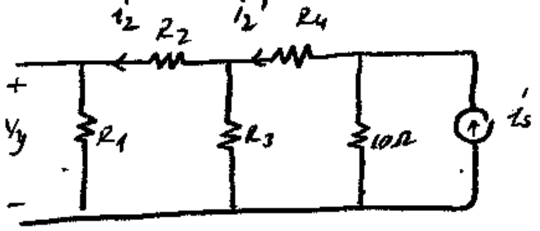
$$0 = \hat{V}_1 + i_2 \quad [\, i_0 = V_0 \,]$$

$$\hat{V}_1 = -i_2 \longrightarrow i_x = V_y$$

Ex Two sets of measurements are made:



$i_1 = 0{,}6\, i_s$
$i_1' = 0{,}3\, i_s$



$i_2 = 0{,}2\, i_s$
$i_2' = 0{,}5\, i_s$

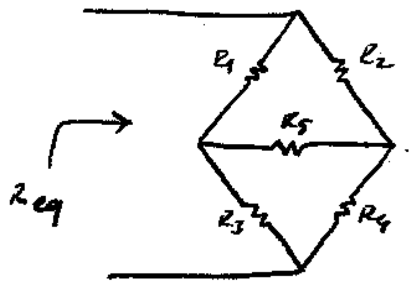
From Reciprocity

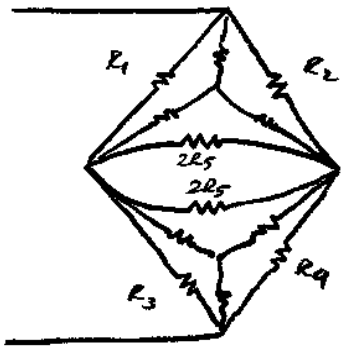
$$V_x = V_y$$

$$i_1' \cdot 10\Omega = i_2 \cdot R_1$$

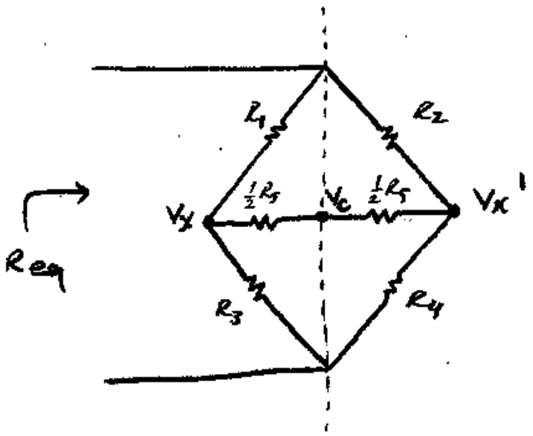
$$0{,}3\, i_s \cdot 10\Omega = 0{,}2\, i_s \cdot R_1 \longrightarrow \boxed{R_1 = 15\Omega}$$

Symmetric Circuits





① $R_1 = R_2$ & $R_3 = R_4$

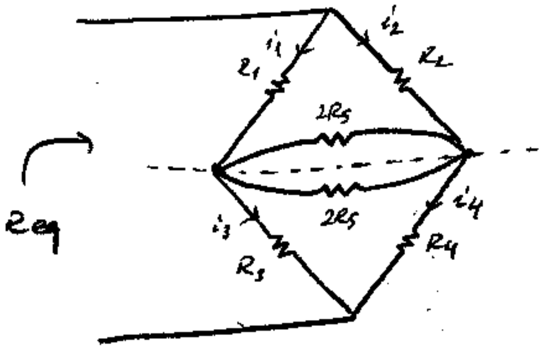


$V_x = V_x'$

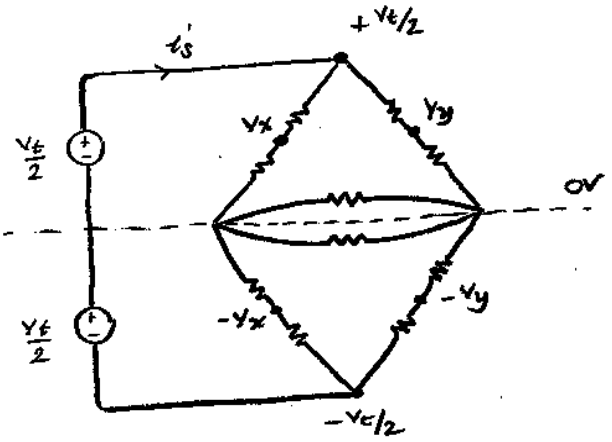
write the node analysis for V_x, V_x' & V_c

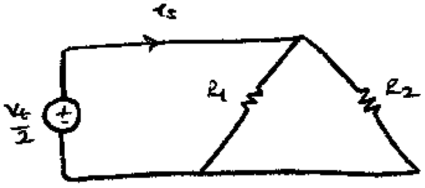
If you exchange $V_x \leftrightarrow V_x'$; you have the same equation system.

② $R_1 = R_3$ & $R_2 = R_4$



$i_1 = i_3$ & $i_2 = i_4$

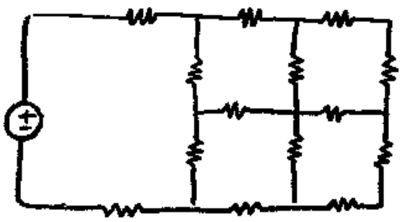




$$i_s = \frac{V_t}{2(R_1 \| R_2)}$$

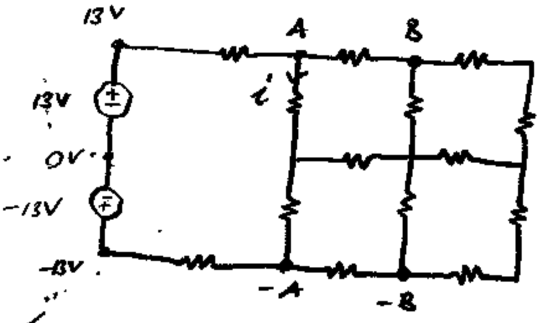
$$R_{eq} = \frac{V_t}{i_s} = 2(R_1 \| R_2)$$

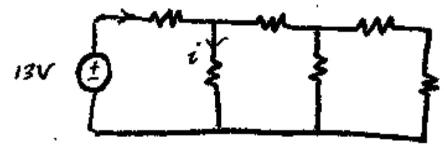
Ex//



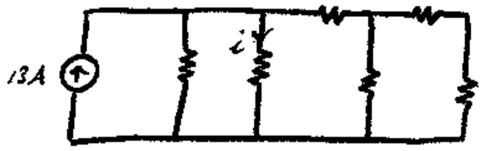
all resistances are 1kΩ

→



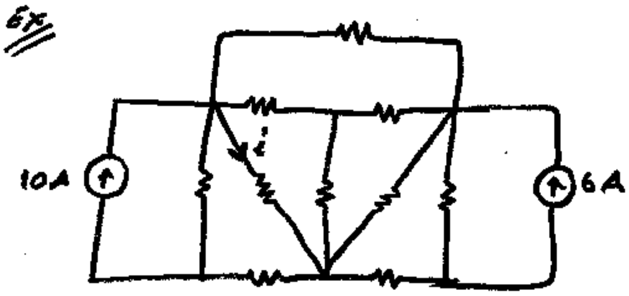


→



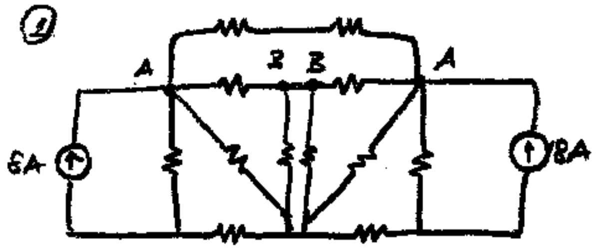
→ $i = 5A$

Ex//



All resistances 1Ω

①



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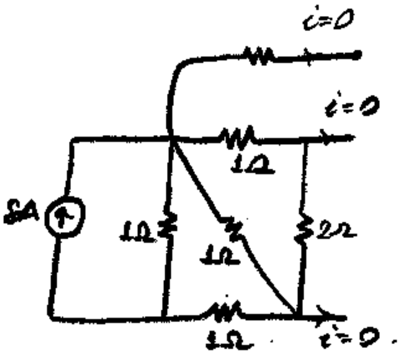

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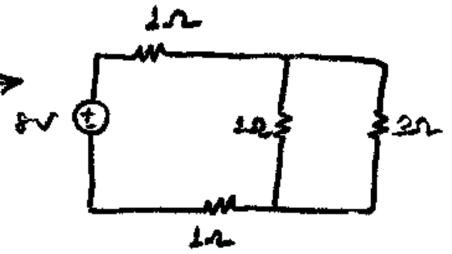

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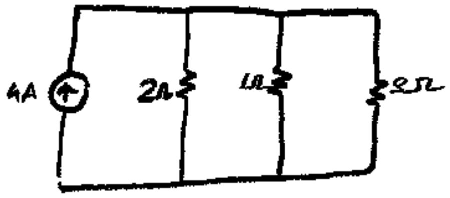
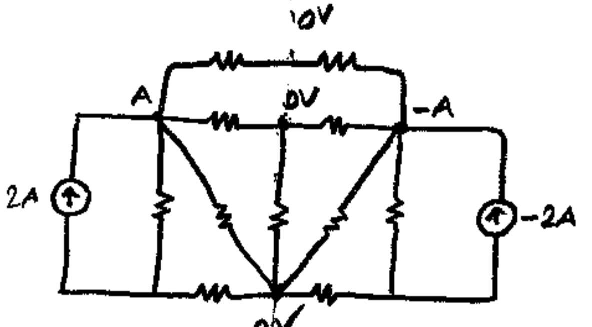


$$i_1 = 4 \cdot \frac{1}{\frac{1}{2}+\frac{1}{1}+\frac{1}{3}} = \frac{24}{11}$$

②



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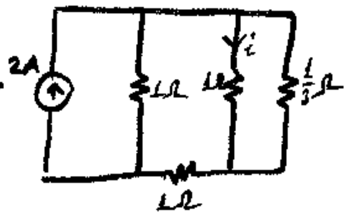

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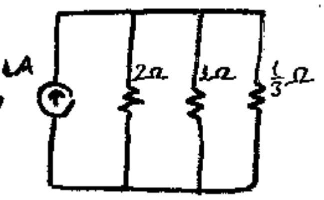


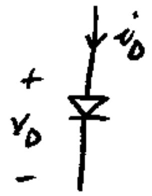
$$i_2 = 1 \cdot \frac{1}{\frac{1}{2}+1+3} = \frac{2}{9}$$

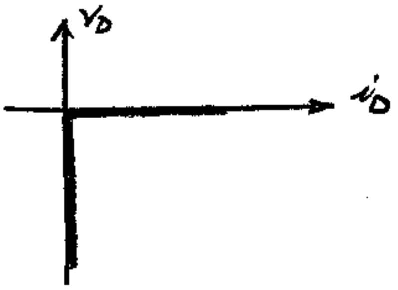
$$i = i_1 + i_2 = \frac{24}{11} + \frac{2}{9} = \frac{238}{99}$$

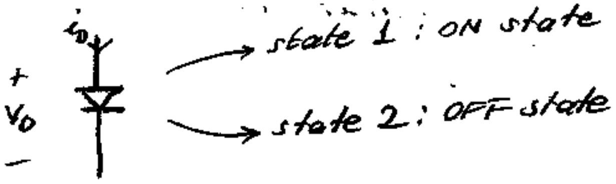
Diodes (Nonlinear Resistors)

→ current controlled component $(v = f(i))$

→ voltage controlled component $(i = g(v))$







ON (conduction) state :

$i_D \geq 0$ (diode conducts)

$V_D = 0$ (short circuit)

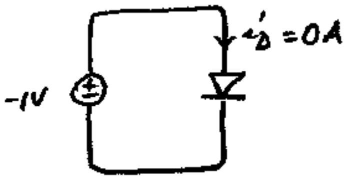
OFF state :

$V_D < 0 \longrightarrow i_D = 0$

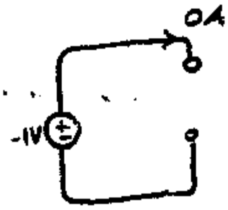
Open-circuit

No current conduction

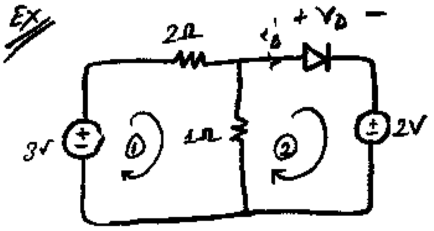
$V_D < 0$



≡



Ex



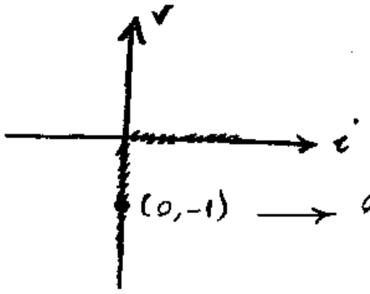
KVL around mesh ②

$+2 + (i_2 - i_1) \cdot 1 + f(i_2) = 0$

$-3 + 2i_1 + (i_1 - i_2) \cdot 1 = 0$

Assume diode is off ⟶ $V_D < 0,\ i_2 = 0$

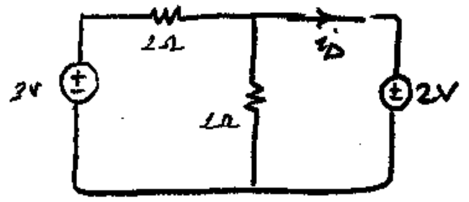
$V_D = 1V - 2V = -1V < 0$ ⇒ assumption holds , analysis is complete



I-V characteristic of diode

→ operation point of diode

Assume otherwise, diode is ON ($v_D = 0$, $i_D \geq 0$)



$$i_D = \frac{3}{2} - \frac{2}{\frac{2}{3}} = -1,5A < 0$$

Assumption does not hold.

Bilateral Components

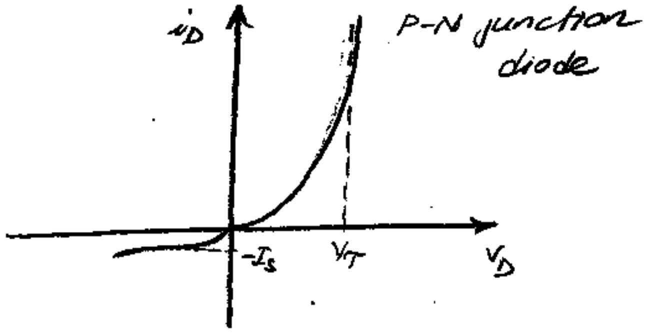
LTI resistors are bilateral component, in other words, the way you connect the component in between two nodes does not matter.

Diodes are **unilateral** components

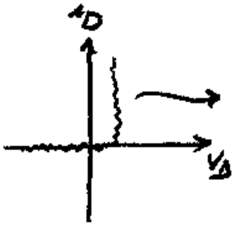
Components have symmetry across the origin in their I-V characteristics

Practical Diodes

$i_D = f(v_D)$

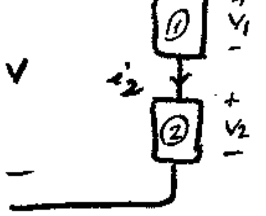


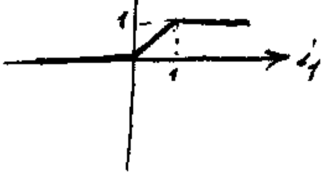
$$i_D = i_s\left(e^{V_T/V_D} - 1\right)$$

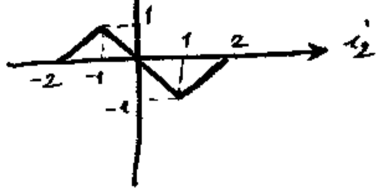


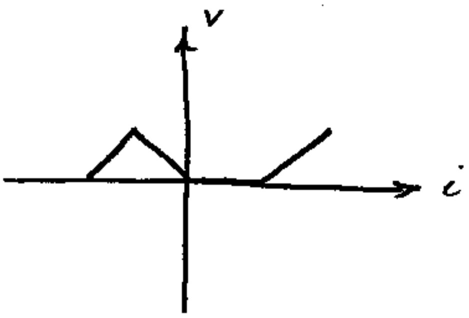
→ 0th order approximation

Series and Parallel Combination of Ideal Diodes (Graphical Solutions)



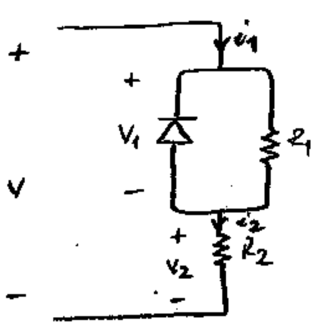


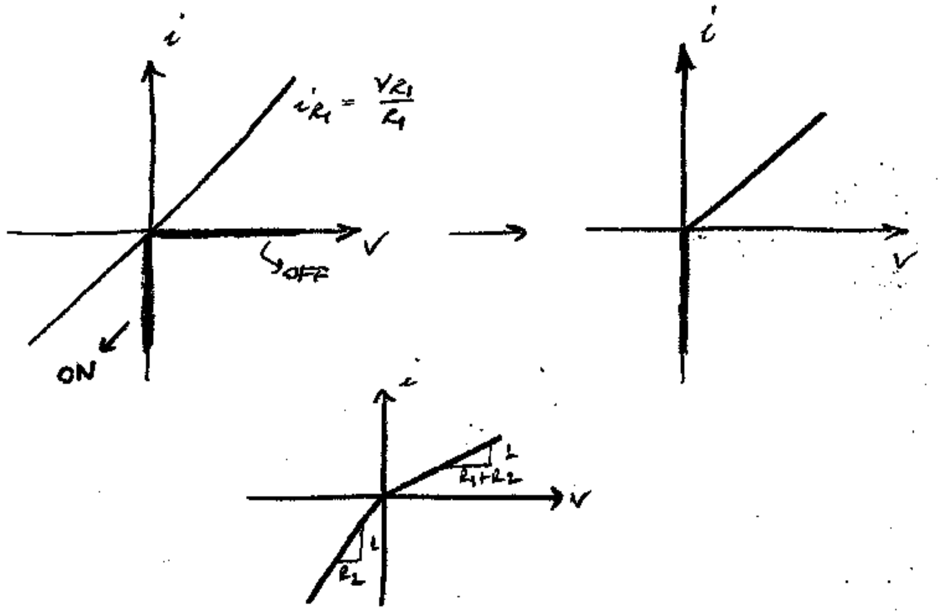




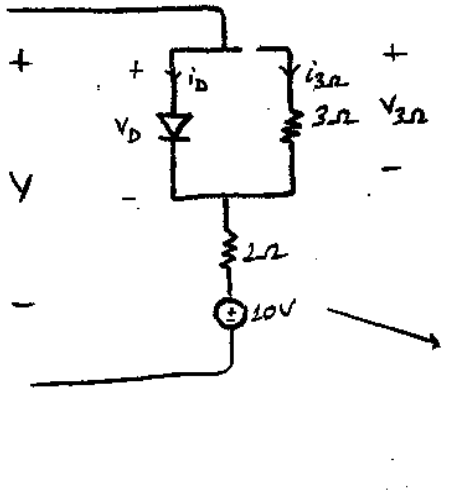
Diodes: Series & Parallel combination

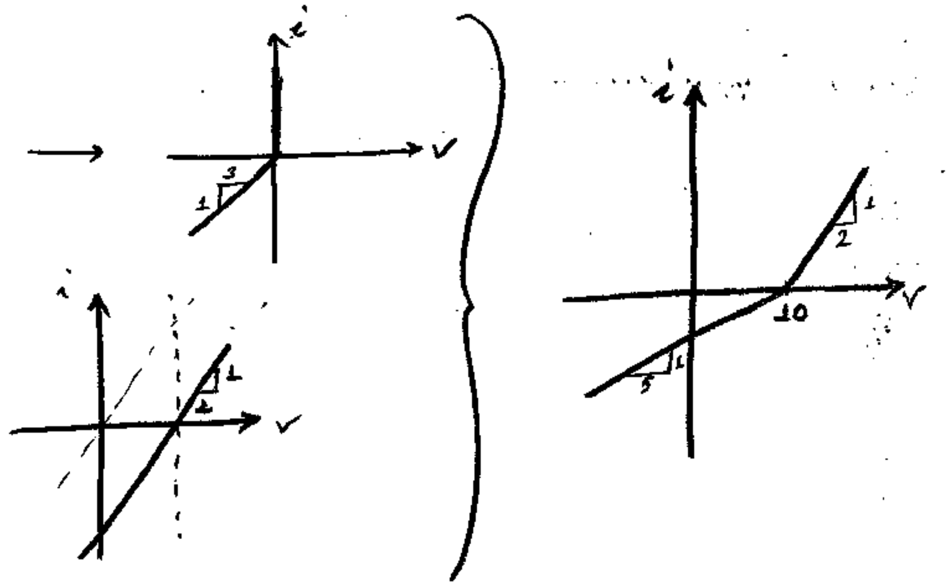
Analysis:





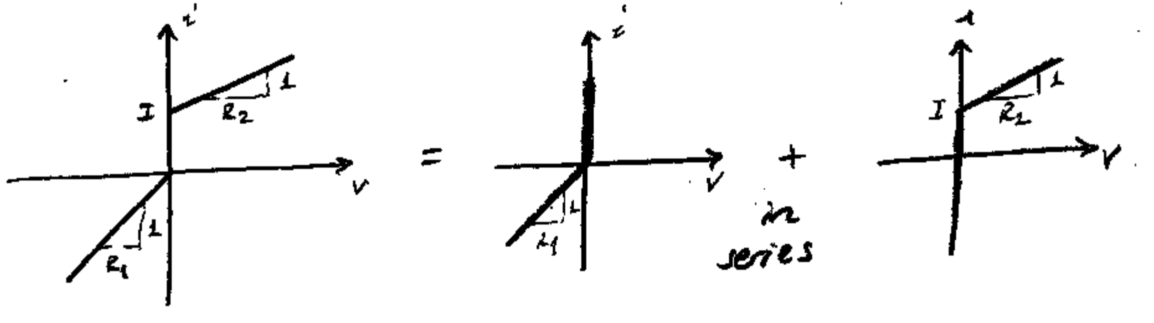
Ex

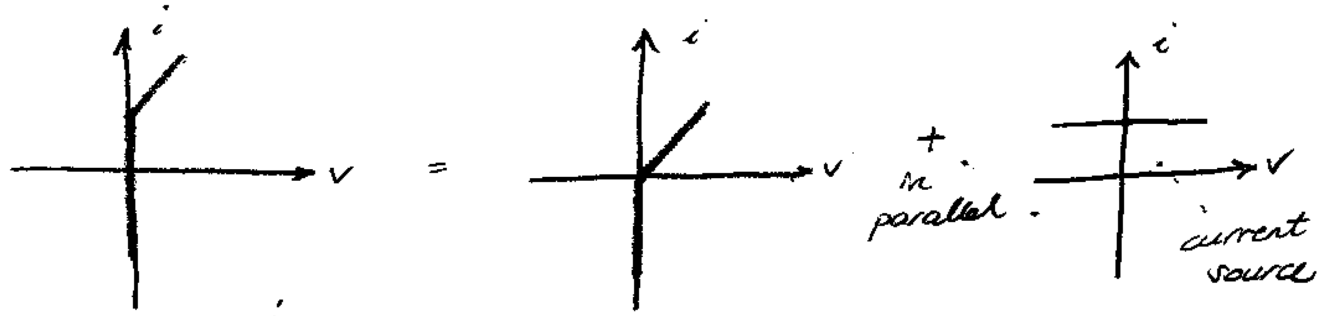


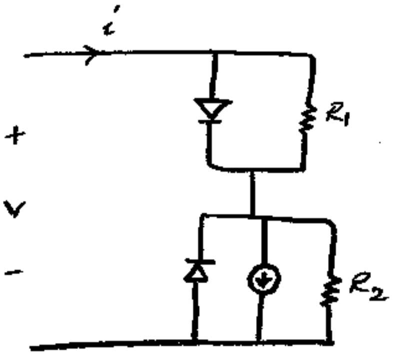


Synthesis:

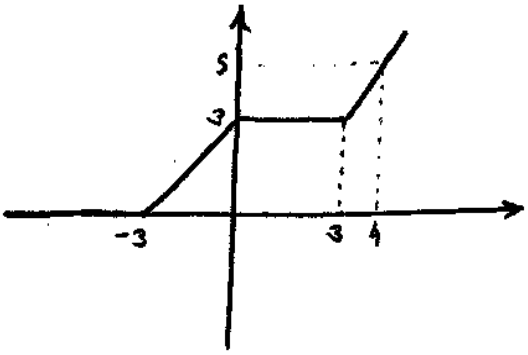
Design a one-port with R, voltage source, current source and diodes such that

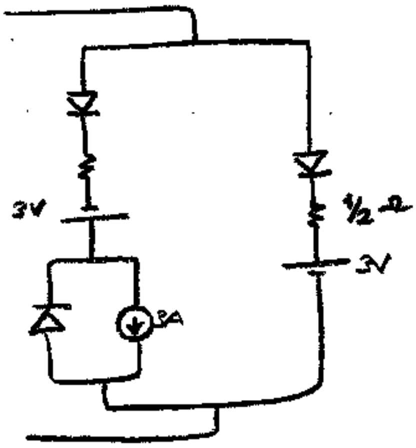




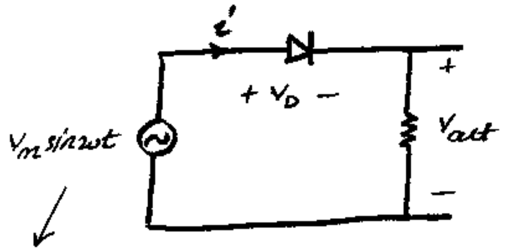


Ex



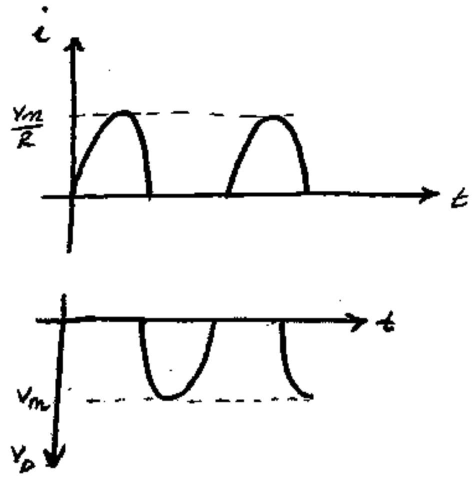


Diode Applications

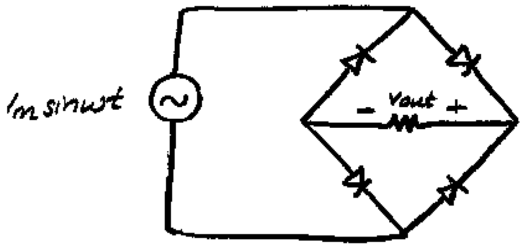


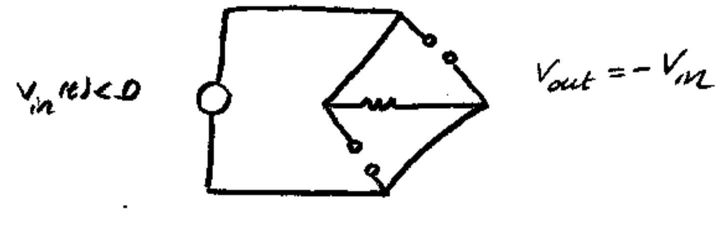
↓

V_{input} Half wave rectifier



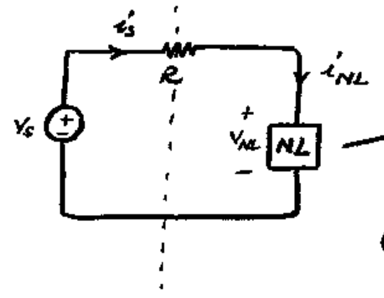
Full-wave rectifier





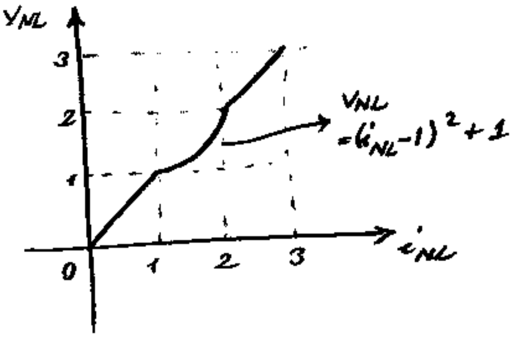
Circuits with a Single NonLinear Element

Load Line:

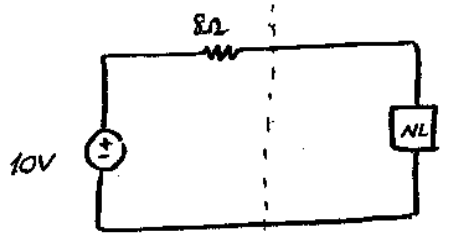


→ NonLinear memoriless component (nonlinear resistor)

$$-V_s + R \cdot i_{NL} + V_{NL}(i_{NL}) = 0$$



$V_{NL} = i_{NL}$, $i_{NL} < 1$

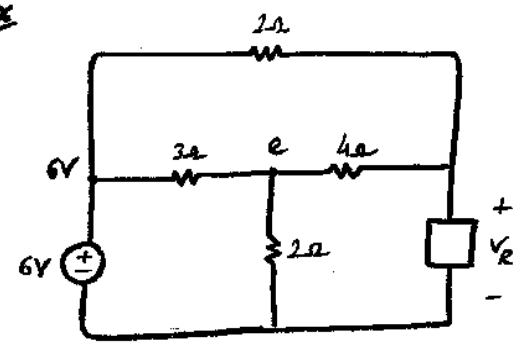


Left: Right

$0 = -10 + 8 i_{NL} + V_{NL}$ $V_{NL} = f(i_{NL})$ $V_{NL} = \frac{10}{9} V$

$V_{NL} = 10 - 8 i_{NL}$

Ex



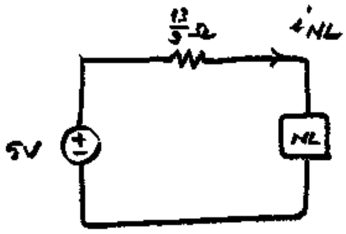
$$i_R = \begin{cases} 0{,}03\, V_R^2 & , V_R \geq 0 \\ 0 & , V_R < 0 \end{cases}$$

$$R_{Th} = [(2 \parallel 3) + 4] \parallel 2 = \frac{13}{9} \Omega$$

$V_{oc} = ?$

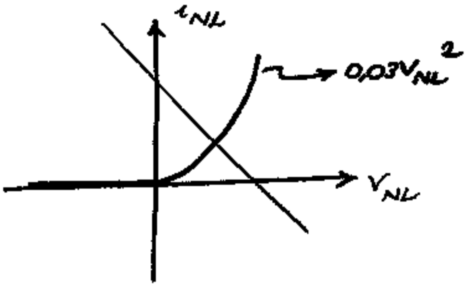
$$\frac{e-6}{6} + \frac{e-6}{3} + \frac{e}{2} = 0 \longrightarrow e = 3V$$

$$\frac{V_{oc}-6}{2} + \frac{V_{oc}-3}{4} = 0 \longrightarrow V_{oc} = 5V$$



$$-5 + \frac{13}{9} i_{NL} + V_{NL} = 0$$

$$i_{NL} = \frac{45}{13} - \frac{9}{13} V_{NL}$$



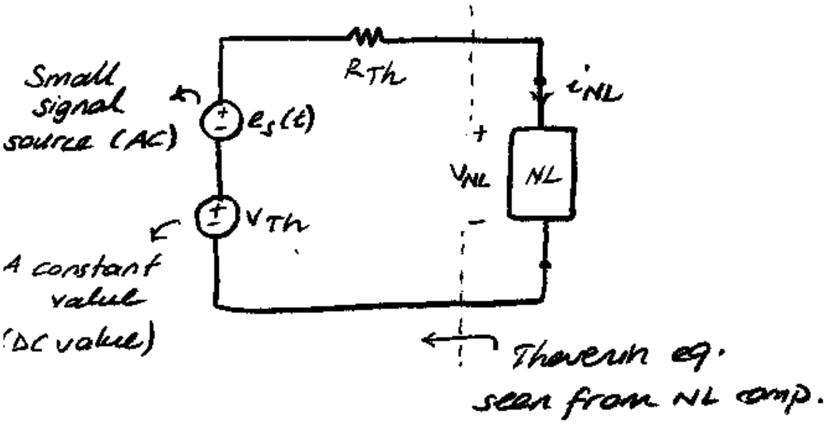
$$i_{NL} = \frac{45}{13} - \frac{9}{13} V_{NL} = 0{,}03\, V_{NL}^2$$

$$V_{NL} = \{4{,}22 , \cancel{-27{,}29}\}$$

$$\boxed{V_{NL} = 4{,}22\, V}$$

Small Signal Analysis

Let's assume that we have a circuit with a single non-linear element. The Thevenin equivalent of this circuit is:



$$-(V_{Th}+e_s(t)) + R\cdot \underset{\swarrow \atop g(V_{NL})}{i_{NL}} + V_{NL}=0$$

$$-(V_{Th}+e_s(t)) + R\cdot g(V_{NL}) + V_{NL}=0$$

$\downarrow$

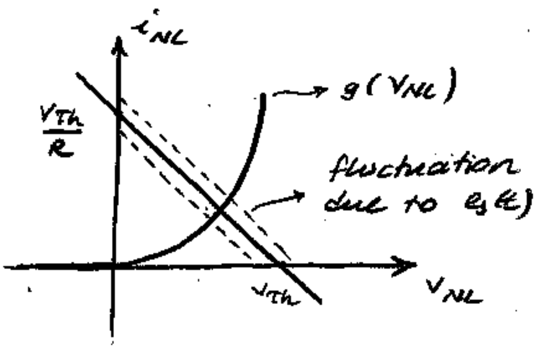
KVL should be satisfied for all t.

due to small signal assumption: $e_s(t)$ is negligible compared to V_{th}.

$$-V_{Th} + R\cdot g(V_{NL}) + V_{NL} = 0$$

using load lines, solution can be found.

$$\frac{V_{Th}-V_{NL}}{R} = g(V_{NL}) = i_{NL}$$



Then, the DC operating point found from "load line" analysis, gives us the solution when $e_s(t)=0$.
since $e_s(t)$ is small, we can expand the $i_{NL}=g(V_{NL})$,
$g(x)$ is function around the operating point using Taylor series;

$$g(x) = g(x_0) + \frac{g'(x_0)}{1!}(x-x_0) + \frac{g''(x_0)}{2!}(x-x_0)^2 + \dots$$

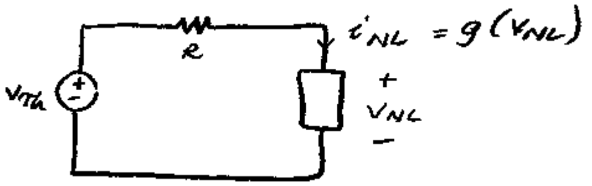
Ex/ $g(x)=x^2$ $\longrightarrow$ $g'(x_0)$ gives us the slope of curve. Then the slope can be interpreted as an LTI resistor. That is,

$$\underset{\downarrow}{g_{NL}^{small}}(v) = g'(x_0)$$

Small signal model for NL element.

Summarizing:

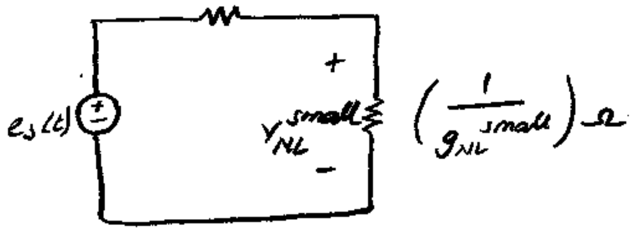
① Find DC operating point (using load line)



② Find small signal conductance parameters

$$g_{NL}^{small} = g'(v_{NL}^{DC}) \rightarrow \text{DC operating point}$$

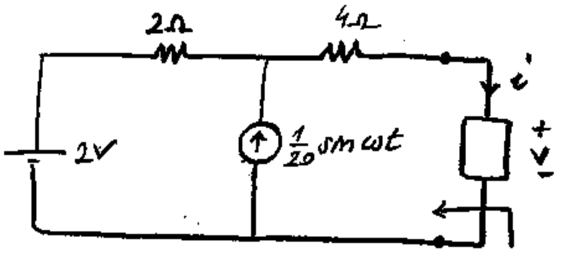
③



Find $v_{NL}^{small}(t)$.

④ Find solution $v_{NL}(t) = v_{NL}^{DC} + v_{NL}^{small}(t)$.

Ex

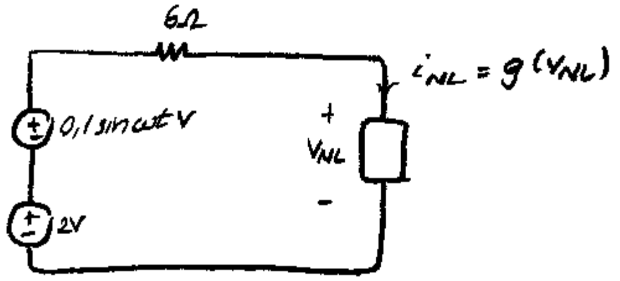


$$i = \begin{cases} v^2, & v>0 \\ 0, & v<0 \end{cases}$$

Thevenin eq: $R_{Th} = 4+2 = 6\Omega$

$V_{oc} = 2 + 0{,}1\sin\omega t$ V

①

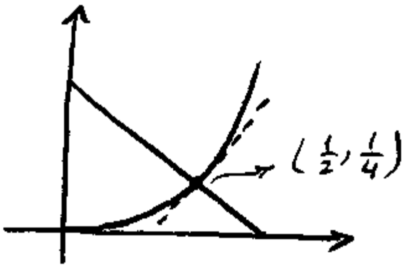


DC operating point:

$$-2 + 6\, i_{NL} + v_{NL} = 0$$

$$-2 + 6\, v_{NL}^2 + v_{NL} = 0$$

$$v_{NL} = \left\{ \frac{1}{2}, \frac{-2}{3} \right\} \Rightarrow \boxed{v_{NL} = \frac{1}{2}V}$$



DC operating point:

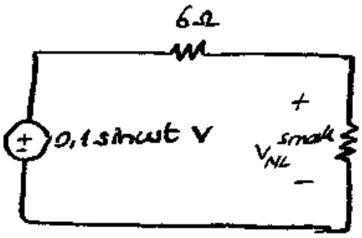
$$(v_{NL}^{DC}, i_{NL}^{DC}) = \left(\frac{1}{2}, \frac{1}{4}\right)$$

② $i_{NL}' = V_{NL}^2 = g(V_{NL})$

$$i_{NL}' \cong \frac{1}{4} + \frac{g'(1/2)}{1!}\left(V_{NL} - \frac{1}{2}\right) + \dots$$

$$i_{NL}' \cong \frac{1}{4} + \left(V_{NL} - \frac{1}{2}\right) + \dots$$

③

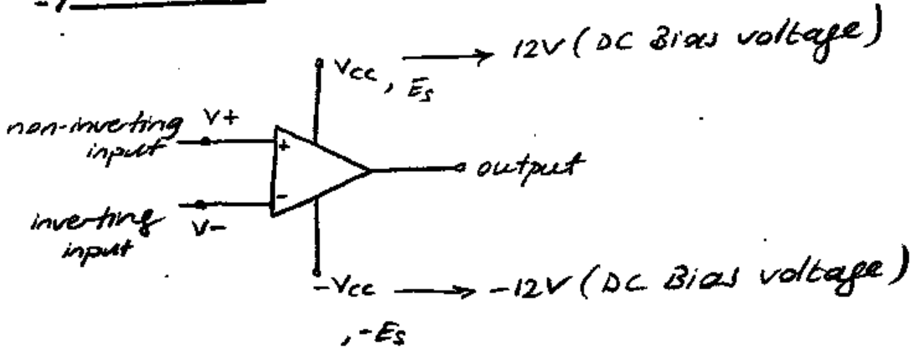


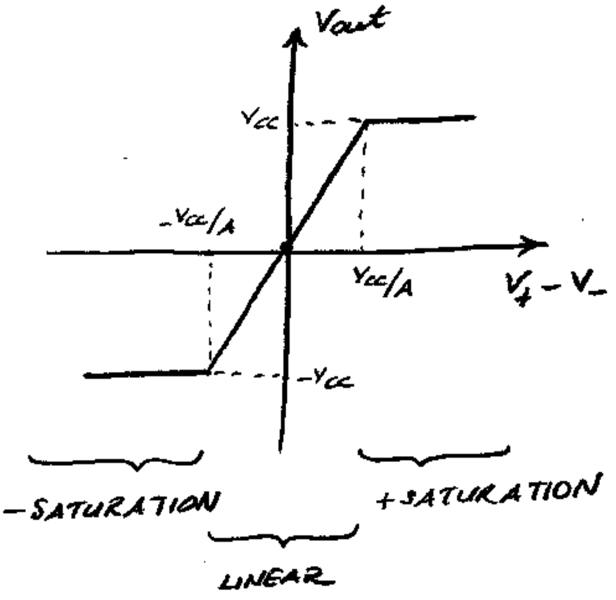
$\frac{1}{g_{NL}^{small}} = 1\Omega \rightarrow$ slope at $\left(\frac{1}{2}, \frac{1}{4}\right) = 1$

$$V_{NL}^{small}(t) = \frac{0,1\sin\omega t}{7}$$

④ $V_{NL}(t) = \underset{\substack{\downarrow \\ DC}}{\frac{1}{2}} + \underset{\substack{\downarrow \\ AC}}{\frac{1}{70}\sin\omega t}$

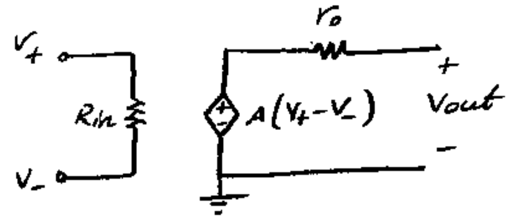
Operational Amplifiers





A : open loop gain
(typically $10^5 - 10^8$)

A little more detailed model



R_{in} : input resistance ~ $(10^6 - 10^{12}\,\Omega)$

r_o : output resistance ~ $(10 - 100\,\Omega)$

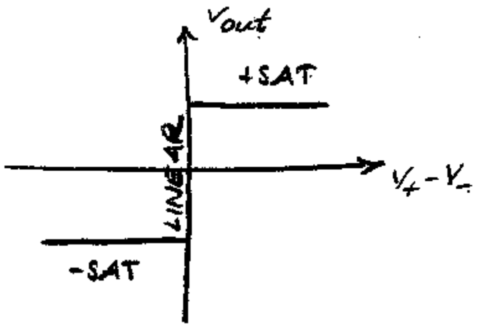
↓

Model for only linear Region.

+SAT Region : $V_{out} = +E_s\ (+V_{cc})$ or $(V_+ - V_-) \geq \frac{E_{SAT}}{A}$

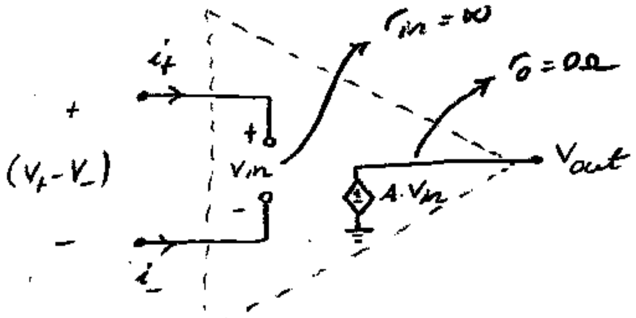
−SAT Region : $V_{out} = -E_s\ (-V_{cc})$ or $(V_+ - V_-) \leq \frac{-E_{SAT}}{A}$

Ideal Op-Amp :



$R_{in} = \infty$, $R_{out} = 0$ (Linear Region)

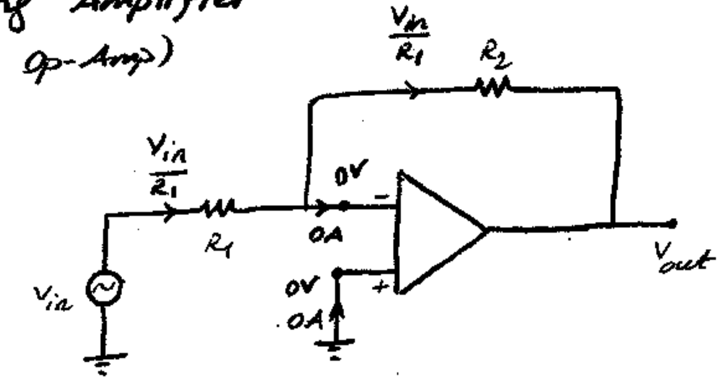
Using ideal Op-Amp assumptions we have in linear region :



① $i_+ = i_- = 0A$ → linear region when $A = \infty$, $i_+ = i_- = 0A$

② $V_+ = V_-$ (for ideal op-amp in linear region)

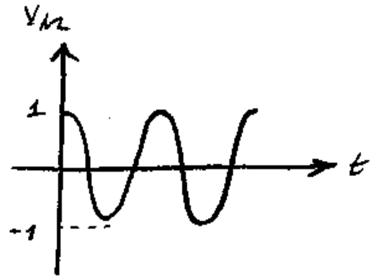
Inverting Amplifier
(Ideal Op-Amp)

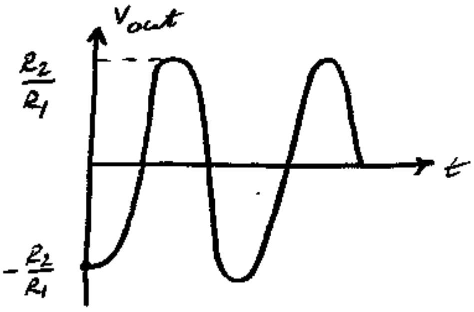


Assume linear region
(Assume V_{in} is adjusted so that Op-Amp is guaranteed to be in Linear region).

$V_{out} = 0 - V_{R_2}$

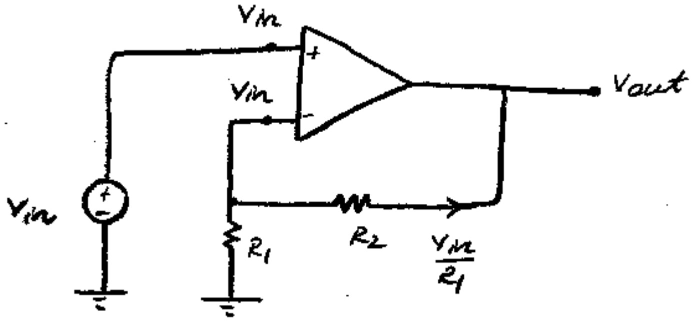
$V_{out}/\ldots = -\frac{R_2}{\ldots}$





Non-Inverting Op-Amp

Assume Linear (Vin is sufficiently small)



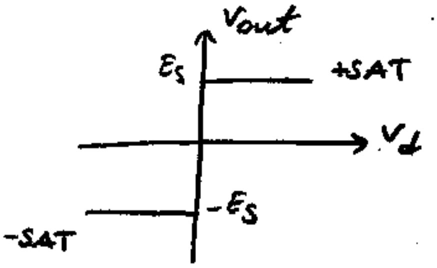
$$V_{R_2} = \frac{V_{in}}{R_1} \cdot R_2$$

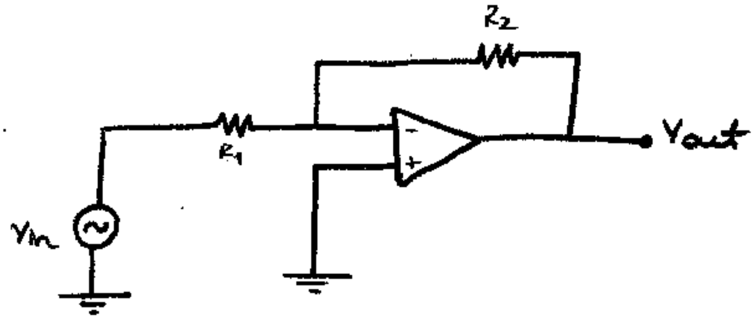
$$V_{out} = V_{in} + V_{R_2}$$

$$V_{out} = V_{in}\left(1+\frac{R_2}{R_1}\right)$$

$$\frac{V_{out}}{V_{in}} = \frac{R_1+R_2}{R_1}$$ → positive gain

Inverting Op-Amp ...





Ideal Model ($A=\infty$, $R_{in}=\infty$, $r_o=0$)

② Assume +SAT; ($V_d > 0$, $V_{out} = E_s$)

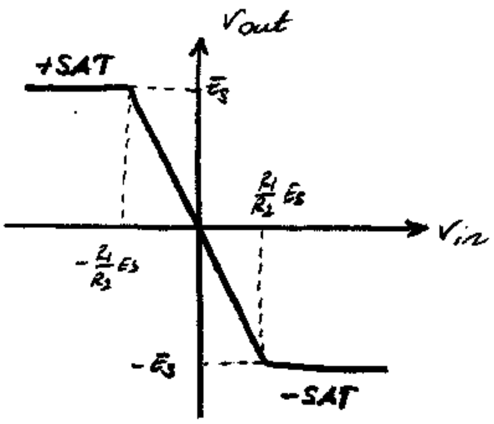
$$V_d = \underset{\substack{\downarrow \\ 0}}{V_+} - V_- > 0 \longrightarrow V_- < 0$$

Node equation at V_- :

$$\frac{V_- - V_{in}}{R_1} + \frac{V_- - V_{out}}{R_2} = 0 \longrightarrow \left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_- = \frac{V_{in}}{R_1} + \frac{E_s}{R_2}$$

$$\boxed{V_- = \frac{R_2}{R_1+R_2}V_{in} + \frac{R_1}{R_1+R_2}E_s} < 0$$

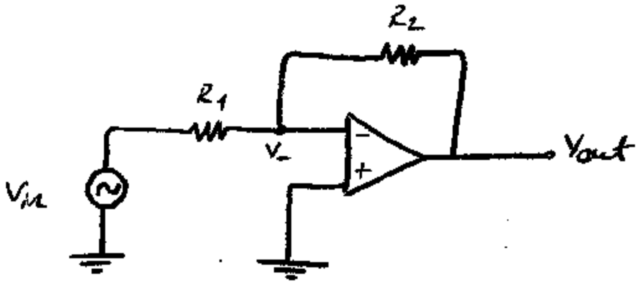
$$V_{in} < -\frac{R_1}{R_2}E_s$$

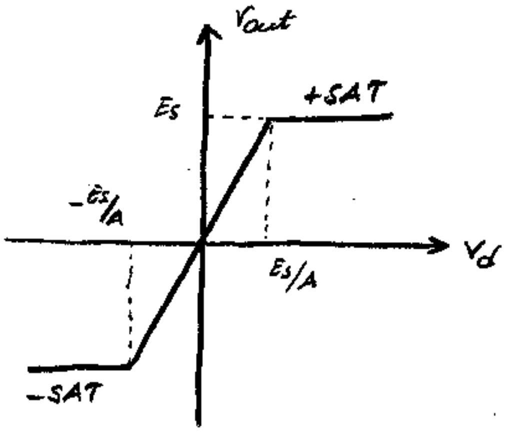


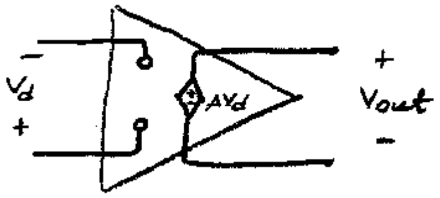
Linear: $V_{out} = -\frac{R_2}{R_1} V_{in}$

Inverting Amplifier (Finite Gain Model)

A: Finite, $R_{in} = \infty$, $r_o = 0$







→ Linear model, finite gain

$|V_{out}| < E_s$; $|V_d| < E_s/A \longrightarrow |V_-| < \frac{E_s}{A}$

$i_- = i_+ = 0$ $(R_{in} = \infty)$

KCL at V_- ;

$$\frac{V_- - V_{in}}{R_1} + \frac{V_- - \overset{\curvearrowright}{V_{out}}}{R_2} = 0 \qquad V_{out} = A \cdot V_d = A \cdot (-V_-)$$

$$R_2 (V_- - V_{in}) + R_1 (A+1) V_- = 0$$

$$\boxed{V_- = \frac{R_2}{R_2 + (A+1) R_1} V_{in}}$$

For the validity of linear region assumption

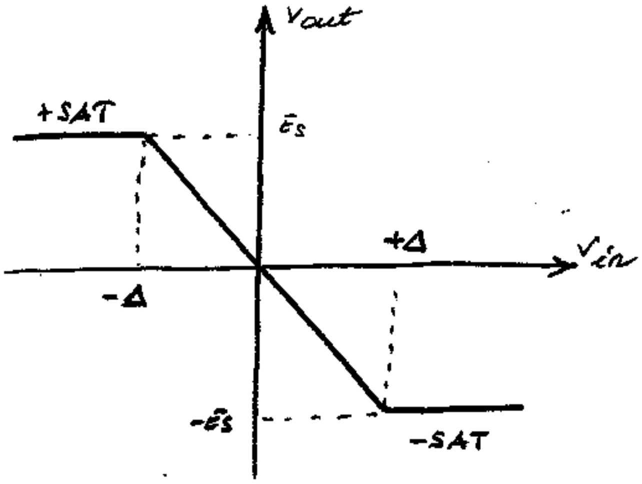
$$|V_-| < \frac{E_s}{A} \rightarrow |V_{in}| < \frac{E_s}{A} \cdot \frac{R_2 + (A+1) R_1}{R_2}$$

If $A \rightarrow \infty$, $|V_{in}| < \frac{R_1}{R_2} E_s$

$V_{out} = A V_d = -A V_-$

$$V_{out} = \frac{-AR_2}{R_2 + (A+1)R_1} V_{in}$$

If $A \to \infty$, $V_{out} = -\frac{R_2}{R_1} V_{in}$



$$\Delta = \frac{E_s}{A} \cdot \frac{R_2 + (A+1)R_1}{R_2}$$

2) Assume + SAT ($V_{out} = E_s$)

$V_d > \frac{E_s}{A}$ $R_{in} = \infty$, $i_- = i_+ = 0A$

Node equation at V_- :

$\longrightarrow V_{out} = E_s$

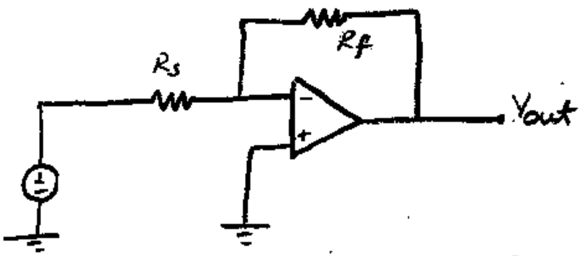
$$\frac{V_- - V_{in}}{R_1} + \frac{V_- - V_{out}}{R_2} = 0$$

$$V_- = \frac{R_2 V_{in} + R_1 E_s}{R_1 + R_2}$$

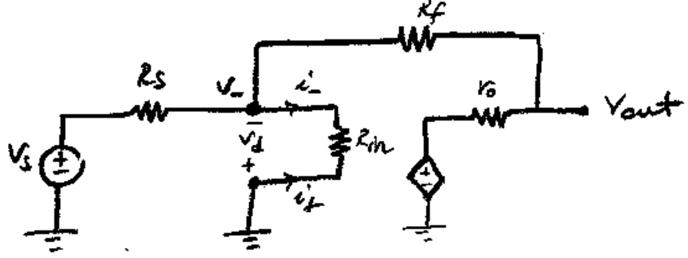
$$V_d > E_s/A \longrightarrow V_- < -E_s/A$$

$$V_{in} < -\frac{E_s}{A} \frac{(R_1+R_2)}{R_2} - \frac{R_1}{R_2} E_s$$

~~Non~~-Inverting Op-Amp
(More realistic Model)



① Assume linear region of operation



for the validity of linear region

$$|V_{out}| < E_s$$

$i_- \neq 0$, $i_+ \neq 0$

Node eqn. at V_- : $\dfrac{V_- - V_s}{R_s} + \dfrac{V_-}{R_{in}} + \dfrac{V_- - V_{out}}{R_f} = 0$

Node eqn at V_{out} : $\dfrac{V_{out} - V_-}{R_f} + \dfrac{V_{out} - A V_d}{r_o} = 0$ $\quad V_d \to -V_-$

$$V_{out} = \frac{-A + r_o/R_f}{\frac{R_s}{R_f}\left(1 + A + \frac{r_o}{R_{in}}\right) + \left(\frac{R_s}{R_{in}} + 1\right) + \frac{r_o}{R_f}} V_s$$

$A \simeq 10^6$ or higher → open loop gain

$r_o \approx 10\text{-}100\Omega$'s

$R_{in} \approx M\Omega$'s

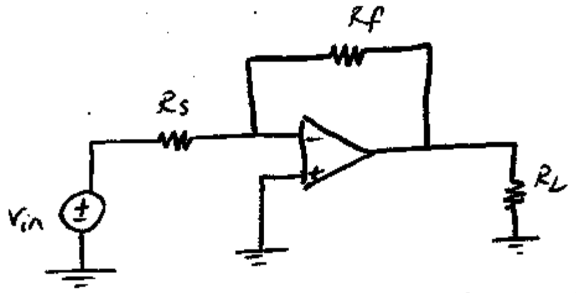
$(R_s, R_f) = O(10k\Omega)$ → order of $10k\Omega$'s

As expected, as $A \to \infty$, $R_{in} \to \infty$, $r_o \to 0$

$$V_{out} = -\frac{R_f}{R_s}$$

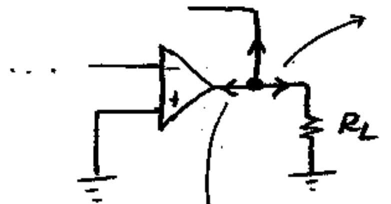
Important Note:

① using the model with finite A, R_{in} and $r_o \neq 0$, we can also analyze the following circuit:



→ Load connected

For the earlier, less accurate model, it is not possible to include the effect of R_L, since



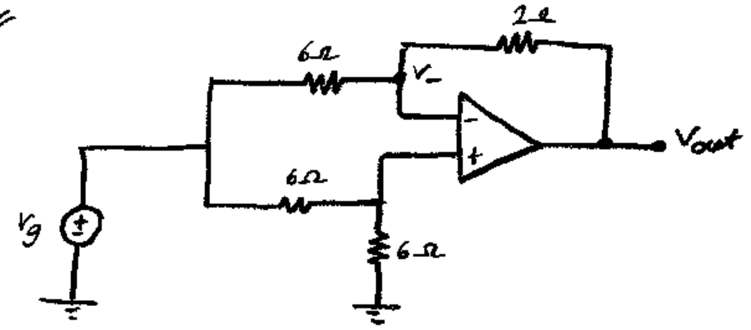
Not possible to write a KCL at output node

since $r_o = 0$, the effect of R_L is not visible when models with $r_o = 0$ are used.

Nodal analysis for Op-Amp Circuits

If ideal model ($A = \infty$) or finite gain model (A: finite; $R_{in} = \infty$, $r_o = 0$) is used, then writing node equations except at the output node can easily lead to the solution

Ex



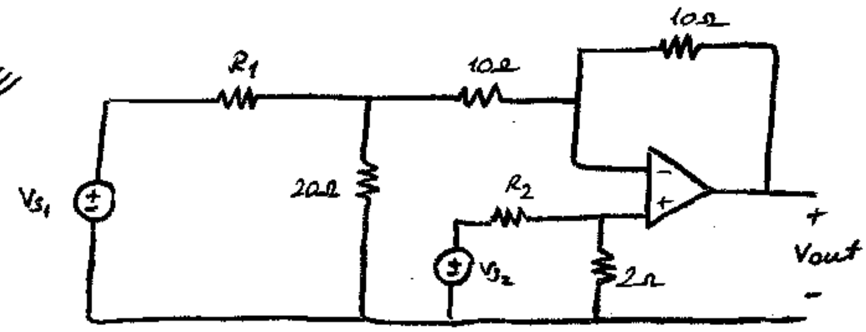
$v_g(t) = 6\cos(2t)$ volts (Assume linear region operation & A→∞)

KCL at V_- : $\frac{V_- - V_g}{6} + \frac{V_- - V_{out}}{2} = 0$ → $V_{out} = \frac{V_g}{3}$

KCL at V_+ : $\frac{V_-}{6} + \frac{V_- - V_g}{6} = 0 \rightarrow V_- = \frac{V_g}{2}$

$V_{out} = 2\cos(2t)$ volts.

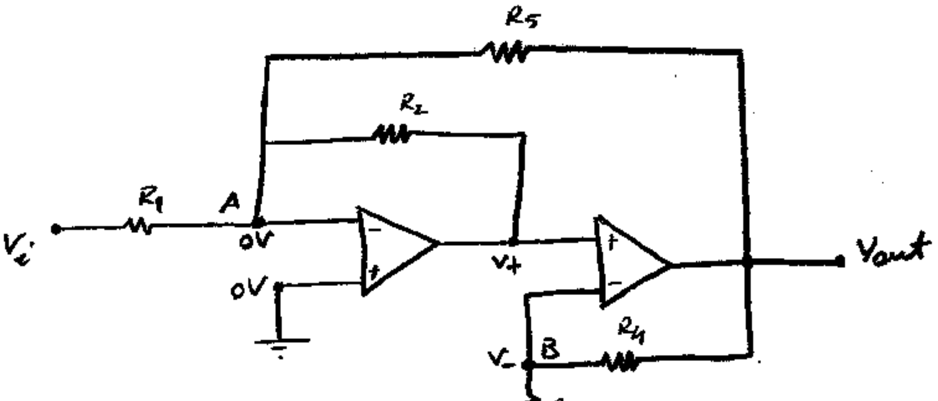
Ex



Assuming ideal model and linear operation

Find R_1 and R_2 such that $V_o = V_{s_2} - \frac{V_{s1}}{4}$

Ex



Assume ideal model and linear region; find V_{out}

KCL at A : $\frac{0 - V_i}{R_1} + \frac{0 - V_{out}}{R_5} + \frac{0 - V_-}{R_2} = 0$

at B : $\frac{V_- - V_{out}}{R_4} + \frac{V_- - 0}{R_3} = 0$

∴ $V_- = -\frac{R_3}{R_1} \cdot \frac{R_5 \cdot R_2 \cdot V_i}{R_2(R_3+R_4)+R_3R_5}$

$V_{out} = \frac{-R_5 R_2 (R_3+R_4)}{R_2(R_3+R_4)+R_3R_5} \cdot \frac{V_i}{R_1}$

Let's check the conditions for V_{in} so that both op-Amps are in linear region

Op-Amp I : $V_{out}(1) = V_-(II)$ $\quad |V_{out}(1)| < E_s$

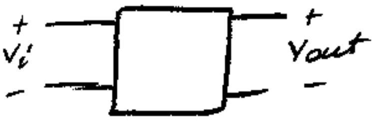
Op-Amp II : $V_{out}(II) = V_{out}$ $\quad |V_{out}(2)| < E_s$

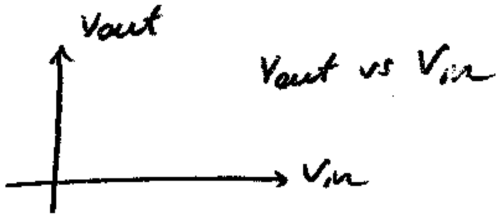
Both conditions should be satisfied at the same time for the validity of the analysis.

$V_{in} \in [A^I, B^I] \qquad V_{in} \in [A^{II}, B^{II}]$

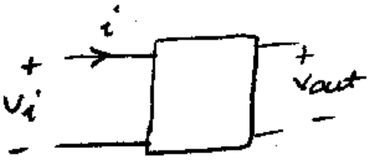
V_{in} should be the intersection of these two intervals

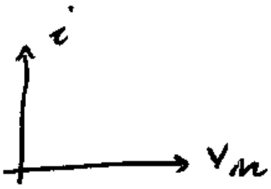
→ Transfer characteristics



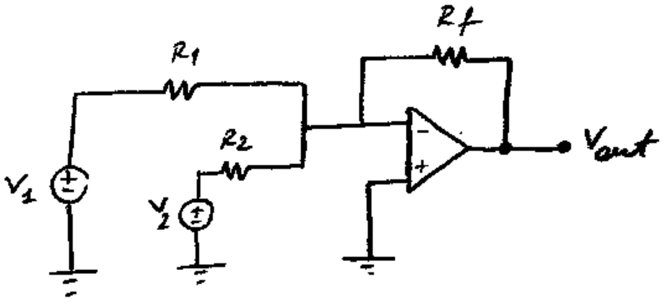


→ Input Characteristics

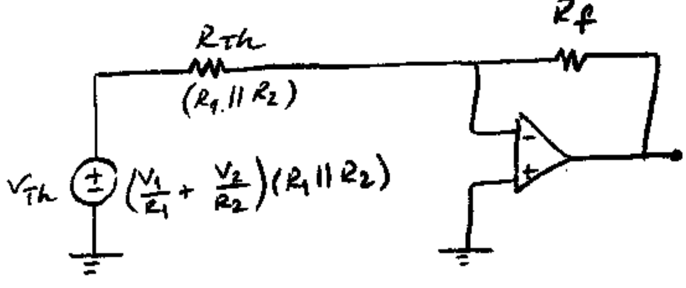




Summer Amplifier
(Assume linear Region)



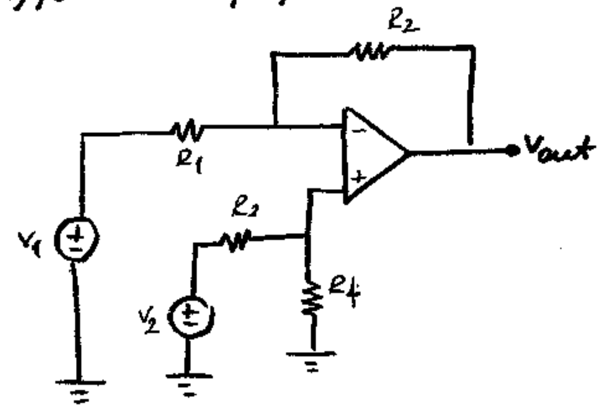
Thevenin equivalent



$$V_{out} = -\frac{R_f}{R_{Th}} \cdot V_{Th}$$

$$V_{out} = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)$$

Difference Amplifier

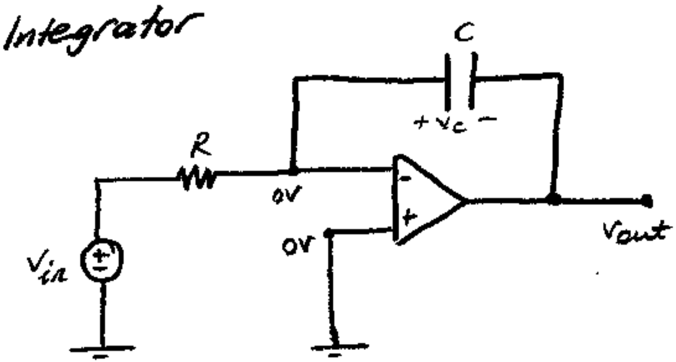


$$V_{out} = \left(\frac{R_1+R_2}{R_1}\right)\cdot\left(\frac{R_4}{R_3+R_4}\right)V_2 - \frac{R_2}{R_1}V_1$$

if $\frac{R_3}{R_4} = \frac{R_1}{R_2}$

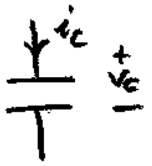
$$\longrightarrow V_{out} = \frac{R_2}{R_1}(V_2 - V_1)$$

Integrator



$$V_{out} = -V_c$$

$$i_c = C\cdot\frac{dV_c}{dt}$$



KCL at V_- : $\frac{0-V_{in}}{R} + i_c = 0$

$$\frac{V_{in}}{R} = C\ \frac{dV_c}{dt}$$

$$\frac{dV_c}{dt} = \frac{1}{RC}V_{in} \longrightarrow V_c(t) - V_c(0) = \frac{1}{RC}\cdot\int_0^t V_{in}(\tau)d\tau$$

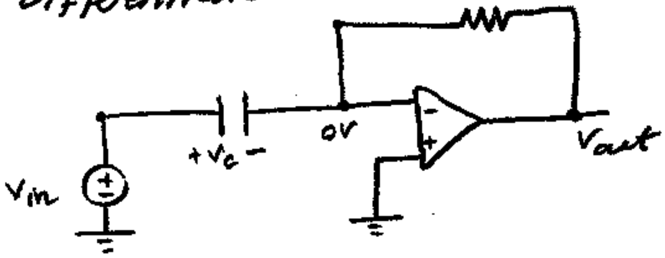
$t=0$ value is not given ; assume $V_c(-\infty) = 0V$

$$V_c(t) = \frac{1}{RC}\int_{-\infty}^{t} V_{in}(\tau)d\tau$$

↙

$-V_{out}(t)$

Differentiator

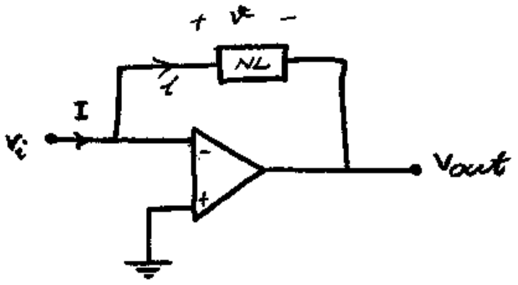


$$i_c = C\cdot\dot{V}_c(t)$$

$$i_c = C\cdot\dot{V}_{in}(t)$$

$$V_{out}(t) = -R\cdot i_c$$
$$= -R\cdot C\cdot\dot{V}_{in}(t)$$

$$V_{out}(t) = -RC\cdot\frac{d}{dt}\left(V_{in}(t)\right)$$



+SAT: $V_{out} = +E_s$, $V_+ > V_-$ } $V_- < 0$, $V_i' < 0$

$\upsilon = V_i' - V_{out}$

$\upsilon < -E_s$

-SAT: $V_{out} = -E_s$, $V_+ < V_-$ } $V_- > 0$, $V_i' > 0$

$\upsilon > E_s$

Linear:

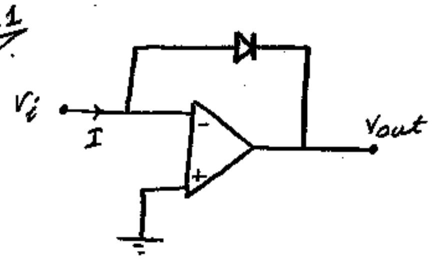
$V_+ = V_- = 0$, $-E_s < V_{out} < E_s$ } $V_i' = 0$

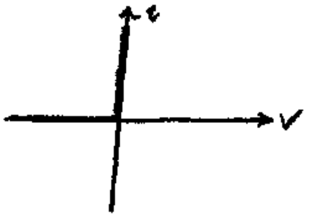
$V_{out} = V_- - \upsilon$

$V_{out} = -\upsilon$

$-E_s < \upsilon < E_s$

EX1





+SAT (op-Amp) $\longrightarrow$ $\upsilon < -E_s$ $\longrightarrow$ diode characteristics $i = 0$

$V_{out} = +E_s$

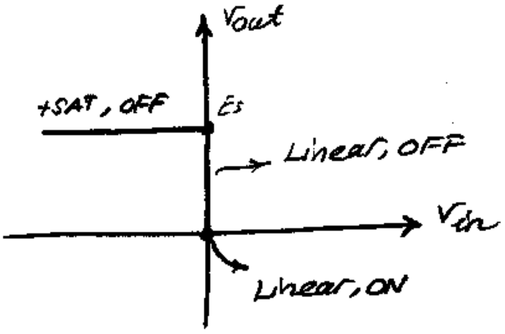
$V_{in} < 0$

-SAT (Op-Amp) $\longrightarrow$ $\upsilon > +E_s$ $\longrightarrow$ not possible to have a current i so that $\upsilon > E_s$ $\Longrightarrow$ Op-Amp does not enter into -SAT region

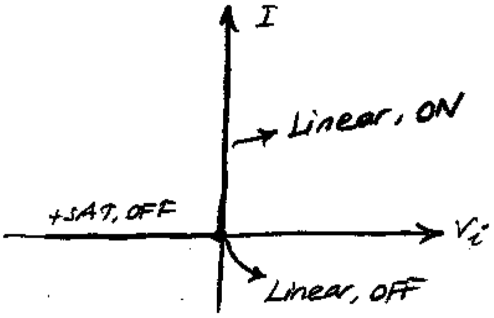
Linear (Op-Amp) $\longrightarrow$ $-E_s \leq \upsilon \leq E_s$ $\longrightarrow$ only possible for (from diode ch.)

① $-E_s \leq \upsilon < 0 \longrightarrow i = 0$

② $\upsilon = 0 \longrightarrow i > 0$

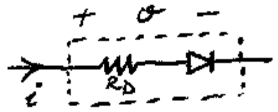


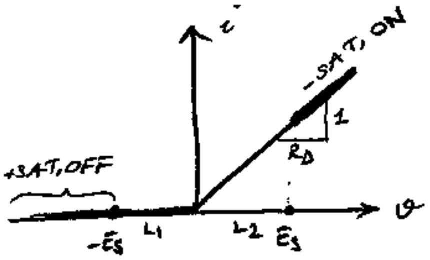
$\longrightarrow$ Transfer characteristics



Driving point (?)

Ex2





+SAT : $\vartheta < -E_s \rightarrow i = 0$

$V_{out} = E_s$

-SAT : $\vartheta > E_s \longrightarrow i = \frac{\vartheta}{R_D} = \frac{V_i + E_s}{R_D}$

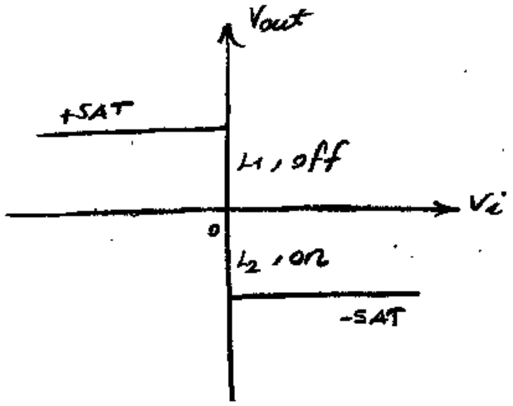
$V_{out} = -E_s$

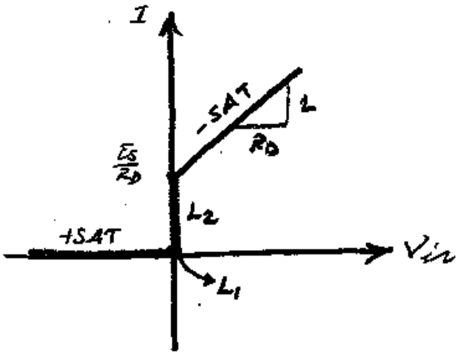
Linear : $-E_s \leq V_{out} \leq E_s$

$V_i = 0$

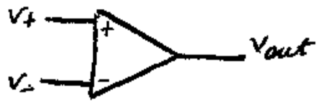
$$\begin{tikzcd}
-E_s \leq \vartheta \leq E_s \arrow[r] \arrow[dr] & -E_s \leq \vartheta \leq 0 \arrow[r, "L_1"] & i = 0 \\
& 0 \leq \vartheta \leq E_s \arrow[r, "L_2"] \arrow[d, leftrightarrow] & i = \frac{\vartheta}{R_D} = \frac{-V_{out}}{R_D} \arrow[d] \\
& -E_s \leq V_{out} \leq 0 & -E_s < V_{out} < 0
\end{tikzcd}$$

$0 \leq i \leq E_s/R_D$





The Common Mode Rejection Ratio (CMRR)



Ideal Op-Amp : (Linear Region)

$V_{out} = A(V_+ - V_-) = AV_+ - AV_-$

Practical Op-Amp

$V_{out} = A_+ V_+ - A_- V_-$ and $A_+ \neq A_-$

$$\left.\begin{aligned} V_{common} &: V_c = \frac{V_+ + V_-}{2} \\ V_{dif} &: V_d = V_+ - V_- \end{aligned}\right\} \quad \begin{aligned} V_+ &= V_c + \frac{V_d}{2} \\ V_- &= V_c - \frac{V_d}{2} \end{aligned}$$

Then practical op-amps:

$$\begin{aligned} V_{out} &= A_+ V_+ - A_- V_- \\ &= A_+ \left(V_c + \frac{V_d}{2} \right) - A_- \left(V_c - \frac{V_d}{2} \right) \\ &= (A_+ - A_-) V_c + \left(\frac{A_+ + A_-}{2} \right) V_d . \end{aligned}$$

$A_c \triangleq A_+ - A_-$ (common mode gain)

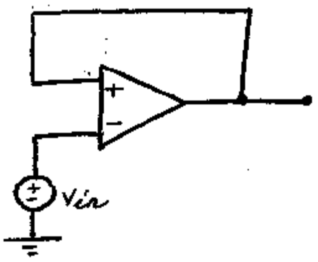
$A_d \triangleq \frac{A_+ + A_-}{2}$ (differential mode gain)

Ideally : $A_c = 0 \quad A_d = A$

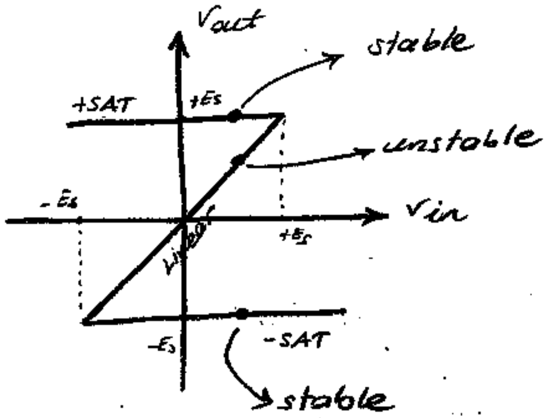
$CMRR \triangleq \left| \frac{A_d}{A_c} \right|$ (ideally infinity)

$(CMRR)_{dB} = 20 \log_{10} \left| \frac{A_d}{A_c} \right|$

Positive - Negative Feedback

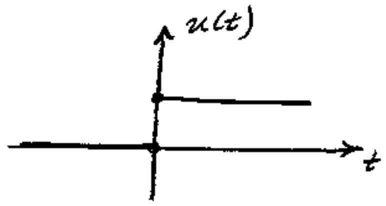


Positive feedback circuit



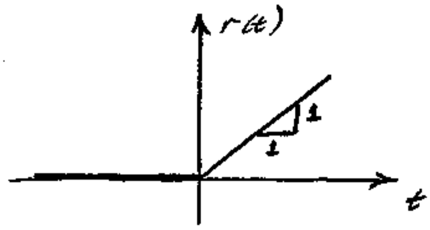
waveforms:

* unit step , ramp, impulse



$$u(t) = \begin{cases} 1 , & t > 0 \\ 0 , & t < 0 \end{cases}$$

→ Ramp function



$$r(t) = t \cdot u(t) = \begin{cases} t , & t > 0 \\ 0 , & t < 0 \end{cases}$$

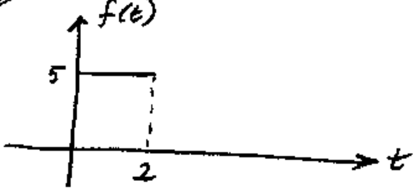
Properties :

① $\frac{d}{dt} r(t) = u(t) = \begin{cases} 1 , & t > 0 \\ 0 , & t < 0 \end{cases}$

Note that not defined at $t = 0$.

② $r(t) = \int_{-\infty}^{t} u(\tau) d\tau$

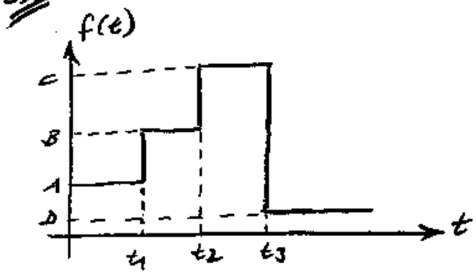
Ex



Express f(t) as a linear combination of unit-step functions.

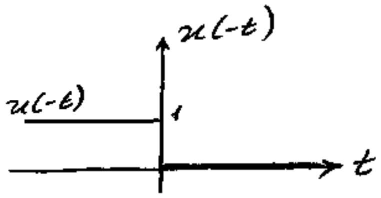
$$f(t) = 5u(t) - 5u(t-2)$$

Ex

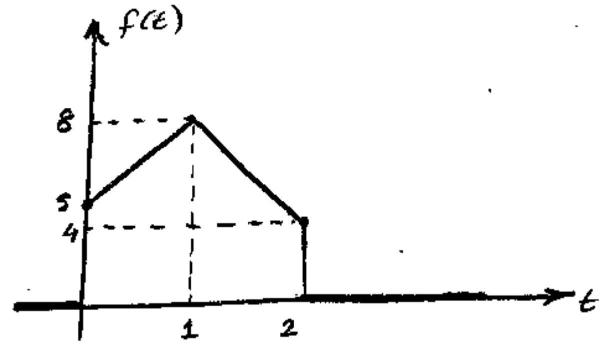


$$f(t) = A \cdot u(t) + (B-A)u(t-t_1) + (C-B)u(t-t_2)$$
$$+ (D-C)u(t-t_3)$$

$u(t^2) = 1 \quad , \quad t \in R$



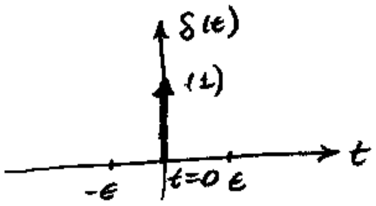
Ex

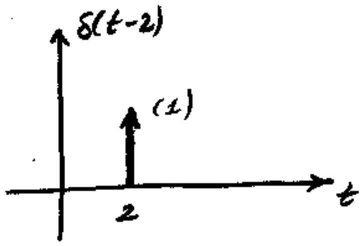


$$f(t) = 5u(t) + 3r(t) - 7r(t-1)$$
$$+4r(t-2) - 4u(t-2)$$

Impulse function:

$\delta(t)$: impulse function





① Sifting property: $\int_{-\epsilon}^{\epsilon} \delta(\tau)d\tau = 1 \quad , \forall \epsilon > 0$

$\delta(t)$: generalized function
(function at the limit)

② $$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

③ $$\frac{d}{dt} u(t) = \delta(t)$$

$$\Rightarrow \quad \begin{tikzcd} \delta(t) \arrow[r, shift left, "\int_{-\infty}^{t}"] & u(t) \arrow[l, shift left, "d/dt"] \arrow[r, shift left, "\int_{-\infty}^{t}"] & r(t) \arrow[l, shift left, "d/dt"] \end{tikzcd}$$

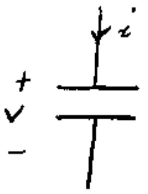
Sinusoidal waveforms

$$s(t) = \underset{\hookrightarrow \text{amplitude}}{A} \cdot \cos(2\pi f t + \underset{\hookrightarrow \text{phase (in radians)}}{\phi})$$

$\omega = 2\pi f$: angular frequency (rad/sec)

$T = 1/f$: period (sec)

Capacitors



c (Farad)

$$Q = C \cdot V$$
$$Q(t) = C(t) \cdot V(t)$$
$$i(t) = \frac{dC(t)}{dt} \cdot V(t) + C(t) \cdot \frac{dV(t)}{dt}$$

LTI capacitors : $C(t) = C$

$$i_c(t) = C \cdot \frac{dV_c(t)}{dt}$$

$P_c(t) = V_c(t) \cdot i_c(t) \longrightarrow$ instantenous power (watt)

Energy : $$\int_{-\infty}^{t} P_c(\tau)\, d\tau = \int_{-\infty}^{t} V_c(\tau)\, i_c(\tau)\, d\tau$$

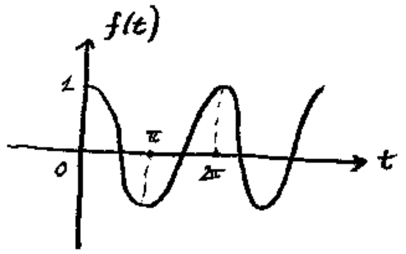
$$= \int_{-\infty}^{t} C \cdot V_c(\tau) \frac{dV_c(\tau)}{d\tau} \cdot d\tau$$

$$= C \cdot \int_{-\infty}^{t} V_c(\tau)\, dV_c(\tau)$$

$$= \frac{C \cdot [V_c(t)]^2}{2} \Bigg|_{t=-\infty}^{t} = \frac{1}{2} C \cdot [V_c(t)]^2$$

assuming $V_c(-\infty) = 0V$

Ex



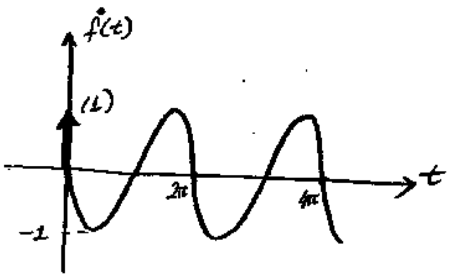
$$f(t) = \cos(t)\cdot u(t)$$

$$\dot{f}(t) = -\sin(t)\cdot u(t) + \cos(t)\cdot \dot{u}(t)$$

$$\dot{f}(t) = -\sin(t)\cdot u(t) + \cos(t)\cdot \delta(t)$$

$$\dot{f}(t) = -\sin(t)\cdot u(t) + \cos(0)\cdot \delta(t)$$

$$\dot{f}(t) = -\sin(t)\cdot u(t) + \delta(t)$$



→ Most Important Fact:

Capacitor voltage is a continuous function if input (current through capacitor) is bounded (not infinite)

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\,d\tau \xleftrightarrow{iff} i_c(t) = C\cdot\frac{dV_c(t)}{dt}$$

$$= \underbrace{\frac{1}{C}\int_{-\infty}^{t_0} i_c(\tau)\,d\tau}_{V_c(t_0)} + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau \qquad (t_0 < t)$$

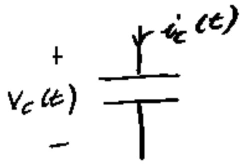
$$V_c(t) = V_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau$$

$$V_c(t_0^+) = V_c(t_0) + \frac{1}{C}\int_{t_0}^{t_0^+} i_c(\tau)\,d\tau \quad , \quad t_0^+ = t_0 + \epsilon \ , \ \epsilon > 0 \text{ and arbitrarily small}$$

$$V_c(t_0^+) = V_c(t_0) + \frac{1}{C}\cdot\epsilon\cdot M \qquad |i_c(\tau)| \le M \rightarrow \text{bounded, finite}$$

$$V_c(t_0^+) = V_c(t_0) \longrightarrow \text{continuous}$$

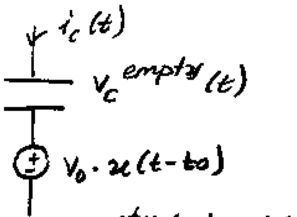
Initial Condition Models for capacitors:



$$V_c(t) = V_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau \quad , \quad t > t_0$$

$V_c(t_0) = V_0$

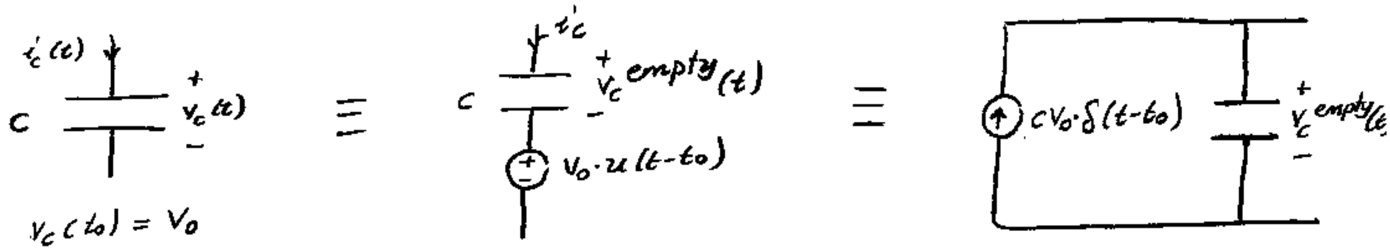
$E_{cap}(t_0) = \frac{1}{2} C V_0^2$



$V_c^{empty}(t_0)=0V$

$$V_c^{empty}(t) + V_0\cdot u(t-t_0) \overset{?}{=} V_c(t)$$

$$\left(\cancel{V_c(t_0)}^{0} + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)d\tau\right) + V_0\cdot u(t-t_0) = V_c(t) \rightarrow t > t_0$$

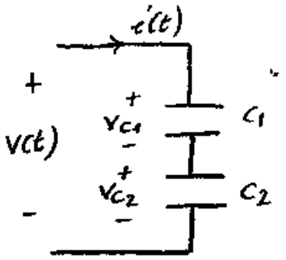


$V_c(t_0) = V_0$

$V_c^{empty}(t_0) = 0V$

$V_c^{empty}(t_0) = 0V$

Series and Parallel Combination of Capacitors



$$V(t) = V_{C1}(t) + V_{C2}(t)$$

$$= \frac{1}{C_1}\int_0^t i(\tau)d\tau + \frac{1}{C_2}\int_0^t i(\tau)d\tau$$

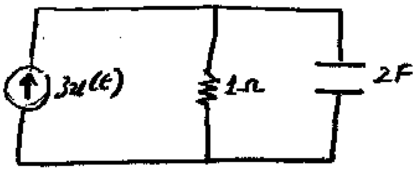
$$= \left(\frac{1}{C_1}+\frac{1}{C_2}\right)\int_0^t i(\tau)d\tau = \frac{1}{C_{eq}}\int_0^t i(\tau)d\tau$$

$$C_{eq} = \left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}$$

Voltage division:

$$V_1 = \frac{C_{eq}}{C_1}\cdot V(t) = \frac{1/C_1}{1/C_1 + 1/C_2}\,V(t) = \frac{C_2}{C_1+C_2}\,V(t)$$

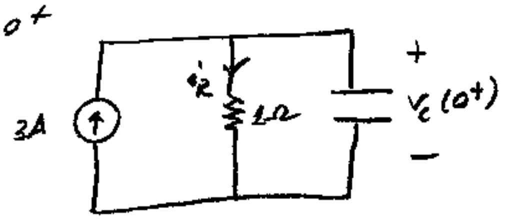
Ex



$V_c(0^-) = 0V$

Find $t=0^+$ solution

at $t=0^+$

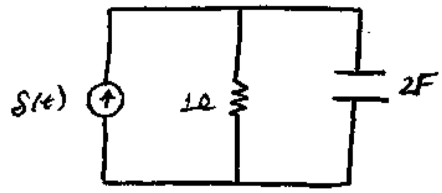


$V_c(0^+) = V_c(0^-) = 0V$

(continuity of capacitor voltage for finite input)

$$i_R(0^+) = \frac{0V}{1\Omega} = 0A$$

Ex

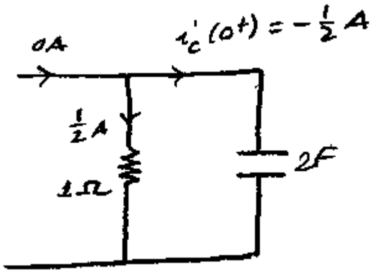


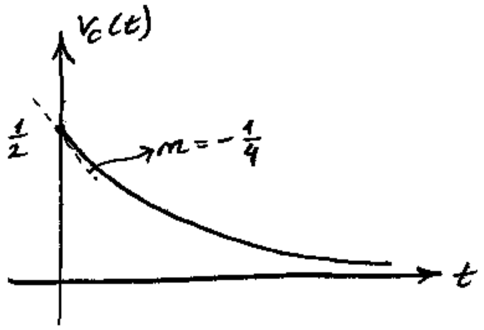
$i_c'(t) = \delta(t)$

$i_R'(t) = 0A$

$$V_c(0^+) = \frac{1}{C}\int_{0^-}^{0^+} \delta(t)\,dt = \frac{1}{C} = \frac{1}{2}V$$

Then at $t = 0^+$





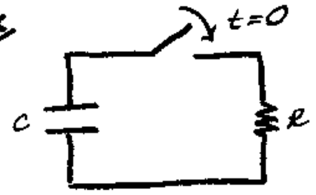
$C\cdot\dot{V}_c = i_c'$

$\dot{V}_c = -\frac{1}{4}$

δ(t): impulse input

$V_c(t)$ is not const. at $t=0$

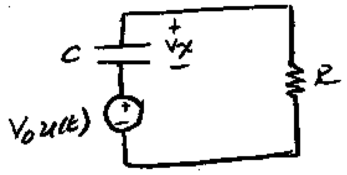
Ex



$V_c(0^-) = V_0$

Analyze the circuit for $t \geq 0$.

At $t \geq 0$



$V_x(t) + V_0\, u(t) + i_c'(t)\cdot R = 0.$

$V_x(t) + V_0\, u(t) + RC\cdot\frac{d}{dt}V_x(t) = 0$

$\left(D + \frac{1}{RC}\right) V_x(t) = \frac{-V_0\, u(t)}{RC}, \quad t \geq 0$

$V_x(0^-) = 0V;$

$V_x(t) = \alpha\cdot e^{-t/RC} + A$

$A = -V_0$

$t=0 \rightarrow V_x(0) = 0 \rightarrow \alpha = V_0$

$V_x(t) = V_0\left(e^{-t/RC} - 1\right), \quad t \geq 0$

$\rightarrow V_c(t) = V_0\cdot u(t) + V_x(t), \quad t \geq 0$

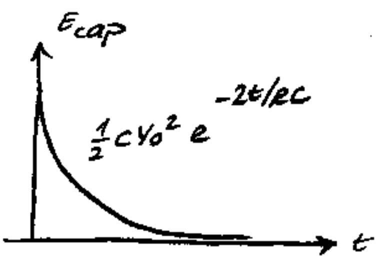
$V_c(t) = V_0\cdot e^{-t/RC}\cdot u(t)$

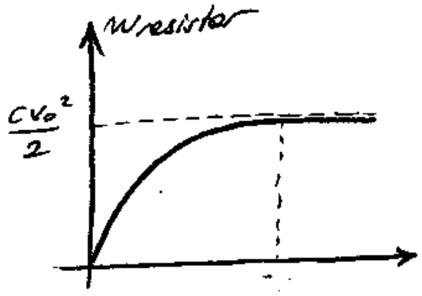
$\rightarrow E_{cap} = \frac{1}{2}C V_c^2(t) \longrightarrow E_{cap} = \frac{1}{2}C\cdot V_0^2\cdot e^{-2t/RC}\cdot u(t)$

Energy dissipiated on resistor,

$$W_{resistor} = \int_0^t R\cdot i_c^2(\tau)d\tau = \int_0^t \frac{V_R^2(\tau)}{R}\cdot d\tau = \frac{V_0^2}{R}\int_0^t e^{-2\tau/RC}d\tau$$

$$= \frac{V_0^2}{R}\cdot\frac{RC}{-2}\cdot e^{-2\tau/RC}\Big|_{\tau=0}^{t} = \frac{1}{2}CV_0^2\left(1-e^{-2t/RC}\right)$$





Inductors:

$$\text{flux} \xleftrightarrow{\text{dual}} \text{charge}$$

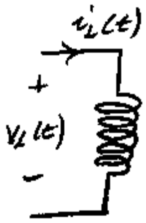
$\phi(t) = L(t)\ i(t) \longrightarrow$ weber's Law

$$\frac{d\phi(t)}{dt} = \frac{dL(t)}{dt}\cdot i(t) + L(t)\cdot\frac{di(t)}{dt} = V(t)$$

LTI inductors:

$$V_L(t) = L\cdot\frac{di_L(t)}{dt}$$

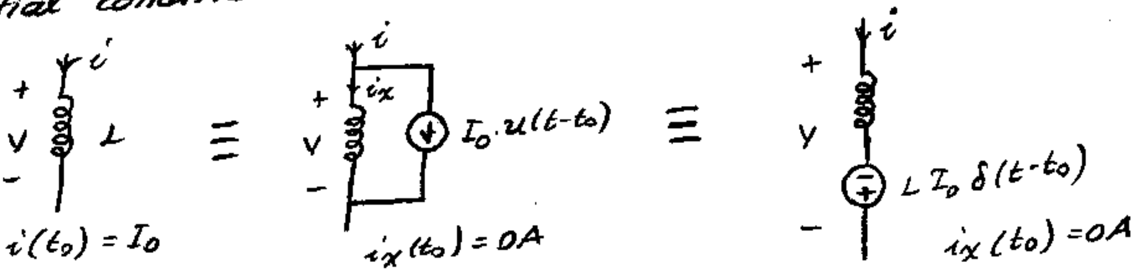
$$E_{ind}(t) = \frac{1}{2}\cdot L\cdot[i_L(t)]^2$$



$i_L(0^-) = i_0$

$$i_L(t) = i_0 + \frac{1}{L}\int_0^t V_L(\tau)d\tau \quad , \quad t \geq 0$$

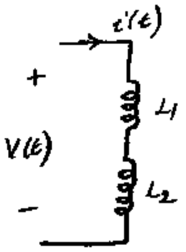
Initial condition models



$$v_L(t) = L\cdot\frac{d}{dt}i_L(t) = L\cdot\frac{d}{dt}\left(I_0\cdot u(t-t_0) + \frac{1}{L}\int_{t_0}^t v_x(\tau)d\tau\right)$$

$$= L\cdot I_0\cdot\delta(t-t_0) + v_x(t) \quad , \quad t > t_0$$

Inductors in series:



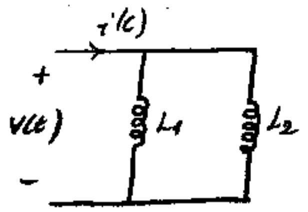
$$v(t) = v_{L_1}(t) + v_{L_2}(t)$$

$$v(t) = L_1 \cdot \frac{d \cdot i(t)}{dt} + L_2 \cdot \frac{d\, i(t)}{dt}$$

$$v(t) = (L_1+L_2) \cdot \frac{d\, i(t)}{dt}$$

↳ Relation of an inductor of (L_1+L_2)H

Inductors in parallel



$$i(t) = i_{L1}(t) + i_{L2}(t)$$

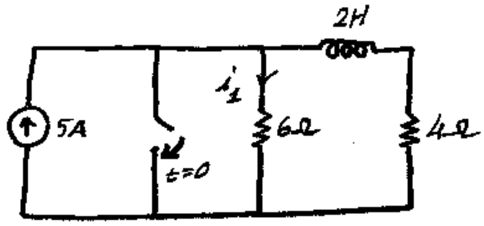
$$= \frac{1}{L_1} \cdot \int_0^t v(\tau)d\tau + \frac{1}{L_2}\int_0^t v(\tau)d\tau$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2}\right) \cdot \int_0^t v(\tau)d\tau$$

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$$

→ Inductors has $i_L(t)$ as a continuous function of time (unless there is an impulse source in the system.)

Ex//

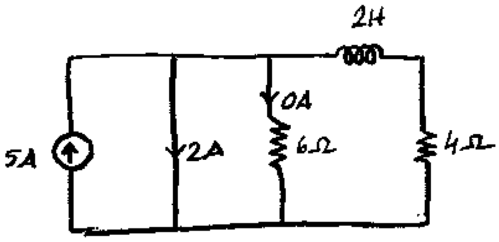


$i_1(0^-) = 2A$

Find $i_L(0^+)$, $i_1(0^+)$, $\left.\frac{d}{dt} i_L(t)\right|_{t=0^+}$

From KCL → $i_L(0^-) = 5-2 = 3A$

At $t=0^+$,



$i_L(0^+) = 3A$

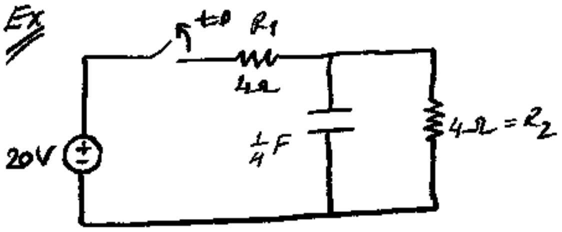
$i_1(0^+) = 0A$ → short circuit

From KVL:

$$L \cdot \frac{d i_L(0^+)}{dt} + 4\, i_L(0^+) = 0$$

$$\frac{d}{dt} i_L(0^+) = \frac{-4}{L} i_L(0^+) = -6 \text{ A/s}$$

Ex



$i_{R_2}(0^-) = 2A$

Find $q(0^-)$, $q(0^+)$, $i_{R_1}(0^-)$, $i_{R_1}(0^+)$

$i_c(0^-)$, $i_c(0^+)$

At $t=0^-$:

$$i_{R_1}(0^-) = \frac{20 - 2.4}{4} = 3A$$

$$q(0^-) = C \cdot V(0^-) = \frac{1}{4} \cdot 8 = 2C$$

$$i_c(0^-) = 3 - 2 = 1A$$

At $t=0^+$:

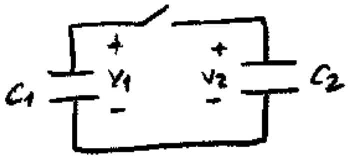
$i_{R_1}(0^+) = 0A$, $i_{R_2}(0^+) = 2A$

$q(0^+) = q(0^-) = 2C$

$i_c(0^-) = -2A$

$C \cdot \dot{V}_c(0^+) = -2 \rightarrow \frac{d}{dt} V_c(0^+) = -8 \, V/s$

Some pathological cases:



$V_{C_1}(0^-) = V_1 \neq V_2 = V_{C_2}(0^-)$, $E(0^-) = \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2$

At $t=0^+$:

$q_{Tot}(0^-) = q_{Tot}(0^+)$

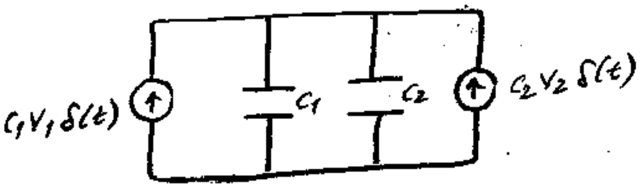
$$C_1 V_1 + C_2 V_2 = (C_1 + C_2) \cdot V_{common}(0^+)$$

$$V_{common}(0^+) = \frac{1}{C_1 + C_2} (C_1 V_1 + C_2 V_2)$$

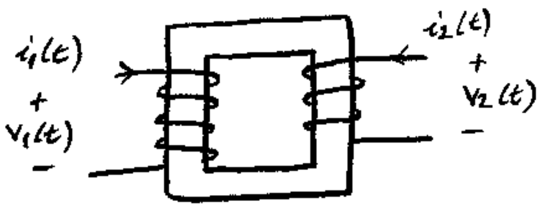
$$E(0^+) = \frac{1}{2}(C_1 + C_2) \frac{1}{(C_1 + C_2)^2} (C_1 V_1 + C_2 V_2)^2$$

$$E(0^-) \neq E(0^+)$$

Second explanation:



Mutual Inductor (coupled Inductor)



$$\phi_1(t) = L_1 \cdot i_1(t) + M_1 \cdot i_2(t)$$

$$\phi_2(t) = L_2 \cdot i_2(t) + M_2 \cdot i_2(t)$$

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

↳ total flux coupled to the primary & secondary sides.

$$\frac{d\phi(t)}{dt} = v(t)$$

$$\Rightarrow \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} di_1(t)/dt \\ di_2(t)/dt \end{bmatrix}$$

① special case $M = 0 \longrightarrow$ Self induction

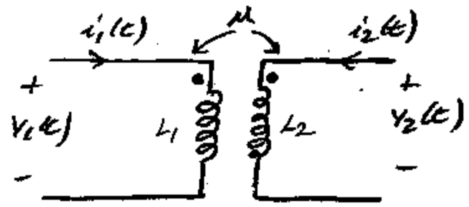
$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \dot{i}_1(t) \\ \dot{i}_2(t) \end{bmatrix}$$

② $M = \sqrt{L_1 \cdot L_2}$ then $k = \frac{M}{\sqrt{L_1 L_2}} = 1 \longrightarrow$ coupling coefficient!

then we say that coils are perfectly coupled

→ no loss in power

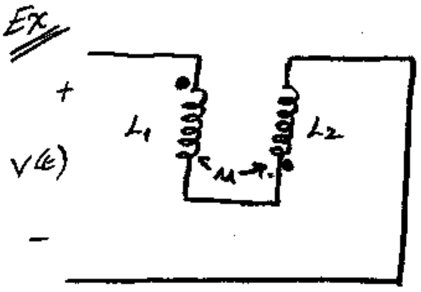
→ ideal transformer



$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \dot{i}_1(t) \\ \dot{i}_2(t) \end{bmatrix}$$

Note the dot convention!

Ex



Express $v(t)$ in terms of $i(t)$

$i(t) = i_1(t) = i_2(t)$

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} \dot{i}_1(t) \\ \dot{i}_2(t) \end{bmatrix}$$

$$v(t) = v_1(t) + v_2(t)$$
$$= L_1 \cdot \dot{i}_1(t) + M\,\dot{i}_2(t) + L_2\,\dot{i}_2(t) + M\,\dot{i}_1(t)$$
$$= (L_1 + L_2 + 2M)\,\dot{i}(t)$$

First Order Circuits

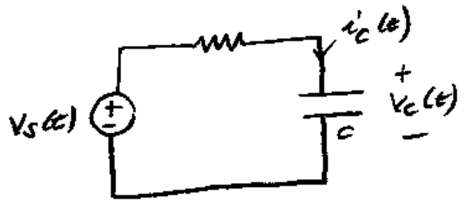
$$(D+\underbrace{\gamma}_{\text{constant real number}})\,\underbrace{x(t)}_{\text{unknown function}} = \underbrace{f(t)}_{\text{forcing term}}$$

$x(0) = x_0 \rightarrow$ the solution at $t=0$!

solution terminology:

Homogeneous, particular solutions
zero-input, zero-state solutions
transient, steady state solutions
complete solutions

RC series

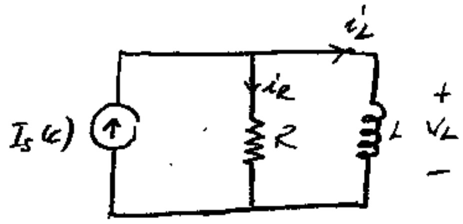


$V_C(0^-) = V_0$

KVL:

$$-V_s(t) + R\cdot i_C(t) + V_C(t) = 0 \qquad t \geq 0$$

$$\left(D+\frac{1}{RC}\right) V_C(t) = \frac{V_S(t)}{RC} \quad ; \; t \geq 0$$



KCL:

$$-I_S(t) + V_L/R + i_L = 0 \;,\quad t \geq 0$$

$$\left(D+\frac{R}{L}\right) i_L(t) = \frac{R}{L}\, i_s(t) \;,\quad t \geq 0$$

① Homogeneous, particular

$$\left(D+\frac{1}{RC}\right) V_C^h(t) = 0 \longrightarrow V_C^h(t) = \alpha e^{-t/RC} \;;\; t \geq 0 \;;\; \alpha \in \mathbb{R}$$

$$\left(D+\frac{1}{RC}\right) V_C^P(t) = V_S(t)/RC \quad ; \; t \geq 0$$

$\rightarrow$ Let's say $V_s(t) = \beta\cdot u(t) \Rightarrow V_c^P(t) = \beta\cdot u(t)$

$\rightarrow v_s(t) = \beta\cdot t\,u(t)$, $V_c^P(t) = A + Bt$

$$\left(B+\frac{A}{RC}\right) + \frac{Bt}{RC} = \frac{\beta t}{RC} \rightarrow \begin{array}{l} B=\beta \\ A=-\beta\cdot RC \end{array}$$

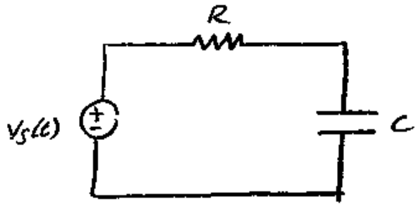
$$V_c^P(t) = \beta\,(t-RC) \;,\quad t \geq 0$$

Complete solution:

$V_c(t) = V_c^h(t) + V_c^P(t) \longrightarrow \alpha$ is set such that $V_c(0) = V_c(0^-) = V_0$

② Zero-input, zero-state solutions

Zero-state:



State: (state vector, state variables)

State variables allow us to exactly characterize the circuit.

$V_c(t)$, $I_c(t)$

Zero-state:

All state variables ($V_c(t)$) are equal to zero at $t=0$.

$V_c(0) = 0V \longrightarrow$ no initial energy in the circuit.

Zero-state solution:

$V_c^{zs}(t) = \dots \quad t \geq 0$

$\rightarrow v_s(t) = \beta u(t) \longrightarrow V_c^{zs}(t) = \beta(1 - e^{-t/RC})u(t) \quad , t \geq 0$

$V_c^{zs}(0) = 0V$

and $V_c^{zs}(t)$ satisfies $(D + \frac{1}{RC})V_c(t) = \frac{\beta}{RC} \quad , t \geq 0$

$\rightarrow v_s(t) = \beta t u(t)$

$V_c^{zs}(t) = \beta RC e^{-t/RC} + \beta(t - RC)$

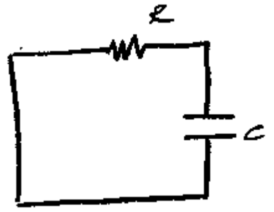
$V_c^{zs}(0) = 0V$

and $V_c^{zs}(t)$ satisfies $(D + \frac{1}{RC})V_c(t) = \frac{\beta t}{RC} \quad , t \geq 0$

Zero-input solution:

The solution when $V_s(t) = 0V$

(all independent sources are killed, i.e., no forcing term)



$V_c(0^-) = V_0$

$(D + \frac{1}{RC})V_c^{zi}(t) = 0$

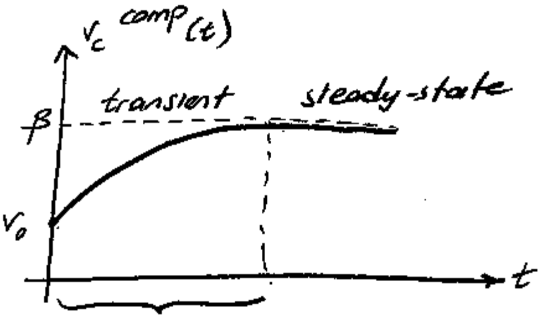
$V_c^{zi}(0) = V_0$

$V_c^{zi}(t) = V_0 \cdot e^{-t/RC}$

Complete solution: $V_c(t) = V_c^{zs}(t) + V_c^{zi}(t)$

③ Transient / steady State

$$V_c^{comp}(t) = \beta(1-e^{-t/RC}) + V_0(e^{-t/RC}),\quad t \geq 0 \qquad V_s(t) = \beta u(t)$$

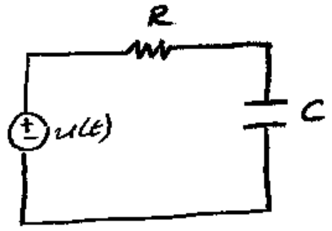


$\Delta t = 5RC$

transient part of solution → 0 as t → ∞

$$V_c^{comp}(t) = \underset{\substack{\downarrow \\ \text{steady state}}}{\beta} + \underbrace{(V_0-\beta)e^{-t/RC}}_{\text{transient}}$$

Step Response



$V_c(0) = V_0$

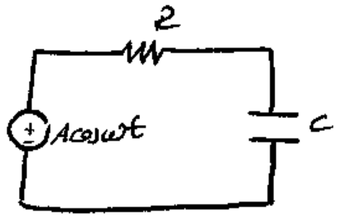
$$V^{step}(t) = 1 + (V_0 - 1)\cdot e^{-t/RC},\quad t \geq 0$$

Ramp Response

$$V_s(t) = r(t) = t\cdot u(t) \rightarrow V^{ramp}(t) = (t-RC) + (V_0+RC)e^{-t/RC},\quad t \geq 0.$$

$$V^{ramp}(0) = V_0$$

Sinusodial Response



$V_c(0^-) = 0V$

$$V^{zi}(t) = V_0\cdot e^{-t/RC},\quad t \geq 0$$

$$V^{zs}(t) = ?$$

$$\left(D + \frac{1}{RC}\right)V_c^{zs}(t) = \frac{V_s(t)}{RC} = \frac{A\cos\omega t}{RC}$$

$$V_{zs}(t) = \alpha\cdot\cos\omega t + \beta\cdot\sin\omega t$$

$$\frac{d}{dt}V_{zs}(t) = \beta\omega\cos\omega t - \alpha\omega\sin\omega t$$

$$V_{zs}(t) = \left(\beta\omega + \frac{\alpha\omega}{RC}\right)\cos\omega t + \left(-\alpha\omega + \frac{\beta\omega}{RC}\right)\sin\omega t$$

$$\begin{bmatrix} 1/RC & \omega \\ -\omega & 1/RC \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} A/RC \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{(1/RC)^2 + \omega^2} \begin{bmatrix} 1/RC & -\omega \\ \omega & 1/RC \end{bmatrix} \begin{bmatrix} A/RC \\ 0 \end{bmatrix}$$

$$V^{zs}(t) = \alpha \cdot \cos\omega t + \beta \cdot \sin\omega t - \alpha \cdot e^{-t/RC}, \quad t \geq 0$$

$$V^{zs}(0) = V_0$$

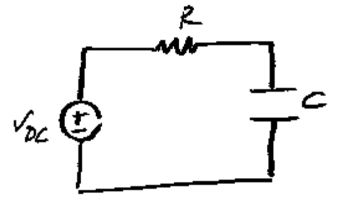
$$V^{sm}(t) = V^{zs}(t) + V^{zi}(t)$$

21.12.2009

Review: RC series (RL parallel)

- → particular, homogeneous
- → zero input, zero state
- → transient, steady state

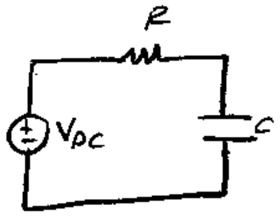
DC Excitation:



$V_C(0^-) = V_0$

complete response

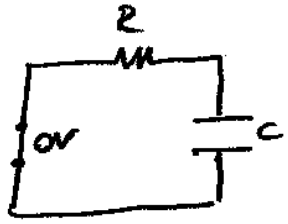
→



$V_C(0^-) = 0$

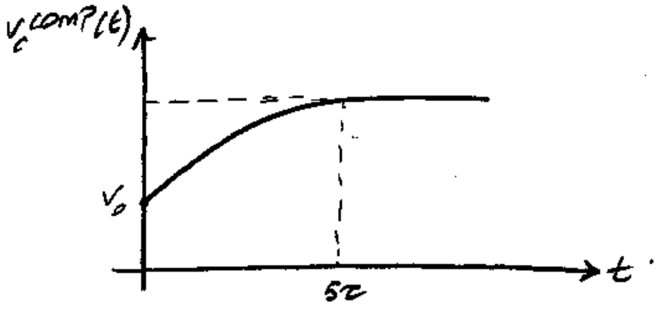
Zero-state response

and



$V_C(0^-) = V_0$

Zero-input response



$$V_C^{comp}(t) = (A + B \cdot e^{-t/RC})\, u(t)$$

for a DC input!

$$V_C(0^+) = V_0$$

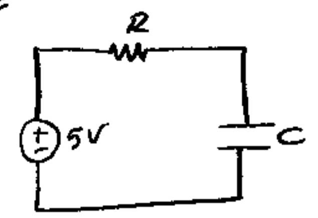
$$V_C(\infty) = V_{DC}$$

$$V_C(t) = V_C(\infty) + (V_C(0) - V_C(\infty))\, e^{-t/RC}, \quad t \geq 0$$

↓

General solution for DC input!

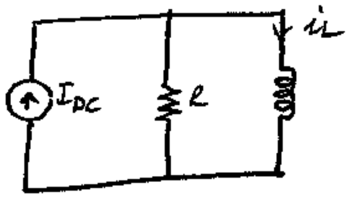
Ex

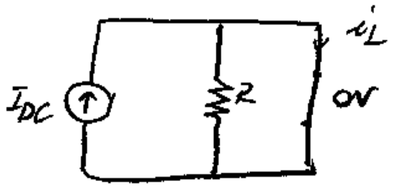


$$V_c(t) = [\,5 - 1\cdot e^{-t/RC}\,]\,u(t)$$

$V_c(0^-) = 4V$

R-L circuit





$i_L(\infty) = I_{DC}$
as $t \to \infty$

$i_L(0^-) = i_0$

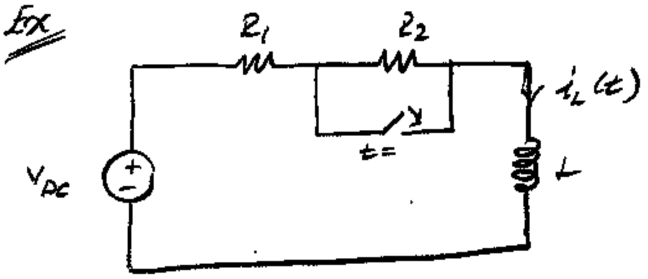
$$i_L(t) = I_L(\infty) + (\,I_L(0) - I_L(\infty)\,) \quad , \; t > 0$$

For DC inputs:

$$\text{Capacitor} \xrightarrow[t\to\infty]{} \text{open circuit}$$

$$\text{Inductor} \longrightarrow \text{short circuit.}$$

Ex

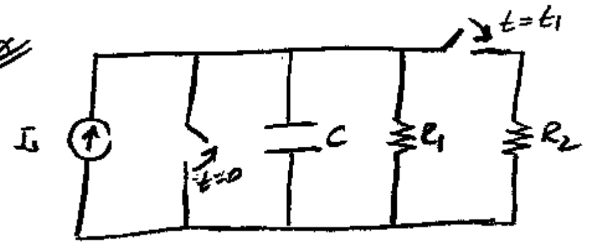


Assume that switch is open for a long time, then is closed at $t=0$.
Find $i_L(t)$ $t>0$.

$$I_L(0^-) = \frac{V_{DC}}{R_1+R_2} \qquad I_L(\infty) = \frac{V_{DC}}{R_1}$$

$$I_L(t) = V_{DC}\left[\frac{1}{R_1} + \left(\frac{1}{R_1+R_2} - \frac{1}{R_1}\right)e^{-tR_1/L}\right]$$

Ex



$V_c(0) = 0V$, $t \geq 0$

$0 < t < t_1$

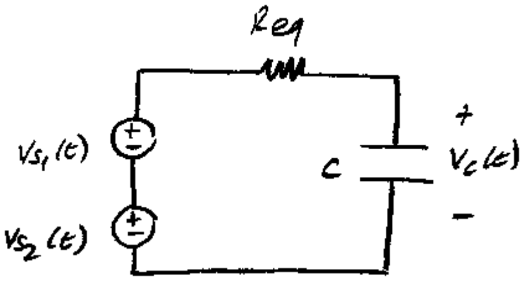
$$V_c(t) = V_c(\infty) + [V_c(0) - V_c(\infty)]\cdot e^{-t/\tau}$$

$$V_c(t) = I_s R_1\,[\,1 - e^{-t/R_1\cdot C}\,]$$

$t > t_1$

$$V_c(t) = I_s(R_1 \| R_2) + \left[\,I_s R_1(1 - e^{-t_1/R_1\cdot C}) - I_s(R_1\|R_2)\right] \times e^{-(t-t_1)/(R_1\|R_2)C}$$

Linearity of Zero-State Response

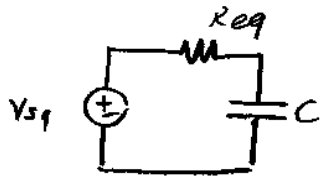


$$\left(D+\frac{1}{R_{eq}C}\right)V_c(t) = \frac{V_{s_1}(t)+V_{s_2}(t)}{R_{eq}\cdot C}$$

$V_c(0^-) = V_0$

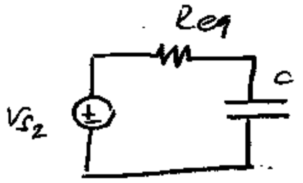
$V_c(0^-) = V_0$

zero-state response : $V_c(0) = 0$!!



$V_c^{\textcircled{1}}(t)$

$V_c^{\textcircled{1}}(0) = 0V$



$V_c^{\textcircled{2}}(t)$

$V_c^{\textcircled{2}}(0) = 0V$

$$V_c^{zs}(t) = V_c^{\textcircled{1}zs}(t) + V_c^{\textcircled{2}zs}(t)$$

From ① $\left(D+\frac{1}{R_{eq}C}\right)V_c^{\textcircled{1}}(t) = \frac{V_{s1}}{R_{eq}C}$

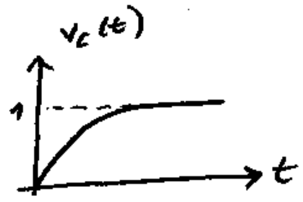
From ② $\left(D+\frac{1}{R_{eq}C}\right)V_c^{\textcircled{2}}(t) = \frac{V_{s2}}{R_{eq}C}$

$$\Rightarrow \left(D+\frac{1}{R_{eq}\cdot C}\right)\left[\underbrace{V_c^{\textcircled{1}}(t)+V_c^{\textcircled{2}}(t)}_{\text{solution of original differential equation}}\right] = \frac{V_{s1}(t)+V_{s2}(t)}{R_{eq}\cdot C}$$

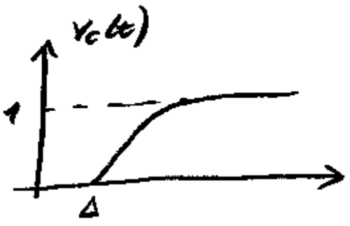
$$V_c(t) = V_c^{\textcircled{1}}(t) + V_c^{\textcircled{2}}(t)$$

Time Invariancy

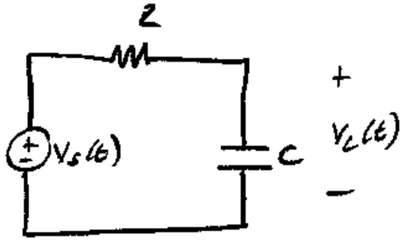
$V_{input} = u(t)$ ⇒

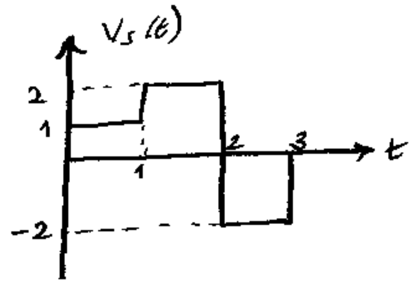


$V_{input} = u(t-\Delta)$ ⇒



Ex

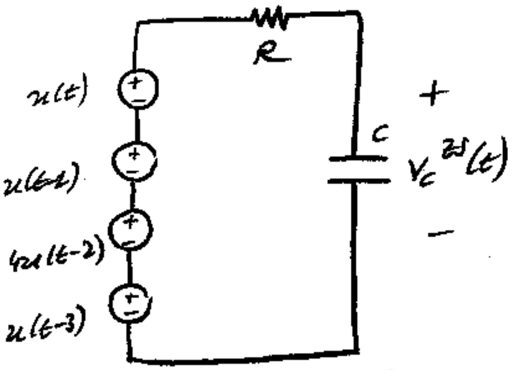




Find $V_c(t)$

$V_c(0) = 0V$

$V_s(t) = u(t) + u(t-1) - 4u(t-2) + 2u(t-3)$



$V_c(0) = 0V$

$V_c^{①}(t) = (1-e^{-t/RC})u(t)$

$V_c^{②}(t) = V_c^{①}(t-1)$

$= (1-e^{-(t-1)/RC})u(t-1)$ → Due to invariancy

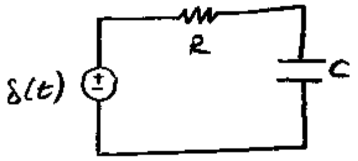
$V_c^{③}(t) = -4 \cdot V_c^{①}(t-2) \cdot u(t-2)$

$V_c^{④}(t) = 2\, V_c^{①}(t-3) \cdot u(t-3)$

$$\Rightarrow V_c(t) = (1-e^{-t/RC})u(t) + (1-e^{-(t-1)/RC})u(t-1) - 4 \cdot (1-e^{-(t-2)/RC}) \cdot u(t-2) + 2 \cdot (1-e^{-(t-3)/RC})u(t-3)$$

Impulse Response

(Always analyzed under zero-state conditions!)



$V_c(0^-) = 0V$

* When we consider impulse / unit / ramp response, we always talk about <u>zero state</u> solution, i.e., all initial conditions are zero.

Since $V_s(t) = \delta(t)$, $V_c(0^-) \neq V_c(0^+)$

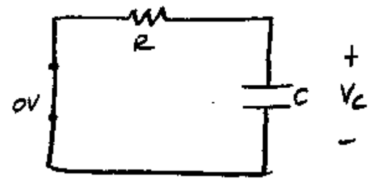
$0^- < t < 0^+$

$V_c(0^-) = 0V$

$i_c(t) = \frac{\delta(t)}{R}$, $0^- < t < 0^+$

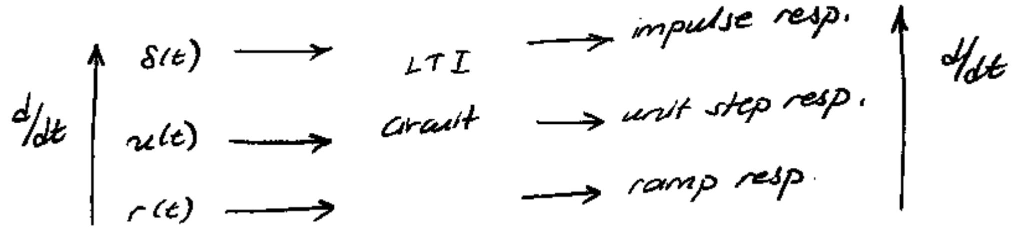
$$V_c(0^+) = V_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(\tau)d\tau = \frac{1}{RC}\int_{0^-}^{0^+} \delta(t)d\tau = \frac{1}{RC}$$

$t > 0^+$



$V_c(0^+) = \frac{1}{RC}$

$V_c(t) = \frac{1}{RC} \cdot e^{-t/\tau}, \quad \tau = RC, \quad t > 0^+$



unit-step response:

$$y^{\text{unit-step}}(t) = \int_{-\infty}^{t} y^{\text{impulse}}(t) \cdot dt$$

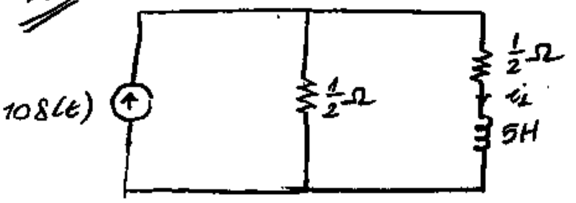
$$= \int_{-\infty}^{t} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t) \cdot dt \longrightarrow y^{\text{unit-step}}(t) = 0, \ t < 0$$

$$= \int_{0}^{t} \frac{1}{RC} \cdot e^{-t/RC} \cdot dt$$

$$= -e^{-t/RC} \Big|_{t=0}^{t} = 1 - e^{-t/RC}, \ t \geq 0.$$

$$y^{\text{unitstep}}(t) = (1 - e^{-t/RC}) \cdot u(t)$$

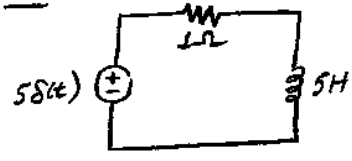
Ex



$i_L(0^-) = -2A$

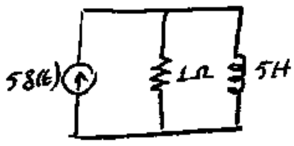
Find $i_L(t) \quad t \geq 0$.

ZS:



$i_L(0^-) = 0A$

$\equiv$



$i_L(0^+) = ?$

$i_L(0^-) = 0A$

$0^- < t < 0^+$

$i_{1\Omega}(t) = 5\delta(t) \longrightarrow V_{1\Omega}(t) = 5\delta(t)$

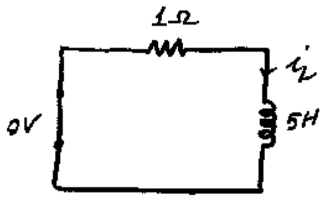
$$i_L(0^+) = \underset{0A}{\cancel{i_L(0^-)}} + \frac{1}{L}\int_{0^-}^{0^+} V_L(t)dt = \frac{1}{5}\int_{0^-}^{0^+} 5\delta(t)dt$$

$$= 1A$$

$\Rightarrow i^{ZS}(t) = 1 \cdot e^{-t/\tau}, \ \tau = \frac{L}{R} = 5s \quad t > 0^+$

or ⇒ $i_L(t) = \underset{0}{\cancel{i_L(\infty)}} + \left(i_L(0^+) - \underset{0}{\cancel{i_L(\infty)}} \right) e^{-t/\tau} = 1 \cdot e^{-t/\tau}$

ZI



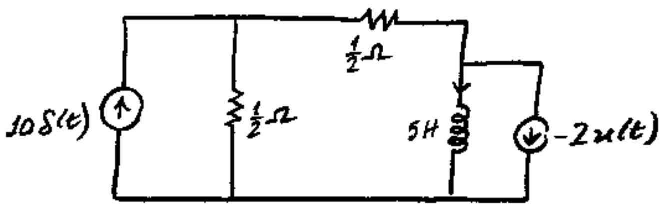
$i_L(0^-) = -2A$

$$i_L^{zs}(t) = -2 \cdot e^{-t/\tau} \cdot u(t)$$

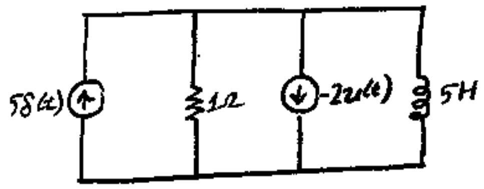
$$\tau = L/R = 5s$$

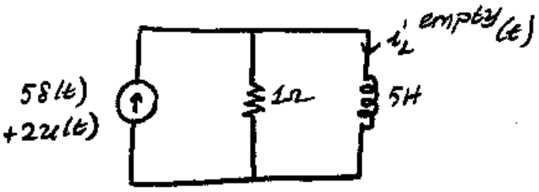
Complete: $i_L^{complete}(t) = i_L^{zs}(t) + i_L^{zi}(t) = -e^{-t/\tau} \cdot u(t)$

2nd way: with initial condition model.



$i_L^{emp.}(0^-) = 0A$





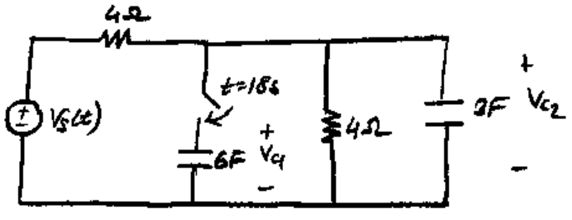
$$i_L^{unitstep}(t) = (1 - e^{-t/\tau}) u(t) \quad , \tau = 5$$

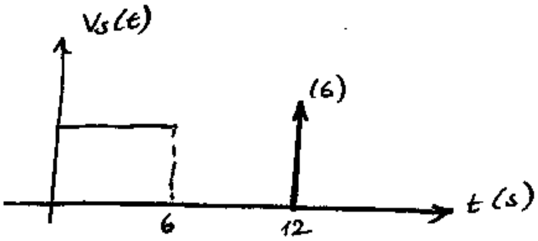
$$i_L^{empty}(t) = (2 + 5D)(1 - e^{-t/\tau})$$

$$i_L^{empty}(t) = (2 - e^{-t/\tau}) u(t)$$

$$i_L(t) = i_L^{empty}(t) - 2u(t) = -e^{-t/\tau} u(t).$$

Ex.



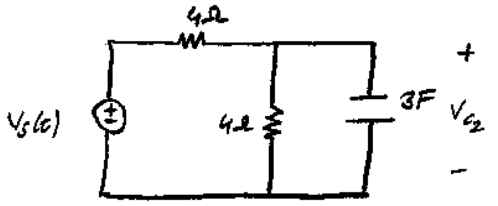


$V_{c_1}(0^-) = -2V$

$V_{c_2}(0^-) = 7V$

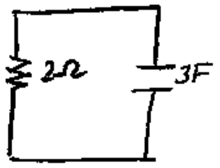
$V_{c_2}(t) = ?$

For $t < 18$ (Before switch closes)



$V_{c_2}(0^-) = 7V$

ZI ⇒

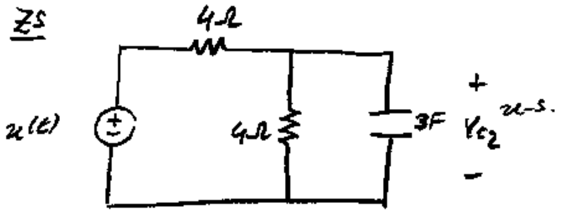


$V_{c_2}(0^-) = 7V$

$$V_{c_2}^{zi}(t) = 7 \cdot e^{-t/\tau}$$

$$\tau = 6s$$

ZS



$$V_{c_2}^{\text{unitstep}}(t) = \left(\frac{1}{2} - \frac{1}{2}\cdot e^{-t/\tau}\right)u(t)$$

$\tau = 6s$

$$V_{c_2}^{zs}(t) = 6\,V_{c_2}^{u.s.}(t) - 6\,V_{c_2}^{u.s.}(t-6) + 6\cdot\frac{d}{dt}\left[V_{c_2}^{u.s.}(t-12)\right]$$

$$= 3\left(1-e^{-t/\tau}\right)\cdot u(t) - 3\left(1-e^{-(t-6)/\tau}\right)u(t-6)$$

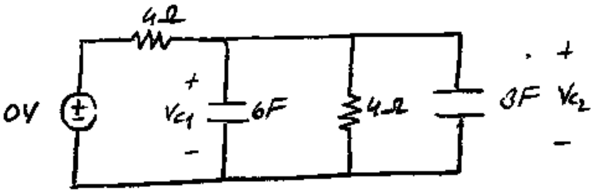
$$+\underbrace{3\,\frac{d}{dt}\left[\left(1-e^{-(t-12)/\tau}\right)\cdot u(t-12)\right]}_{\frac{1}{\tau}\,e^{-(t-12)/\tau}\cdot u(t-12) + \underbrace{\left(1-e^{-(t-12)/\tau}\right)}_{0 \text{ at } t=12}\cdot\delta(t-12)}$$

$$V_{c_1}^{zs}(t) = 3\left(1-e^{-t/\tau}\right)u(t) - 3\left(1-e^{-(t-6)/\tau}\right)u(t-6) + \frac{1}{2}\,e^{-(t-12)/\tau}\cdot u(t-12)$$

At $t = 18^-$

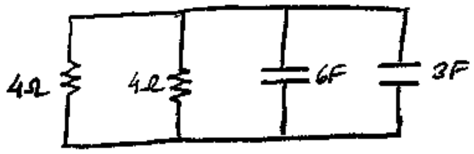
$$V_{c_2}(t) = V_{c_2}^{zi}(t) + V_{c_2}^{zs}(t)\Big\downarrow_{t=18^-} = 2.14\text{ V.}$$

$t > 18$



$V_{c_2}(18^-) = 2{,}14\text{ V}$

$V_{c_1}(18^-) = V_{c_1}(0^-) = -2\text{V}$



total charge at $t = 18^-$

$$Q_T(18^-) = 6F\cdot V_{c_1}(18^-) + 3F\cdot V_{c_2}(18^-)$$

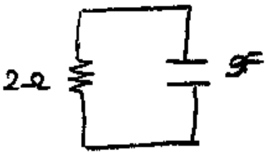
total charge at $t = 18^+$

$$Q_T(18^+) = (6F + 3F)\cdot V_{com}(18^+)$$

$$Q_T(18^-) = Q_{Tot}(18^+)$$

$$\Rightarrow V_{common}(18^+) = \frac{6\cdot(-2) + 3\cdot 2{,}14}{9} = -1{,}07\text{V}$$

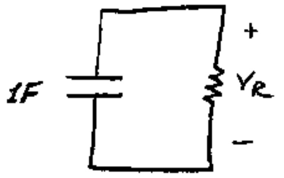
$t \geqslant 18^+$



$$V_{common}(t) = -1{,}07\,e^{-(t-18)/18}\cdot u(t-18)\ , \quad t \geqslant 18$$

Time Varying circuits with Nonlinear elements

Ex



$V_c(0) = 1V$

a) $R = 1\Omega$

b) $R(t) = \frac{1}{1+0,5\cos t}\ \Omega$

c) Nonlinear element: $i_R = v_R^2$

a) $V_c(t) \cdot 1 \cdot e^{-t/\tau} \cdot u(t)$, $\tau = R \cdot C = 1s$

b) $V_c(t) + R(t) \cdot i_c(t) = 0$

$$V_c(t) + \frac{C}{1+0,5\cos t} \cdot \frac{dV_c(t)}{dt} = 0 \quad ; \quad V_c(0) = 1V$$

$$\frac{dV_c}{dt} = -(1+0,5\cos t) \cdot V_c$$

$$\frac{dV_c}{V_c} = -(1+0,5\cos t)\,dt \longrightarrow \ln(V_c(t)) = -t - 0,5\sin t + c_1$$

$$V_c(t) = c_0 \cdot e^{-(t+0,5\sin t)}$$

At $t=0 \longrightarrow V_c(0) = c_0 = 1V \longrightarrow c_0 = 1$

$$V_c(t) = e^{-(t+0,5\sin t)} u(t)$$

c) $i_c(t) = C \cdot \frac{dV_c(t)}{dt} = -i_R(t) = -V_R^2(t) = -V_c^2(t)$

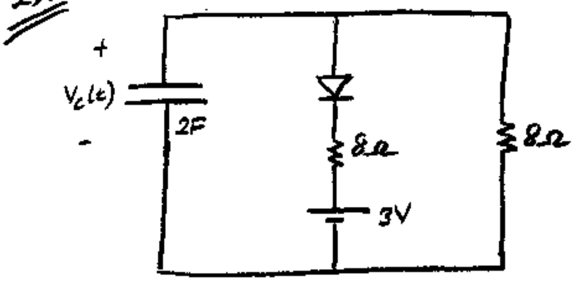
$$\frac{dV_c(t)}{V_c^2(t)} = -\frac{1}{C}\,dt$$

$$\frac{1}{V_c(t)} = t + c_0 \longrightarrow V_c(t) = \frac{1}{t+c_0}$$

At $t=0 \longrightarrow V_c(0) = \frac{1}{c_0} = 1 \longrightarrow c_0 = 1$

$$V_c(t) = \frac{1}{t+1}\ u(t)$$

Ex

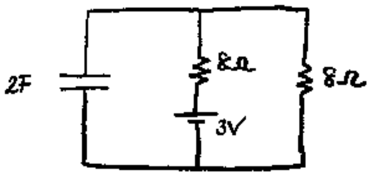


$V_c(0^-) = 10V$

$V_c(t) = ?$

a) when $V_c(t) \geq 3V$, diode is on.

At $t=0^+$, $V_c(0^+) = 10V$ → diode is on



$\tau = 2F \cdot (8\Omega \parallel 8\Omega) = 8s$

$V_c(\infty) = \frac{3}{2}V$

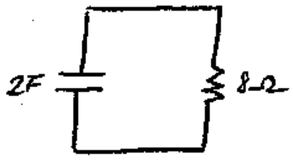
$V_c(t) = \left(\frac{3}{2} + \frac{17}{2} \cdot e^{-t/8}\right) u(t)$

b) when $V_c(t) < 3V \longrightarrow$ diode is OFF

$V_c(t_x) = \frac{3}{2} + \frac{17}{2} \cdot e^{-tx/8} = 3$

$e^{-tx/8} = \frac{3}{17} \longrightarrow t_x = 8 \cdot \ln \frac{17}{3}$ s

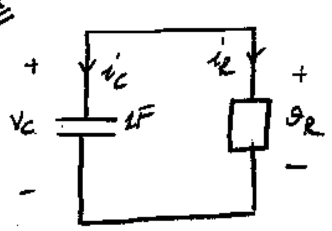
After $t_x = 8 \cdot \ln 17/3$ sec.

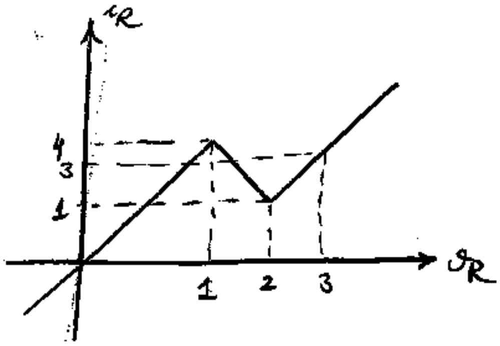


$V_c(t) = 3 \cdot e^{-(t-tx)/16} \cdot u(t-tx)$

$V_c(t\dot{x}) = 3V$

Ex





Find $V_c(t)$

$V_c(0) = 4V$

$v_R(0^+) = V_c(0^+) = 4V$

$C \cdot \frac{dV_c}{dt} = -i_R \longrightarrow$ $i_R > 0$ when $v_R > 0$

$i_R < 0$ when $v_R < 0$

1F

$\frac{dv_c(t)}{dt}$ is a negative number when $V_R(t) = v_c(t) > 0$

when $V_c(t) > 0$; then $V_c(t)$ is decreasing

a) $2 < V_c(t) < 4 \longrightarrow i_R = 2\left(v_R - \frac{3}{2}\right)$

$\frac{dV_c(t)}{dt} = -2\left(V_c(t) - \frac{3}{2}\right)$, $V_c(0^+) = 4V$

$(D+2)\,V_c = 3 \longrightarrow v_c(t) = \alpha \cdot e^{-2t} + \frac{3}{2}$

$V_c(0^+) = \alpha + \frac{3}{2} = 4 \longrightarrow \alpha = \frac{5}{2}$

$\Rightarrow V_c(t) = \frac{5 \cdot e^{-2t} + 3}{2}\, u(t)$

$V_c(t_x) = 2V \longrightarrow \frac{5e^{-2t}+3}{2} = 2 \longrightarrow t_x = \frac{\ln 5}{2} s \cong 0{,}8 s$

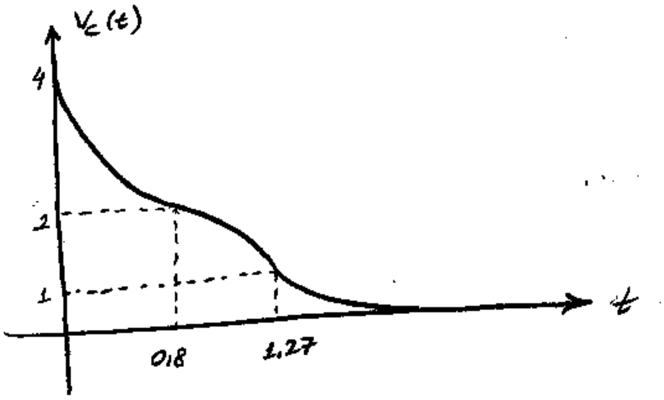
b). $1 < V_c(t) < 2$; $\frac{dV_c(t)}{dt} = -\left(-3\left(V_c - \frac{7}{3}\right)\right)$, $v_c(t_x^+) = 2V$!

$\longrightarrow V_c(t) = -\frac{1}{3} e^{3(t-t_x)} + \frac{7}{3}$, $t > t_x$

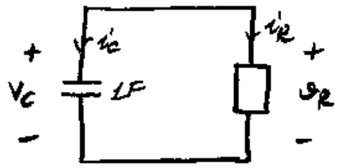
$V_c(t_y) = 1V \longrightarrow t_y = 1{,}27$ sec.

c) $0 < V_c < 1$; $\frac{dV_c(t)}{dt} = -4V_c$; $V_c(t_y) = 1V$

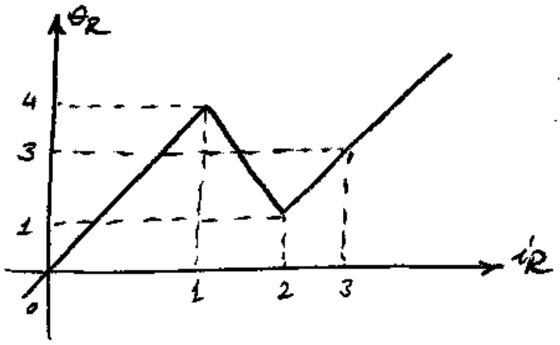
$\longrightarrow V_c(t) = e^{-4(t-t_y)}$, $t > t_y$



Ex



$v_c(0) = 4.1V$



Current controlled

$\frac{dV_c(t)}{dt} = -i_R \longrightarrow i_R > 0 \longrightarrow V_c(t)$ decreases ★!▷

a) $i_R > 2 \longrightarrow \frac{dV_c(t)}{dt} = -\frac{v_c+3}{2}$; $V_c(0) = 4.1V$

$V_c(t) = \alpha \cdot e^{-t/2} - 3$

$\rightarrow V_c(t) = 7.1\, e^{-t/2} - 3$

$V_c(t_x) = 1V \longrightarrow t_x = 2 \cdot \ln\frac{7}{4}$

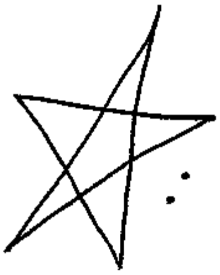
b) $V_c(t_x^+) = 1V$

Since i_R is positive $V_c(t)$ cannot increase ★★.

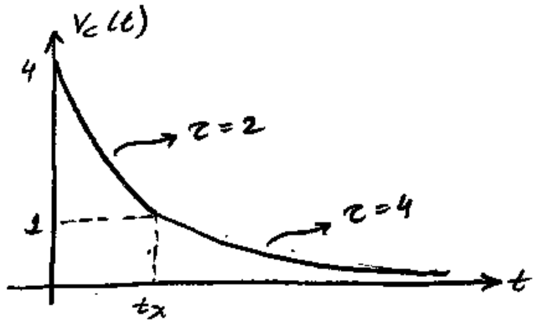
It jumps to the leftmost segment ★★★

$$i_R < 0{,}25 \longrightarrow \frac{dV_c}{dt} = -\frac{1}{4}\,\vartheta_c(t) \;,\quad \vartheta_c(t_x) = 1V$$

$$\vartheta_c(t) = e^{-(t-t_x)/4} \;,\quad t > t_x$$

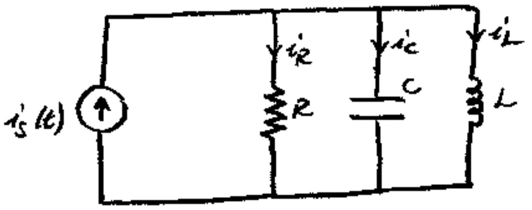


Current Controlled



Second Order Circuits

Parallel RLC



$i_L(0^-) = I_0$

$V_c(0^-) = V_0$

$$i_s(t) = i_R(t) + i_c(t) + i_L(t)$$

$$i_s(t) = i_R(t) + i_L(t) + C \cdot \frac{dV_c(t)}{dt}$$

$$i_s(t) = \frac{V_c(t)}{R} + i_L(t) + C \cdot \frac{dV_c(t)}{dt}$$

$$\downarrow$$

$$i_L(0^-) + \frac{1}{L}\underbrace{\int_0^t V_c(\tau)\,d\tau}_{D^{-1} V_c(t)}$$

$$\frac{d\, i_s(t)}{dt} = \frac{1}{R}\frac{dV_c(t)}{dt} + \frac{1}{L} V_c(t) + C \cdot \frac{d^2 V_c(t)}{dt}$$

$$\left(CD^2 + \frac{1}{R}D + \frac{1}{L}\right) V_c(t) = \frac{d\, i_s(t)}{dt} \longrightarrow \text{forcing term}$$

$V_c(0) = V_0$

$\frac{dV_c(0)}{dt} = ?$ → need to be found from given initial conditions

$$\left(D^2 + \frac{1}{RC} D + \frac{1}{LC}\right) V_c(t) = \frac{1}{C} \cdot \frac{d\, i_s(t)}{dt} \longrightarrow \text{Parallel RLC}$$

Write a first order matrix diff. eqn. for RLC circuit:

state variables: $v_c(t)$, $I_L(t)$

$$\frac{d v_c(t)}{dt} = \alpha\, v_c(t) + \beta \cdot I_L(t) + \gamma \cdot i_s(t) \longrightarrow$$ a linear combination of state variables and input

$$C \cdot \dot{v}_c = i_s - i_R - i_L$$

$$\Rightarrow \frac{d v_c(t)}{dt} = -\frac{1}{RC} v_c(t) - \frac{1}{C} I_L(t) + \frac{1}{C} i_s(t) \longrightarrow (1)$$

$$L \cdot \frac{d I_L(t)}{dt} = v_L(t) = v_c(t)$$

$$\frac{d I_c(t)}{dt} = \frac{v_c(t)}{L} \longrightarrow (2)$$ second state eqn.

$$\begin{bmatrix} d v_c(t)/dt \\ d I_L(t)/dt \end{bmatrix} = \begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} v_c(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i_s(t)$$

$$v_c(0^-) = V_0$$
$$I_L(0^-) = I_0$$

Type of Zero Input Solutions

$$(D^2 + 2\alpha D + w_0^2)\, v_c(t) = 0 \longrightarrow$$ zero-input (no forcing term)

Parallel RLC $\rightarrow$ $2\alpha = 1/RC$

$$w_0^2 = 1/LC$$

characteristic polynomial

$$s^2 + 2\alpha s + w_0^2 = 0$$

$$s_{1,2} = \frac{-2\alpha \mp \sqrt{4\alpha^2 - 4w_0^2}}{2} \Rightarrow s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2}$$

$$v_c(t) = c_1 \cdot e^{s_1 t} + c_2 \cdot e^{s_2 t}$$

<u>3 cases for s_1 and s_2</u>

1) Real and distinct ($\Delta > 0$)
2) Real and the same ($\Delta = 0$)
3) Imaginary roots ($\Delta < 0$)

① Real and distinct → OVERDAMPED system

$$v_c(t) = c_1 \cdot e^{s_1 t} + c_2 \cdot e^{s_2 t}$$

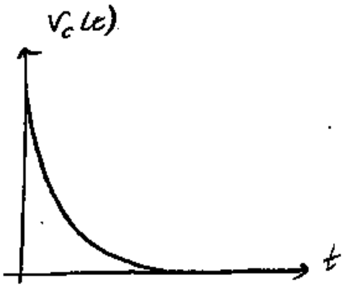
② Real and same → CRITICALLY DAMPED system

$$V_c(t) = c_1 \cdot e^{s_0 t} + c_2 \cdot t \cdot e^{s_0 t}$$

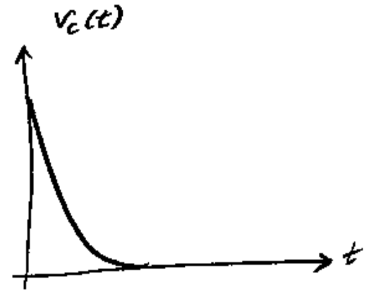
③ Imaginary roots → UNDERDAMPED

$$s_{1,2} = \alpha \mp \mu j \qquad j^2 = -1$$

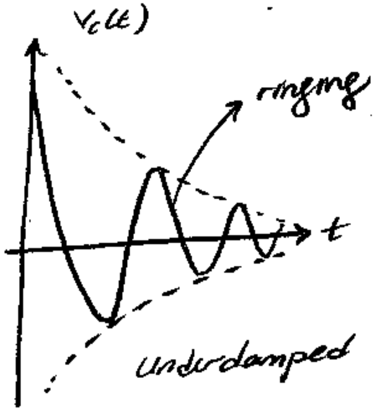
$$\begin{aligned} V_c(t) &= c_1 \cdot e^{s_1 t} + c_1^* \cdot e^{s_1^* t} \\ &= 2 \cdot Re\left\{ c_1 \cdot e^{s_1 t} \right\} = 2\, Re\left\{ c_1\, e^{-\alpha t} \cdot e^{\mu j t} \right\} \\ &= 2 e^{-\alpha t} \cdot Re\left\{ \underset{\substack{\downarrow \\ c_R + J c_i}}{c_1} \cdot e^{\mu j t} \right\} = 2 \cdot e^{-\alpha t}\, Re\left\{ (c_R + J \cdot c_i)(\cos \mu t + J \cdot \sin \mu t) \right\} \\ &= 2 \cdot e^{-\alpha t} \cdot Re\left\{ (c_R \cos \mu t - c_i \cdot \sin \mu t) + J(c_i \cos \mu t + c_R \cdot \sin \mu t) \right\} \\ &= 2 e^{-\alpha t} \cdot (c_R \cdot \cos \mu t - c_i \sin \mu t) \end{aligned}$$



Overdamped
↓
combination of two decaying exponents

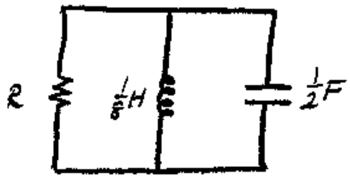


Critically damped



Underdamped

Ex



$i_L(0^-) = -4A$
$v_c(0^-) = 5V$

a) $R = \frac{1}{5}\Omega$

$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) v_c(t) = \frac{1}{C} D\, i_s(t)$$

$$\underset{\displaystyle (D^2 + 2\alpha D + \omega_0^2)}{\overset{\downarrow}{(D^2 + 10D + 16)}}\, v_c(t) = 0$$

α: damping ratio
ω_0 : natural frequency

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

a) $R = \frac{1}{5}\Omega \longrightarrow$ two distinct roots, overdamped response $(|\alpha| > \omega_0)$

$$\left.\begin{array}{l} |\alpha| = 5 \\ \omega_0 = 4 \end{array}\right\} |\alpha| > \omega_0 \longrightarrow \text{overdamped response}$$

$$v_c(t) = (A \cdot e^{-2t} + B \cdot e^{-8t})u(t) \quad , \quad t \geq 0$$

$$V_c(0) = 5V \longrightarrow A + B = 5$$

$$\dot{v}_c(0^+) = \frac{i_c(0^+)}{C} = \frac{-i_L(0^+) - i_R(0^+)}{C} = \frac{4 - 25}{1/2} = -42$$

$$\longrightarrow A + 4B = 21$$

$$\Longrightarrow A = \frac{-1}{3} \,, \quad B = \frac{16}{3}$$

$$V_c(t) = \left(-\frac{1}{3} e^{-2t} + \frac{16}{3} \cdot e^{-8t}\right)u(t)$$

b) Critically Damped: $|\alpha| = \omega_0 \longrightarrow \alpha = 4 \rightarrow R = \frac{1}{4}\Omega$

$$(D^2 + 2\alpha D + \omega_0^2) = (D + \omega_0)^2$$

$$V_c(t) = (5 - 12t)\, e^{-4t}\, u(t)$$

c) $R = \frac{1}{3}\Omega \longrightarrow |\alpha| < \omega_0$ Two complex roots for char. equation

$$s_{1,2} = \{-3 \mp j\sqrt{7}\}$$

$$v_c(t) = c \cdot e^{s_1 t} + c^* \cdot e^{s_2 t} = c \cdot e^{s_1 t} + c^* e^{s_1^* t}$$

$$= \|c\| \cdot e^{J\theta} e^{s_1 t} + \|c^*| \cdot e^{J\theta^*} \cdot e^{s_1^* t}$$

$\angle c = \theta$
$\hookrightarrow$ angle of c

$$= \|c\| \cdot \left[e^{s_1 t + J\theta} + e^{s_1^* t + J\theta^*} \right]$$

$$= \|c\| \cdot \left[e^{-3t + j\sqrt{7}t + J\theta} + e^{-3t - j\sqrt{7}t - J\theta^{\cdot\cdot}} \right]$$

$$= \|c\| \cdot e^{-3t} \left[e^{J(\sqrt{7}t + \theta)} + e^{-J(\sqrt{7}t + \theta)} \right]$$

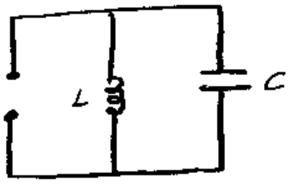
$V_c(t) = \|c\| \cdot e^{-3t} \cdot 2\cos(\sqrt{7}t + \theta)$

$\downarrow$ real number in radians

$V_c(0^+) = 5V$
$\dot{V}_c(0^+) = -22$
$\Big\} \quad V_c(t) = e^{-3t} \cdot 4\sqrt{2}\cos\left(\sqrt{7}t + \tan^{-1}\frac{\sqrt{7}}{5}\right)$

$\rightarrow$ underdamped response.

d) $R = \infty$



$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)V_c(t) = 0$$

Lossless Systems!

$2\alpha \rightarrow 0$ as $R \rightarrow \infty$

$$\left(D^2 + \frac{1}{LC}\right)V_c(t) = 0$$

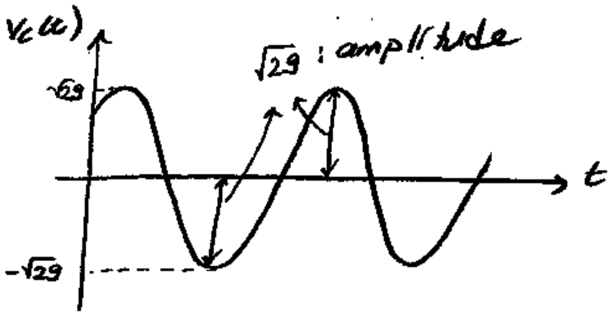
$(D^2+16)V_c(t) = 0$

$V_c(t) = c \cdot e^{-j4t} + d \cdot e^{j4t}$

$= A \cdot \cos(4t) + B\sin(4t)$

$= C \cdot \cos(4t + \theta) \quad ; \quad \theta = \angle c$

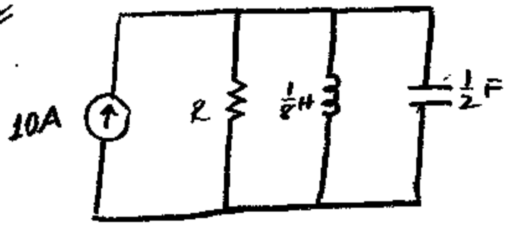
$V_c(t) = 5\cos 4t + 2\sin 4t$



Previously we have examined zero-input solution for parallel R-L-C circuit.

Ramp / Unit-step / Impulse response (Elementary zero-state responses)

Ex

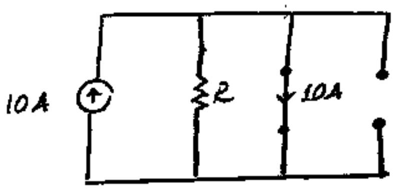


a) $R = \frac{1}{5}\,\Omega$

$V_c^{zi}(t) = -\frac{1}{3}e^{-2t} + \frac{16}{3}e^{-8t}$

$(D^2 + 10D + 16)V_c(t) = \frac{1}{C}\cancel{\dot{i}_s(t)} \rightarrow 0$

As $t \to \infty$



$i_L(\infty) = 10A$

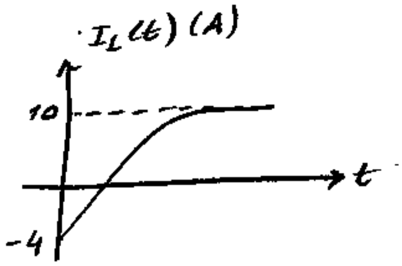
$V_C(\infty) = 0V$

$v_c(t) = \alpha + \alpha \cdot e^{-2t} + \beta \cdot e^{-8t} \longrightarrow V_C(\infty) = 0 \rightarrow \alpha = 0$

$V_c(0^+)$ & $\dot{v}_c(0^+)$ gives α & β.

$v_c(t) = 3 \cdot e^{-2t} + 2 \cdot e^{-8t}$

$$I_L(t) = \underset{\substack{\downarrow \\ t=\infty \text{ analysis}}}{10} - \underbrace{12 e^{-2t} - 2 \cdot e^{-8t}}_{t=0^+ \text{ analysis.}}$$



* Ramp Response

$i_s(t) = r(t)$

$$(D^2 + 2\alpha D + \omega_0^2)\, v_c(t) = \frac{1}{C} \cdot \overbrace{\frac{d}{dt} i_s(t)}^{u(t)} \quad \ldots\ldots \quad (1)$$

$$\left.\begin{aligned} v_c(0^-) &= 0 \\ \dot{v}_c(0^-) &= 0 \end{aligned}\right\} \text{zero-state!}$$

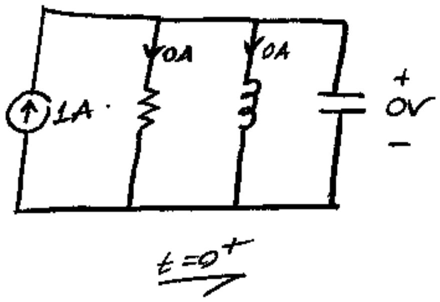
$$\longrightarrow \quad (D^2 + 2\alpha D + \omega_0^2)\, v_c(t) = \frac{1}{C} \quad , \quad t \geq 0$$

$$v_c(t) = A + B \cdot e^{-\lambda_1 t} + C \cdot e^{-\lambda_2 t}$$

$$\left.\begin{aligned} v_c(0^+) &= v_c(0^-) \\ \frac{d}{dt} v_c(0^+) &= \frac{d}{dt} v_c(0^-) \end{aligned}\right\} \text{since no impulse at RHS of (1)}$$

* Unit-Step Response

$$(D^2 + \underset{\substack{\downarrow \\ \text{damping ratio}}}{2\alpha D} + \underset{\substack{\downarrow \\ \text{natural (resonance) frequency}}}{\omega_0^2})\, v_c(t) = \frac{1}{C} \cdot \overbrace{\dot{i}_s(t)}^{\delta(t)}$$



$v_c(0^+) = 0V$

$$\frac{dv_c(0^+)}{dt} = \frac{1}{C}$$

Then for $t \geq 0 \longrightarrow (D^2+2\alpha D+\omega_0^2)\, V_c(t) = 0$

$$V_c(0^+) = 0V, \quad \frac{d}{dt} V_c(0^+) = \frac{1}{C}$$

$$(D^2+2\alpha D+\omega_0^2)\, V_c(t) = \frac{1}{C}\delta(t)$$

<u>claim</u> $\ddot{v}_c(t) = \frac{1}{C}\delta(t)$

$$\int_{0^-}^{0^+} (D^2+2\alpha D+\omega_0^2)\, V_c(t)\,dt = \int_{0^-}^{0^+} \frac{1}{C}\delta(t)\cdot dt$$

$$\underset{\downarrow}{\int_{0^-}^{0^+} \ddot{V}_c(t)\,dt} + 2\alpha \int_{0^-}^{0^+} \dot{V}_c(t)\,dt + \omega_0^2 \int_{0^-}^{0^+} V_c(t)\,dt = \frac{1}{C}\int_{0^-}^{0^+} \delta(t)\,dt$$

has an impulse at $t=0$

$$\left(\dot{V}_c(0^+) - \dot{V}_c(0^-)\right) + 2\alpha\left(V_c(0^+) - V_c(0^-)\right) + \omega_0^2 \int_{0^-}^{0^+} V_c(t)\,dt = \frac{1}{C}$$

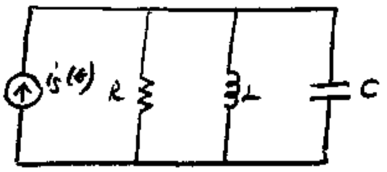
$$\Longrightarrow \dot{V}_c(0^+) + \underset{\searrow}{\cancelto{0}{2\alpha\,(0-0)}} + \underset{\downarrow}{\cancelto{0}{\omega_0^2\,(0-0)}} = \frac{1}{C}$$

does not contain an impulse

$$\Rightarrow \dot{V}_c(0^+) = \frac{1}{C}$$

6.01.2010

Impulse Response

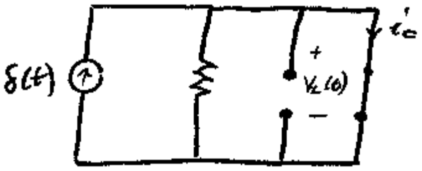


$$\left(D^2+2\alpha D+\omega_0^2\right) V_c(t) = \frac{1}{C}\dot{i}_s(t)$$

$$\begin{array}{lcl} V_c(0^-) = 0V & & V_c(0^-) = 0V \\ & \longleftrightarrow & \\ \dot{V}_c(0^-) = 0 & & I_L(0^-) = 0A \end{array}$$

$$i_s(t) = \delta(t) \longrightarrow \left(D^2+2\alpha D+\omega_0^2\right) V_c(t) = \frac{1}{C}\dot{\delta}(t)$$

$0^- < t < 0^+$



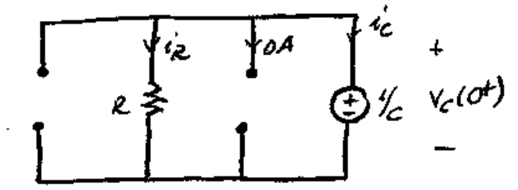
$0^- < t < 0^+$

$$i_c(t) = \delta(t) \longrightarrow V_c(0^+) = \frac{1}{C}\int_{0^-}^{0^+} \delta(t)\,dt = \frac{1}{C}$$

$$V_L(t) = 0V \longrightarrow 0^- < t < 0^+$$

$$I_L(0^+) = \frac{1}{L}\int_{0^-}^{0^+} V_L(t)\,dt = 0.$$

At $t = 0^+$.



$$i_R(0^+) = \frac{1}{CR}$$

$$i_c(0^+) = \frac{-1}{CR}$$

$$C \cdot \dot{v}_c(0^+) = -1/CR$$

$$\Rightarrow \boxed{\dot{v}_c(0^+) = -\frac{1}{C^2R}}$$

For $t > 0^+$

$$(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = 0$$

$$V_c(0^+) = \frac{1}{C}$$

$$\dot{V}_c(0^+) = \frac{-1}{C^2 \cdot R}$$

$$V_c^{\,impulse}(t) = h(t) = A \cdot e^{\lambda_1 t} + B \cdot e^{\lambda_2 t}$$

$$(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = \frac{1}{C}\dot{\delta}(t) \quad \dots \ (1)$$

$$\ddot{V}_c(t) = \frac{1}{C}\dot{\delta}(t) \longrightarrow V_c(t) = \frac{1}{C}u(t).$$

Integrating (1) between 0^- & 0^+

$$\int_{0^-}^{0^+} \ddot{V}_c(t)\,dt + 2\alpha \int_{0^-}^{0^+} \dot{V}_c(t)\,dt + \cancelto{0}{\omega_0^2 \int_{0^-}^{0^+} V_c(t)\,dt} = \frac{1}{C}\int_{0^-}^{0^+} \dot{\delta}(t)\,dt$$

$$\left[\dot{V}_c(0^+) - \cancelto{0}{\dot{V}_c(0^-)}\right] + 2\alpha\left[V_c(0^+) - \cancelto{0}{V_c(0^-)}\right] = 0$$

$$\dot{V}_c(0^+) + \frac{1}{RC} V_c(0^+) = 0$$

$$(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = \frac{1}{C} \cdot D\,\delta(t)$$

$$D^{-1}(D^2 + 2\alpha D + \omega_0^2)\, V_c(t) = \frac{1}{C} D^{-1} D\,\delta(t)$$

$$(D + 2\alpha + \omega_0^2 D^{-1})\, V_c(t) = \frac{1}{C}\delta(t) \quad \dots \ (2)$$

Integrate (2) between 0^- & 0^+

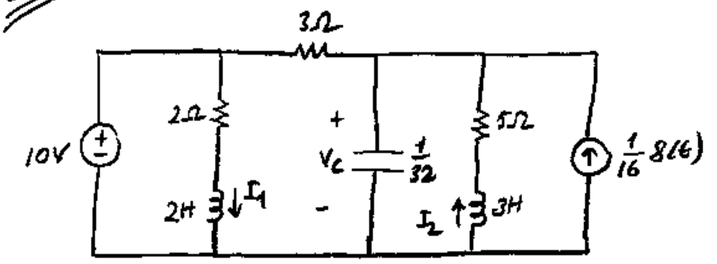
$$\int_{0^-}^{0^+} \dot{V}_c(t)\,dt + 2\alpha \cancelto{0}{\int_{0^-}^{0^+} V_c(t)\,dt} + \omega_0^2 \cancelto{0}{\int_{0^-}^{0^+}\left(\int_{0}^{t} V_c(\tau)\,d\tau\right)dt} = \frac{1}{C}$$

$$V_c(0^+) - V_c(0^-) = \frac{1}{C}$$

$$\boxed{V_c(0^+) = \frac{1}{C}}$$

$$\boxed{\dot{V}_c(0^+) = \frac{-1}{RC^2}}$$

Ex.



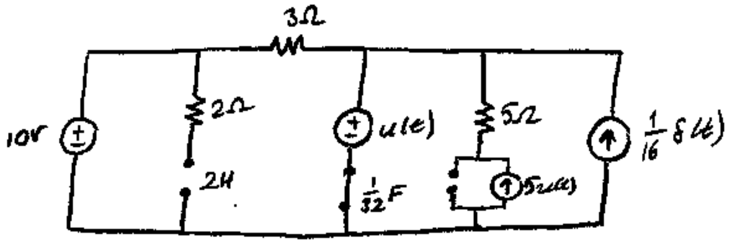
$V_c(0^-) = 1V$

$I_1(0^-) = 0A$

$I_2(0^-) = 5A$

Find all currents at $t=0^+$ and $t \to \infty$

$0^- < t < 0^+$



$$i_c(t) = \frac{10-1}{3} + 5 + \frac{1}{16}\delta(t) = 8 + \frac{1}{16}\delta(t) \quad , \; 0^- < t < 0^+$$

$$v_1^{2H}(t) = 10V \;;$$

$$v_2^{3H}(t) = 0-26 = -26V$$

$$v_C^{emp}(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i_c(t)dt = 2V$$

$$I_L^{emp-2H}(0^+) = \frac{1}{L}\int_{0^-}^{0^+} v^{2H}(t)\,dt = 0$$

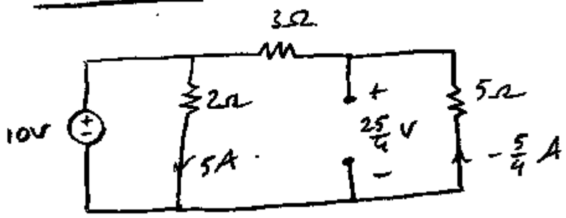
$$I_L^{emp-3H}(0^+) = \frac{1}{L}\int_{0^-}^{0^+} v^{3H}(t)\,dt = 0$$

$V_c(0^+) = 2V + 1V = 3V$

$I_1(0^+) = 0$

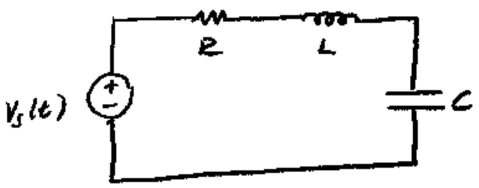
$I_2(0^+) = 5$

As $t \to \infty$



Series RLC

Dual of parallel RLC



$$\left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right) V_c(t) = \frac{1}{C}\cdot D \cdot i_s(t)$$

Dual

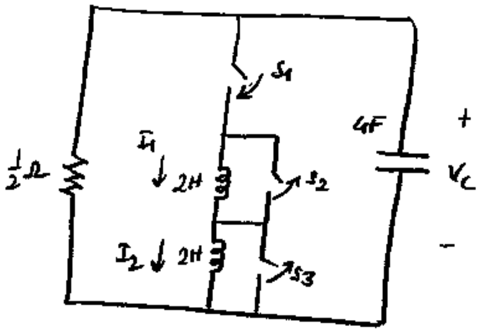
$$\rightarrow \left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) I_L(t) = \frac{1}{L}\, D \cdot V_s(t)$$

$$V_R(t) + V_L(t) + V_C(t) = V_s(t)$$

$$R \cdot I_L(t) + L \cdot \dot{I}_L(t) + \frac{1}{C}\int_{-\infty}^{t} I_L(t)\,dt = V_s(t)$$

$$L \cdot \ddot{I}_L(t) + R \cdot \dot{I}_L(t) + \frac{1}{C} I_L(t) = \dot{V}_s(t)$$

Ex



S_1 closes at $t = 4$ sec

S_2 opens at $t = 8$ sec

S_3 opens at $t = 12$ sec

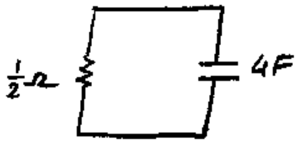
$I_1(0) = 1A$

$I_2(0) = -2A$

$V_C(0) = 5V$

$V_C(t) = ?$

① $0 < t < 4$



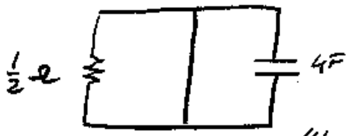
$V_C(0^-) = 5V$

$V_C(t) = 5 \cdot e^{-t/2} \cdot u(t)$

$I_1(t) = 1A$

$I_2(t) = -2A$

② $4 < t < 8$



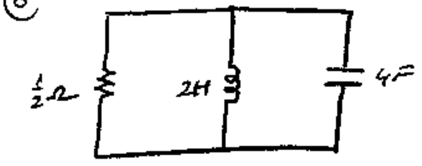
$V_C(4^-) = 5 \cdot e^{-4/2} = 5/e^2$

$V_C(4^+) = 0V$

$I_1(t) = 1A$

$I_2(t) = -2A$

③



$V_C(8^+) = 0V$

$I_1(8^+) = 1A$

$$\left.\begin{aligned} 2\alpha &= \frac{1}{RC} = \frac{1}{2} \longrightarrow \alpha = \frac{1}{4} \\ \omega_0^2 &= \frac{1}{LC} = \frac{1}{8} \longrightarrow \omega_0 = \frac{1}{2\sqrt{2}} \end{aligned}\right\} |\alpha| < \omega_0 \rightarrow \text{underdamped}$$

$$\left(D^2 + \frac{1}{2}D + \frac{1}{8}\right) V_C(t) = 0$$

$$V_C(t) = A \cdot e^{\lambda_1 t} + B \cdot e^{\lambda_2 t}$$

λ_1, λ_2 : complex conjugate

$$\lambda_1, \lambda_2 = -\frac{1}{4} \mp j\frac{1}{4}$$

$$\dot{V}_c(t) = e^{-1/4\,t}\left(A\cdot\cos\left(\tfrac{1}{4}t\right) + B\cdot\sin\left(\tfrac{1}{4}t\right)\right)$$

But instead,

$$V_c(t) = a^{-(t-8)4}\left(A\cos\left(\tfrac{t-8}{4}\right) + B\sin\left(\tfrac{t-8}{4}\right)\right)u(t-8)$$

$A = ?\ ,\ B = ?$

$$V_c(8^+) = 0V \quad , \quad \dot{V}_c(8^+) = \frac{I_c(8^+)}{C} = \frac{1}{4}\cdot(-1) = -\frac{1}{4}$$

$$V_c(t) = e^{-(t-8)/4}\left(\underset{\underset{0}{\downarrow}}{A}\cdot\cos\left(\tfrac{t-8}{4}\right) + \underset{\underset{-1}{\downarrow}}{B}\sin\left(\tfrac{t-8}{4}\right)\right)$$

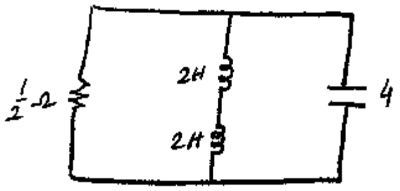
$$V_c(12^-) = e^{-1}(-\sin(1)) = -0{,}31$$

$$I_1(t) = -C\cdot\dot{V}_c(t) - \frac{V_c(t)}{R} = -4\dot{V}_c(t) - 2V_c(t)$$

$$I_1(12^-) = -\frac{1}{e}A$$

$$I_2(t) = -2A$$

④ $t > 12$



$V_c(12^-) = -0{,}31V$
$I_1(12^-) = -1/e\ A$
$I_2(12^-) = -2A$

$$I_{4H} = \frac{-1/e - 2}{2} = -\frac{1}{2e} - 1\ A$$

$$V_c(12^+) = -0{,}31\ V$$

$$\left(D^2 + \tfrac{1}{2}D + \tfrac{1}{16}\right)V_c(t) = 0$$

$$\alpha = \tfrac{1}{4}\ ;\quad \omega_0 = \tfrac{1}{4}$$

$$V_c(t) = A\cdot e^{-t/4} + Bt\cdot e^{-t/4}$$ → critically damped response

$$V_c(t) = \left[\underset{\underset{-0{,}31}{\downarrow}}{A}\cdot e^{-(t-12)/4} + \underset{\underset{0{,}37}{\downarrow}}{B}(t-12)\,e^{-(t-12)/4}\right]u(t-12)$$

$\dot{V}_c(12^+) = 0{,}45 \qquad V_c(12^+) = -0{,}31$