11 Loop and Cut-set Analysis

In the previous chapter we have learned to perform systematically the node analysis of *any* linear time-invariant network. We have also learned to perform the mesh analysis for any such network provided its graph is planar. In this chapter we briefly discuss two generalizations, or perhaps variations, of these methods, namely, the cut-set analysis and the loop analysis. There are two reasons for studying loop and cut-set analysis: first, these methods are useful because they are much more flexible than mesh and node analysis, and, second, they use concepts and teach us points of view that are indispensable for writing state equations.

In Sec. 1, we introduce some new graph-theoretic concepts and prove a fundamental theorem. In Sec. 2, we study loop analysis, and in Sec. 3 we study cut-set analysis. Section 4 is devoted to comments on these methods. In Sec. 5 we establish a basic relation between the loop matrix **B** and the cut-set matrix **Q**.

1 Fundamental Theorem of Graph Theory

In order to develop this theorem we need to indicate precisely what we mean by a *tree*. Let $\mathcal{G}$ be a connected graph and T a subgraph of $\mathcal{G}$. We say that **T is a tree of the connected graph $\mathcal{G}$** if (1) T is a *connected* subgraph, (2) it contains *all the nodes* of $\mathcal{G}$, and (3) it contains *no loops*.

Given a connected graph $\mathcal{G}$ and a tree T, the branches of T are called **tree branches**, and the branches of $\mathcal{G}$ not in T are called **links**. (Some authors call them cotree branches, or chords.)

A graph has usually many trees. In Fig. 1.1 we show a few trees of a connected graph $\mathcal{G}$. To help you understand the definition, in Fig. 1.2 we show a few subgraphs (of the same graph $\mathcal{G}$) which are *not* trees of $\mathcal{G}$. To emphasize the fact that complicated graphs have many trees, remember that if a graph has n_t nodes and has a single branch connecting every pair of nodes, then it has $n_t^{n_t-2}$ trees. For such graphs, when $n_t = 5$, there are 125 trees; when $n_t = 10$, there are 10^8 trees.

Exercise Draw all possible trees for the graph shown in Fig. 1.3.

The following fundamental theorem relates the properties of loops, cut sets and trees.

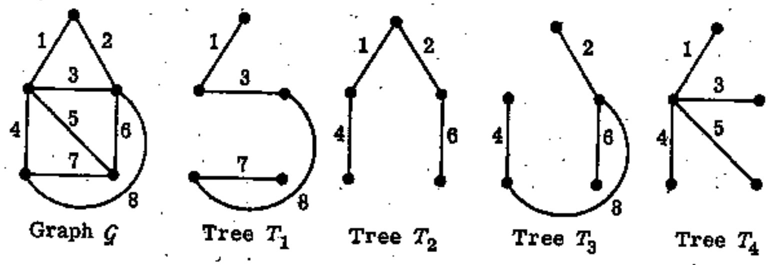


Fig. 1.1 Examples of trees of graph $\mathcal{G}$.

THEOREM Given a connected graph $\mathcal{G}$ of n_t nodes and b branches, and a tree T of $\mathcal{G}$,

1. There is a unique path along the tree between any pair of nodes.
2. There are $n_t - 1$ tree branches and $b - n_t + 1$ links.
3. Every link of T and the unique tree path between its nodes constitute a *unique loop* (this is called the **fundamental loop** associated with the link).
4. Every tree branch of T together with some links defines a *unique* cut set of $\mathcal{G}$. This cut set is called a **fundamental cut set** associated with the tree branch.

Proof

1. Suppose there were two paths along the tree between node ① and node ②. Since some branches of these two paths would constitute a loop, the tree would contain a loop. This contradicts requirement 3 of the definition of a tree.
2. Let T be a tree of $\mathcal{G}$; then T is a subgraph of $\mathcal{G}$ which connects all nodes, and it therefore has n_t nodes. If a node of T has only one tree branch incident with it, this node is called a *terminal node* of T. Since T is a connected subgraph which contains no loops, it has at least two terminal nodes. Let us remove from the tree one of the terminal nodes and its incident tree branch. The remaining subgraph must still have at least two terminal nodes. Let us continue removing terminal nodes and their incident tree branches until only one tree branch is left. This last branch is incident with two nodes. Thus, we have removed one tree branch for every node except for the last branch which was connected with two nodes. Since there were n_t nodes, T must have had $n_t - 1$ branches. Since all branches of $\mathcal{G}$ which are not in T are called links, there are $b - (n_t - 1) = b - n_t + 1$ links.
3. Consider a link l_1 which connects nodes ① and ②. By part 1, there is a unique tree path between ① and ②. This tree path, together with the link l_1, constitutes a loop. There cannot be any other loop since the tree had no loop to start with.
4. Consider the branch b_1 of T as shown in Fig. 1.4. Remove b_1 from T. What remains of T is then made up of two separate (connected) parts, say T_1 and T_2. Since every link connects a node of T to another node of T, let us consider the set L of all the links that connect a node of T_1 to a node of T_2. It is easily verified that the links in L, together with the tree branch b_1, constitute a cut set. All links not in L cannot contribute to another cut set since each one of them with a tree path in either T_1 or T_2 constitutes a loop.

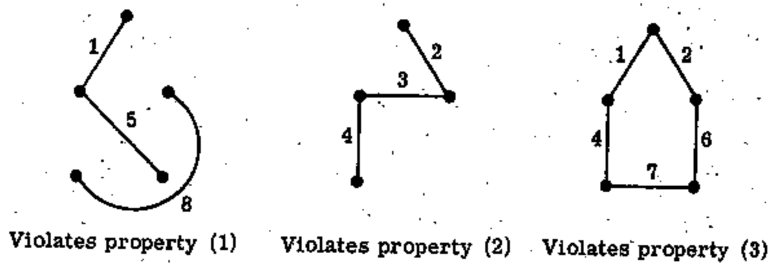


Fig. 1.2 Examples of subgraphs of $\mathcal{G}$ which are *not* trees.

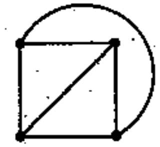


Fig. 1.3 A connected graph with four nodes and six branches.

The theorem can readily be extended to the case in which the graph consists of several separate parts, as shown in the following statement.

COROLLARY Suppose that $\mathcal{G}$ has n_t nodes, b branches, and s separate parts. Let $T_1, T_2, \ldots, T_s$ be trees of each separate part, respectively. The set $\{T_1, T_2, \ldots, T_s\}$ is called **a forest of** $\mathcal{G}$. Then the forest has $n_t - s$ branches,

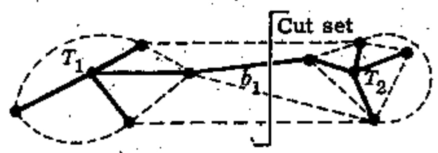


Fig. 1.4 Illustration of properties of a fundamental cut set.

$\mathcal{G}$ has $b - n_t + s$ links, and the remaining statements of the theorem are true.

Exercise Consider the graph $\mathcal{G}$ of Fig. 1.1. List all the fundamental loops and all the fundamental cut sets corresponding to tree T_1. Repeat for trees T_2, T_3, and T_4.

2 Loop Analysis

2.1 Two Basic Facts

Consider a connected graph with b branches and n_t nodes. Pick an arbitrary tree T. There are $n = n_t - 1$ tree branches and $l = b - n$ links. Number the branches as follows: links first from 1 to l, tree branches next from $l + 1$ to b. Every link defines a fundamental loop, i.e., the loop formed by the link and the unique tree path between the nodes of that link. This is illustrated in Fig. 2.1 in terms of a simple graph with $b = 8$, $n_t = 5$, $n = 4$, and $l = 4$.

In order to apply KVL to each fundamental loop we adopt *a reference direction for the loop* which agrees with the reference direction of the link which defines that fundamental loop. This is shown in Fig. 2.1; for example, fundamental loop 1 has the same orientation as link 1, etc. The KVL equations can be written for the four fundamental loops in terms of the branch voltage as follows:

Loop 1: $v_1 - v_5 + v_6 = 0$

Loop 2: $v_2 + v_5 - v_6 + v_7 + v_8 = 0$

Loop 3: $v_3 - v_6 + v_7 + v_8 = 0$

Loop 4: $v_4 - v_6 + v_7 = 0$

In matrix form, the equation gives

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}}_{l \text{ links} \;|\; n \text{ tree branches}}\Bigg\} l \text{ loops} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

More generally, if we apply the KVL to each one of the l fundamental loops, we obtain a system of l linear algebraic equations in b unknowns $v_1, v_2, \ldots, v_b$. The first basic fact of loop analysis is as follows:

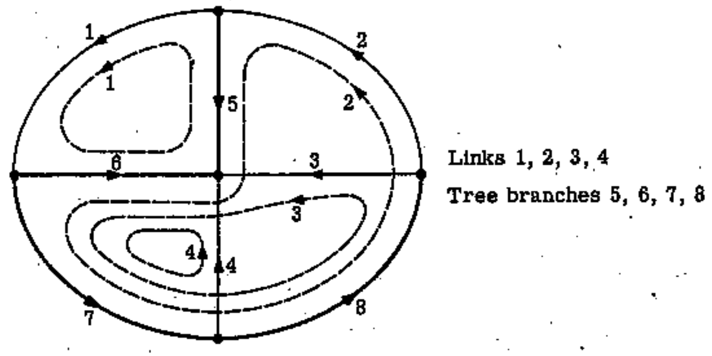


Fig. 2.1 Fundamental loops for the chosen tree of a graph.

The l linear homogeneous algebraic equations in $v_1, \ldots, v_b$ obtained by applying KVL to each fundamental loop constitute a set of l linearly independent equations.

If we recall the convention that the reference direction of the loop agrees with that of the link which defines it, we see that the system of equations obtained from KVL is of the form

(2.1) $\mathbf{Bv} = \mathbf{0}$

where $\mathbf{B}$ is an $l \times b$ matrix called the **fundamental loop matrix**. Furthermore, its (i,k)th element is defined as follows:

(2.2) $$b_{ik} = \begin{cases} 1 & \text{if branch } k \text{ is in loop } i \text{ and their reference directions } \textit{agree} \\ -1 & \text{if branch } k \text{ is in loop } i \text{ and their reference directions } \textit{do not agree} \\ 0 & \text{if branch } k \textit{ is not in } \text{loop } i \end{cases}$$

Since each fundamental loop includes one link only and since the orientations of the loop and the link are picked to be the same, it is clear that if we number the links $1, 2, \ldots, l$ and the tree branches $l + 1, l + 2, \ldots, b$, the matrix $\mathbf{B}$ has the form

(2.3) $$\mathbf{B} = \Big[\underbrace{\mathbf{1}_l}_{l \text{ links}} \;\vdots\; \underbrace{\mathbf{F}}_{n \text{ tree branches}}\Big]\Big\} l \text{ loops}$$

where $\mathbf{1}_l$ designates a unit matrix of order l and $\mathbf{F}$ designates a rectangular matrix of l rows and n columns. It is obvious that the rank of $\mathbf{B}$ is l, since $\mathbf{B}$ includes the unit matrix $\mathbf{1}_l$ and has only l rows. Therefore, we have established the fact that the l fundamental loop equations written in terms of the branch voltages constitute a set of l linearly independent equations.

Exercise For the graph of Fig. 2.1, consider a loop $\mathcal{L}$ that is not a fundamental loop. Show that KVL applied to loop $\mathcal{L}$ gives an equation which depends linearly on the l equations based on the fundamental loops.

Turning now to KCL, we note that KCL implies that any current that comes to a node must leave this node; therefore, we may think of the branch currents as having been formed by currents around loops. Call $i_1, i_2, \ldots, i_l$ the currents in the l links of the tree T. We imagine each of these currents flowing in its respective fundamental loop; thus, each tree branch current is the superposition of one or more loop currents. More precisely, we assert that

(2.4) $$\mathbf{j} = \mathbf{B}^T\mathbf{i}$$

where $\mathbf{B}^T$ is the transpose of the fundamental loop matrix. To prove Eq. (2.4), let us write it in the form $\mathbf{j} = \mathbf{C}\mathbf{i}$, where $\mathbf{C}$ is the appropriate matrix of b rows and l columns which makes the equation true. We wish to show that $\mathbf{C} = \mathbf{B}^T$. Let us consider the branch currents. For those branches which are links of the given tree, the link currents are identical to the fundamental loop currents; that is,

(2.5) $$j_k = i_k \qquad k = 1, 2, \ldots, l$$

The remaining branches belong to the tree; hence they are tree branches. Each tree-branch current is a linear combination of the fundamental loop currents. More specifically, the kth branch current can be written as

(2.6) $$j_k = \sum_{i=1}^{l} c_{ki} i_i \qquad k = l+1, l+2, \ldots, b$$

where c_{ki} is given by the following equation:

(2.7) $$c_{ki} = \begin{cases} 1 & \text{if branch } k \text{ is in loop } i \text{ and their reference directions } \textit{agree} \\ -1 & \text{if branch } k \text{ is in loop } i \text{ and their reference directions } \textit{do not agree} \\ 0 & \text{if branch } k \textit{ is not in } \text{loop } i \end{cases}$$

It is obvious that Eq. (2.7) considers all branches, since for a link, branch k is only in loop k, and their reference directions coincide; hence, as in Eq. (2.5), all $c_{kk} = 1$. Comparing Eq. (2.7) with Eq. (2.2), we conclude that $c_{ki} = b_{ik}$; hence, the matrix $\mathbf{C} = (c_{ki})$ as specified by $\mathbf{j} = \mathbf{C}\mathbf{i}$ is the transpose of $\mathbf{B}$; that is, $\mathbf{C} = \mathbf{B}^T$. If we partition the matrix $\mathbf{B}^T$ in Eq. (2.4) according to whether a branch is a link or a tree branch, we obtain

(2.8) $$\mathbf{j} = \mathbf{B}^T\mathbf{i} = \left[\frac{\mathbf{1}_l}{\mathbf{F}^T}\right]\mathbf{i}$$

This equation will be useful for later applications.

Let us consider our example of Fig. 2.1. We can write the following equations according to Eq. (2.7):

$$j_1 = i_1$$
$$j_2 = i_2$$
$$j_3 = i_3$$
$$j_4 = i_4$$
$$j_5 = -i_1 + i_2$$
$$j_6 = i_1 - i_2 - i_3 - i_4$$
$$j_7 = i_2 + i_3 + i_4$$
$$j_8 = i_2 + i_3$$

In matrix form the equation is

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Summary KVL is expressed by $\mathbf{B}\mathbf{v} = \mathbf{0}$, and KCL by $\mathbf{j} = \mathbf{B}^T\mathbf{i}$ where $\mathbf{i}$ is the loop current vector. As a result of our choice of reference directions, the fundamental loop matrix $\mathbf{B}$ is of the form (2.3). These equations are valid irrespective of the nature of the branches.

Exercise 1 Prove Tellegen's theorem by using Eqs. (2.1) and (2.4).

Exercise 2 Consider the graph $\mathcal{G}$ of Fig. 1.1. Assign reference directions to each branch. Determine $\mathbf{B}$ for the tree T_1.

Exercise 3 Mesh analysis is not always a special case of loop analysis; give an example of a special case. (Hint: This will be the case if for each mesh current there is one branch that is traversed by only that mesh current.)

2.2 Loop Analysis for Linear Time-invariant Networks

In this section we shall restrict our consideration to linear time-invariant networks. We shall introduce branch equations and obtain by elimination

l linear network equations in terms of the l fundamental loop currents. For simplicity, we shall consider networks with resistors. The extension to the general case is exactly the same as the generalization discussed in Chap. 10.

The branch equations are written in matrix form as follows:

$$\mathbf{v} = \mathbf{R}\mathbf{j} + \mathbf{v}_s - \mathbf{R}\mathbf{j}_s \tag{2.9}$$

As before, $\mathbf{R}$ is a diagonal branch resistance matrix of dimension b, and $\mathbf{v}_s$ and $\mathbf{j}_s$ are voltage source and current source vectors, respectively. Combining Eqs. (2.1), (2.8), and (2.9), we obtain

$$\mathbf{B}\mathbf{R}\mathbf{B}^T\mathbf{i} = -\mathbf{B}\mathbf{v}_s + \mathbf{B}\mathbf{R}\mathbf{j}_s \tag{2.10}$$

or

$$\mathbf{Z}_l\mathbf{i} = \mathbf{e}_s \tag{2.11}$$

where

$$\mathbf{Z}_l \triangleq \mathbf{B}\mathbf{R}\mathbf{B}^T \qquad \mathbf{e}_s \triangleq -\mathbf{B}\mathbf{v}_s + \mathbf{B}\mathbf{R}\mathbf{j}_s \tag{2.12}$$

$\mathbf{Z}_l$ is called the **loop impedance matrix** of order l, and $\mathbf{e}_s$ is the **loop voltage source vector.** The loop impedance matrix has properties similar to those of the mesh impedance matrix discussed in the previous chapter. The matrix $\mathbf{Z}_l$ is symmetric. This is immediately seen once it is observed that in Eq. (2.12) $\mathbf{R}$ is a symmetric matrix. Let us rewrite Eq. (2.11) as follows:

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1l} \\ z_{21} & z_{22} & \cdots & z_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ z_{l1} & z_{l2} & \cdots & z_{ll} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_l \end{bmatrix} = \begin{bmatrix} e_{s1} \\ e_{s2} \\ \vdots \\ e_{sl} \end{bmatrix} \tag{2.13}$$

Example Let us consider the network of Fig. 2.2. The graph of the network is that of Fig. 2.1; hence the fundamental loop matrix has been obtained before. The branch equation is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} R_1 & & & & & & & \\ & R_2 & & & & & 0 & \\ & & R_3 & & & & & \\ & & & R_4 & & & & \\ & & & & R_5 & & & \\ & 0 & & & & R_6 & & \\ & & & & & & R_7 & \\ & & & & & & & R_8 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} + \begin{bmatrix} v_{s1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_8 j_{s8} \end{bmatrix}$$

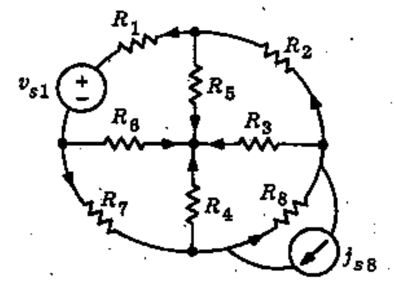


Fig. 2.2 Example of loop analysis.

Using Eq. (2.12), we can obtain the loop impedance matrix

$$\mathbf{Z}_l = \mathbf{B}\mathbf{R}\mathbf{B}^T$$
$$= \begin{bmatrix} R_1+R_5+R_6 & -R_5-R_6 & -R_6 & -R_6 \\ -R_5-R_6 & R_2+R_5+R_6+R_7+R_8 & R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7+R_8 & R_3+R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7 & R_6+R_7 & R_4+R_6+R_7 \end{bmatrix}$$

The loop equations are

$$\begin{bmatrix} R_1+R_5+R_6 & -R_5-R_6 & -R_6 & -R_6 \\ -R_5-R_6 & R_2+R_5+R_6+R_7+R_8 & R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7+R_8 & R_3+R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7 & R_6+R_7 & R_4+R_6+R_7 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} -v_{s1} \\ -R_8 j_{s8} \\ -R_8 j_{s8} \\ 0 \end{bmatrix}$$

Exercise Assume that the network shown in Fig. 2.2 is in the sinusoidal steady state and that its kth branch has an impedance $Z_k(j\omega)$. In terms of phasors, write the loop equations corresponding to the given tree.

2.3 Properties of the Loop Impedance Matrix

It is clear that the analysis of a resistive network and sinusoidal steady-state analysis of a similar network are very closely related. The main difference is in the appearance of phasors and impedances.

The following properties of the loop impedance matrix $\mathbf{Z}_l(j\omega)$ follow from the relation

$$\mathbf{Z}_l(j\omega) = \mathbf{B}\mathbf{Z}_b(j\omega)\mathbf{B}^T$$

1. If the network has no coupling elements, the branch impedance matrix $\mathbf{Z}_b(j\omega)$ is diagonal, and the loop impedance matrix is symmetric.
2. Also, if the network has no coupling elements, the loop impedance matrix $\mathbf{Z}_l(j\omega)$ can be written by inspection.
   a. The ith diagonal element of $\mathbf{Z}_l(j\omega)$, z_{ii}, is equal to the sum of the impedances in loop i; z_{ii} is called the **self-impedance of loop** i.
   b. The (i,k) element of $\mathbf{Z}_l(j\omega)$, z_{ik}, is equal to plus or minus the sum of the impedances of the branches common to loop i and to loop k; the plus sign applies if, in the branches common to loop i and loop k, the loop reference directions agree, and the minus sign applies when they are opposite.
3. If all current sources are converted, by Thévenin's theorem, into voltage sources, then the forcing term e_{si} is the algebraic sum of all the source voltages in loop i.
4. If the network is resistive and if all its resistances are positive, then $\det(\mathbf{Z}_l) > 0$.

Exercise 1 Write in a few sentences the circuit-theoretic consequences of property 4.

Exercise 2 Give an example of a linear time-invariant network made of *passive* elements such that for some tree and some frequency ω_0, $\det[\mathbf{Z}_l(j\omega_0)] = 0$. Can you give an example which includes a resistor?

Exercise 3 In the network of Fig. 2.2, pick the tree consisting of branches 1, 2, 3, and 4. Write the loop equations by inspection.

3 Cut-set Analysis

3.1 Two Basic Facts of Cut-set Analysis

Cut-set analysis is the dual of loop analysis. First, we pick a tree; call it T. Next we number branches; as before, the links from range 1 to l, and the tree branches range from $l + 1$ to b. We know that every tree branch defines (for the given tree) a *unique* fundamental cut set. That cut set is made up of links and of one tree branch, namely the tree branch which defines the cut set. In Fig. 3.1 we show the same graph $\mathcal{G}$ and the same tree T as in the previous section. The four fundamental cut sets are also shown.

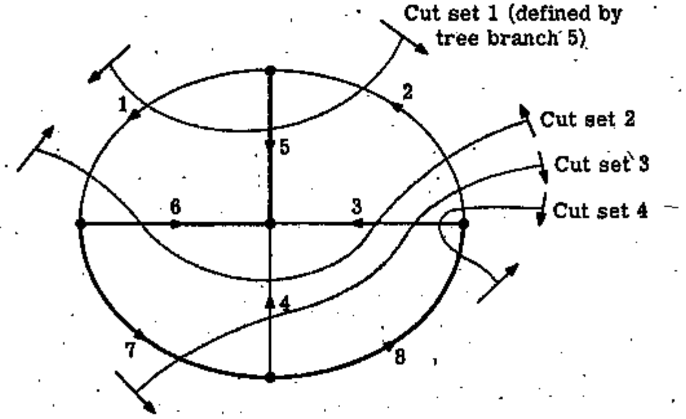


Fig. 3.1 Fundamental cut sets for the chosen tree of a given graph.

Let us number the cut sets as follows: cut set 1 is associated with tree branch 5, cut set 2 with tree branch 6, etc. By analogy to previous conventions, for each fundamental cut set we adopt a *reference direction for the cut set* which agrees with that of the tree branch defining the cut set. Under these conditions, if we apply KCL to the four cut sets, we obtain

Cut set 1: $j_1 - j_2 + j_5 = 0$
Cut set 2: $-j_1 + j_2 + j_3 + j_4 + j_6 = 0$
Cut set 3: $-j_2 - j_3 - j_4 + j_7 = 0$
Cut set 4: $-j_2 - j_3 + j_8 = 0$

In matrix form, the equation is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

More generally, if we apply KCL to each one of the n fundamental cut sets, we obtain a system of n linear homogeneous equations in n unknowns $j_1, j_2, \ldots, j_b$. The first basic fact of cut-set analysis is summarized in the following statement:

The n linear homogeneous algebraic equations in $j_1, j_2, \ldots, j_b$ obtained by applying KCL to each fundamental cut set constitute a set of n linearly independent equations.

Recalling the convention of sign for cut sets, we see that the KCL equations are of the form

(3.1) $$\mathbf{Qj} = \mathbf{0}$$

where $\mathbf{Q}$ is an $n \times b$ matrix defined by

(3.2) $$q_{ik} = \begin{cases} 1 & \text{if branch } k \text{ belongs to cut set } (i) \text{ and has the } \textit{same} \text{ reference direction} \\ -1 & \text{if branch } k \text{ belongs to cut set } (i) \text{ and has the } \textit{opposite} \text{ reference direction} \\ 0 & \text{if branch } k \textit{ does not belong} \text{ to cut set } (i) \end{cases}$$

$\mathbf{Q} = [q_{ik}]$ is called the **fundamental cut-set matrix.** As before we note that it is of the form

(3.3) $$\mathbf{Q} = [\underbrace{\mathbf{E}}_{l \text{ links}} \;\vdots\; \underbrace{\mathbf{1}_n}_{n \text{ tree branches}}] \quad n \text{ cut sets}$$

where $\mathbf{E}$ is an appropriate $n \times l$ matrix with elements $-1, +1, 0$, and $\mathbf{1}_n$ is the $n \times n$ unit matrix. Obviously, $\mathbf{Q}$ has a rank n since it includes the unit matrix $\mathbf{1}_n$. Hence, the n fundamental cut-set equations in terms of the branch currents are linearly independent.

Turning now to KVL, we note that each branch voltage can be expressed as a linear combination of the tree-branch voltages. For convenience, let us label the tree-branch voltages by $e_1, e_2, \ldots, e_n$. For the example in Fig. 3.1, from KVL we obtain the following equations:

$$v_1 = v_5 - v_8 = e_1 - e_2$$
$$v_2 = -v_5 + v_6 - v_7 - v_8 = -e_1 + e_2 - e_3 - e_4$$
$$v_3 = v_6 - v_7 - v_8 = e_2 - e_3 - e_4$$
$$v_4 = v_6 - v_7 = e_2 - e_3$$
$$v_5 = e_1$$
$$v_6 = e_2$$
$$v_7 = e_3$$
$$v_8 = e_4$$

By following the reasoning dual to that of the loop analysis, we can prove the assertion of the second basic fact, namely

(3.4) $$\mathbf{v} = \mathbf{Q}^T\mathbf{e}$$

that is, the branch voltage vector is obtained by forming the product of the cut-set matrix *transposed* and the tree-branch voltage vector.

Summary KCL requires that $\mathbf{Qj} = \mathbf{0}$. KVL is expressed by $\mathbf{v} = \mathbf{Q}^T\mathbf{e}$. As a result of our numbering convention, the fundamental cut-set matrix $\mathbf{Q}$ is of the form of (3.3). These equations are valid irrespective of the nature of the branches.

Exercise 1 Prove Tellegen's theorem by using Eqs. (3.1) and (3.4).

Exercise 2 Node analysis is not always a special case of cut-set analysis. Give an example of such a non-special case.

3.2 Cut-set Analysis for Linear Time-invariant Networks

In cut-set analysis Kirchhoff's laws are expressed by [see (3.1) and (3.4)]

$$\mathbf{Qj} = \mathbf{0}$$
$$\mathbf{v} = \mathbf{Q}^T\mathbf{e}$$

These equations are combined with branch equations to form network equations with the n tree-branch voltages $e_1, e_2, \ldots, e_n$ as network variables.

For the case of linear time-invariant resistive networks, the branch equations are easily written in matrix form. Let us illustrate the procedure with a resistive network. The branch equations are written in matrix form as follows:

(3.5) $$\mathbf{j} = \mathbf{Gv} + \mathbf{j}_s - \mathbf{Gv}_s$$

As before, $\mathbf{G}$ is the diagonal branch conductance matrix of dimension b and $\mathbf{j}_s$ and $\mathbf{v}_s$ are the source vectors. Combining Eqs. (3.1), (3.4), and (3.5), we obtain

(3.6) $$\mathbf{QGQ}^T\mathbf{e} = \mathbf{QGv}_s - \mathbf{Qj}_s$$

or

(3.7) $$\mathbf{Y}_q\mathbf{e} = \mathbf{i}_s$$

where

(3.8) $$\mathbf{Y}_q \triangleq \mathbf{QGQ}^T \qquad \mathbf{i}_s \triangleq \mathbf{QGv}_s - \mathbf{Qj}_s$$

$\mathbf{Y}_q$ is called the **cut-set admittance matrix,** and $\mathbf{i}_s$ is the **cut-set current source vector.**

In scalar form, the cut-set equations are

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} i_{s1} \\ i_{s2} \\ \vdots \\ i_{sn} \end{bmatrix}$$

3.3 Properties of the Cut-set Admittance Matrix

As before, we note that for sinusoidal steady-state analysis the cut-set admittance matrix $\mathbf{Y}_q$ has a number of properties based on the equation

$$\mathbf{Y}_q(j\omega) = \mathbf{Q}\mathbf{Y}_b(j\omega)\mathbf{Q}^T$$

1. If the network has no coupling elements, the branch admittance matrix $\mathbf{Y}_b(j\omega)$ is diagonal, and $\mathbf{Y}_q(j\omega)$ is symmetric.
2. If there are no coupling elements,
   a. The ith diagonal element of $\mathbf{Y}_q(j\omega)$, $y_{ii}(j\omega)$, is equal to the sum of the admittances of the branches of the ith cut set.
   b. The (i,k) element of $\mathbf{Y}_q(j\omega)$, $y_{ik}(j\omega)$, is equal to the sum of all the admittances of branches common to cut set i and cut set k when, in the branches common to their two cut sets, their reference directions agree; otherwise, y_{ik} is the negative of that sum.
3. If all the voltage sources are transformed to current sources, then i_{sk} is the total current-source contribution to cut set k.
4. If the network is resistive and if all branch resistances are positive, then $\det(\mathbf{Y}_q) > 0$.

Example Consider the resistive network shown in Fig. 3.2. The cut-set equations are

$$\begin{bmatrix} G_1+G_2+G_5 & -G_1-G_2 & G_2 & G_2 \\ -G_1-G_2 & G_1+G_2+G_3+G_4+G_6 & -G_2-G_3-G_4 & -G_2-G_3 \\ G_2 & -G_2-G_3-G_4 & G_2+G_3+G_4+G_7 & G_2+G_3 \\ G_2 & -G_2-G_3 & G_2+G_3 & G_2+G_3+G_8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} G_1 v_{s1} \\ -G_1 v_{s1} \\ 0 \\ j_{s8} \end{bmatrix}$$

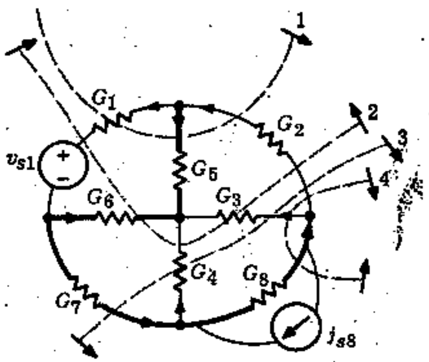


Fig. 3.2 Example of cut-set analysis.

4 Comments on Loop and Cut-set Analysis

Both the loop analysis and cut-set analysis start with choosing a tree for the given graph. Since the number of possible trees for a graph is usually large, the two methods are extremely flexible. It is obvious that they are more general than the mesh analysis and node analysis. For example, consider the graph of Fig. 4.1, where the chosen tree is shown by the emphasized branches. The fundamental loops for the particular tree coincide with the four meshes of the graph. Thus, the mesh currents are identical with the fundamental loop currents. Similarly, as shown in Fig. 4.2, the fundamental cut sets for the particular tree coincide with the sets of branches connected to nodes ①, ②, ③, and ④. If node ⑤ is picked as the datum node, the tree-branch voltages are identical with the node-to-datum voltages. Thus, mesh analysis and node analysis for this particular example are special cases of the loop analysis and cut-set analysis. However, it should be pointed out that for the graph of Fig. 4.3, the meshes are not special cases of fundamental loops; i.e., there exists no tree such that the five meshes are fundamental loops. Similarly, in Fig. 4.2, if node ④ is picked as datum node, there exists no tree which gives tree-branch voltages identical to the node-to-datum voltages.

As far as the relative advantages of cut-set analysis and loop analysis, the conclusion is the same as that between mesh analysis and node analysis. It depends on the graph as well as on the kind and number of sources in the network. For example, if the number of tree branches, n, is much smaller than the number of links, l, the cut-set method is usually more efficient.

It is important to keep in mind the duality among the concepts pertaining to general networks and graphs. Table 10.1 of Chap. 10 should be studied again at this juncture. Whereas in our first study, duality applied

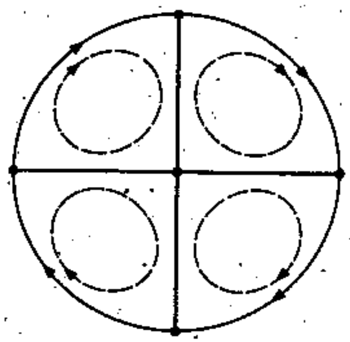


Fig. 4.1 Fundamental loops for the chosen tree are identical with meshes.

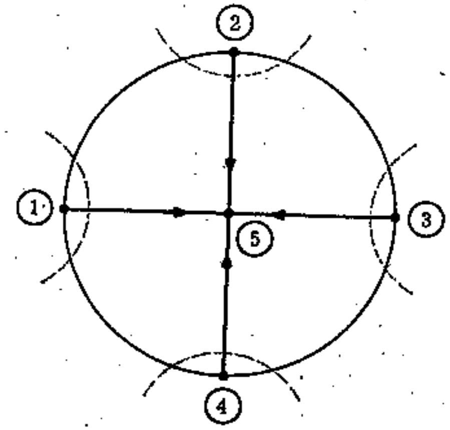


Fig. 4.2 The four fundamental cut sets for the chosen tree coincide with the set of branches connected to nodes ①, ②, ③, and ④.

only to planar graphs and planar networks and we thought in terms of node and mesh analysis, it is now apparent that duality extends to concepts pertaining to nonplanar graphs and networks; for example, cut sets and loops are dual concepts. The entries of Table 10.1 of Chap. 10 should be carefully considered.

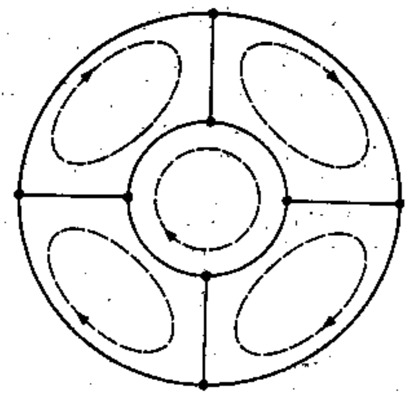


Fig. 4.3. A graph showing that meshes are not special cases of fundamental loops.

5 Relation Between B and Q

If we start with an oriented graph $\mathcal{G}$ and pick any one of its trees, say T, and if we write the fundamental loop matrix **B** and the fundamental cut-set matrix **Q**, we should expect to find a very close connection between these matrices. After all, **B** tells us which branch is in which fundamental loop, and **Q** tells us which branch is in which fundamental cut set. The precise relation between **B** and **Q** is stated in the following theorem.

THEOREM Call **B** the fundamental loop matrix and **Q** the fundamental cut-set matrix of the *same* oriented $\mathcal{G}$, and let both matrices pertain to the *same* tree T; then

(5.1) $$\mathbf{B}\mathbf{Q}^T = \mathbf{0} \quad \text{and} \quad \mathbf{Q}\mathbf{B}^T = \mathbf{0}$$

Furthermore, if we number the links from 1 to l and number the tree branches from $l + 1$ to b, then

(5.2) $$\mathbf{B} = \left[\mathbf{1}_l \middle| \mathbf{F}\right] \quad \text{and} \quad \mathbf{Q} = \left[-\mathbf{F}^T \middle| \mathbf{1}_n\right]$$

Before proving these facts, let us see what the first Eq. (5.1) means. This equation tells us that the product of the $l \times b$ matrix **B** and the $b \times n$ matrix $\mathbf{Q}^T$ is the $l \times n$ zero matrix. In other words, the product of every row of **B** and every column of $\mathbf{Q}^T$ is zero. The second Eq. (5.1) is simply the first one transposed: the product of every row of **Q** by every column of $\mathbf{B}^T$ is zero.

Proof Let the components of the vector $\mathbf{e} = [e_1, e_2, \ldots, e_n]^T$ be *arbitrary*. Since they are the tree-branch voltages of the tree T, the branch voltages of $\mathcal{G}$ are given by

$$\mathbf{v} = \mathbf{Q}^T\mathbf{e}$$

In other words, whatever the n-vector **e** may be, this equation gives us a set of b branch voltages that satisfies KVL. On the other hand, any time a set of branch voltages v_k satisfies KVL, we have

$$\mathbf{B}\mathbf{v} = \mathbf{0}$$

(that is, these v_k's satisfy KVL along all the fundamental loops). Substituting **v**, we obtain

(5.3) $$\mathbf{B}\mathbf{Q}^T\mathbf{e} = \mathbf{0} \quad \text{for all } \mathbf{e}$$

Note very carefully that this equation means that given *any* n-vector **e**, if we multiply it on the left by $\mathbf{B}\mathbf{Q}^T$, we get the zero vector! Observe that the product $\mathbf{B}\mathbf{Q}^T$ is an $l \times n$ matrix. This means that whenever we multiply any n-vector **e** by $\mathbf{B}\mathbf{Q}^T$, we get the zero vector. For example if we choose $\mathbf{e} = \mathbf{e}_1 \triangleq [1, 0, 0, \ldots, 0]^T$, $\mathbf{B}\mathbf{Q}^T\mathbf{e}_1$ is easily seen to be the first column of $\mathbf{BQ}^T$; hence, the first column of $\mathbf{BQ}^T$ is a column of zeros. Similarly, if we choose $\mathbf{e} = \mathbf{e}_2 \triangleq [0, 1, 0, \ldots, 0]^T$, we see that the second column of $\mathbf{BQ}^T$ is a column of zeros, and so forth. Therefore, Eq. (5.3) implies that the matrix $\mathbf{BQ}^T$ has all its elements equal to zero. Therefore, Eqs. (5.1) are established. (The second equation is simply the first one transposed.)

To prove (5.2), let us recall that we noted that **Q** was of the form

(5.4) $\mathbf{Q} = \left[\mathbf{E} \;\middle|\; \mathbf{1}_n\right]$

Therefore,

$$\mathbf{BQ}^t = [\underbrace{\mathbf{1}_l}_{l} \mid \underbrace{\mathbf{F}}_{n}]\begin{bmatrix}\mathbf{E}^T\\ \mathbf{1}_n\end{bmatrix}\begin{matrix}\}l\\ \}n\end{matrix}$$

Using the fact that a product of matrices is performed as rows by columns and noting that $\mathbf{1}_l$ has the same number of columns as $\mathbf{E}^T$ has rows, we conclude that

$$\mathbf{BQ}^T = \mathbf{1}_l\mathbf{E}^T + \mathbf{F}\mathbf{1}_n = \mathbf{E}^T + \mathbf{F} = \mathbf{0}$$

Hence,

$$\mathbf{E}^T = -\mathbf{F}$$

and transposing,

$$\mathbf{E} = -\mathbf{F}^T$$

Using this conclusion into (5.4), we see that

$$\mathbf{Q} = \left[-\mathbf{F}^T \;\middle|\; \mathbf{1}_n\right]$$

Thus, the proof is complete.

The relation between **B** and **Q** expressed by (5.2) is extremely useful since it means that whenever we know one of these matrices, we can write the other one by inspection; or, even better, both matrices **B** and **Q** are uniquely specified by the $l \times n$ matrix **F**.

Exercise 1 Verify that $\mathbf{BQ}^T = \mathbf{0}$ for the graph of Fig. 3.1.

Exercise 2 Prove the first equation (5.1) by referring to the definitions of **B** and **Q**. Note that the (i,k) element of $\mathbf{BQ}^T$ is of the form

$$\sum_{j=1}^{b} q_{ij}b_{kj} = q_{ik}b_{kk} + q_{i(l+1)}b_{k(l+1)}$$

that is, the sum has two nonzero terms.

Summary

- In both the loop analysis and the cut-set analysis we first pick a tree and number all branches. For convenience, we number the links first from 1 to l and number the tree branches from $l + 1$ to b. Then we assign branch orientations.

- In loop analysis we use the fundamental loop currents $i_1, i_2, \ldots, i_l$ as network variables. We write l linearly independent algebraic equations in terms of branch voltages by applying KVL for each fundamental loop. In linear time-invariant networks, taking the branch equations into account, the l equations can be put explicitly in terms of the l fundamental loop currents. In general, the resulting network equations form a system of l integrodifferential equations, in matrix form,

  $$\mathbf{Z}_l(D)\mathbf{i} = \mathbf{e}_s$$

  The solution of this system of linear integrodifferential equations will be treated in succeeding chapters. Once the fundamental loop currents **i** are determined, the branch currents can be found immediately from

  $$\mathbf{j} = \mathbf{B}^T\mathbf{i} \qquad \text{(KCL)}$$

  The b branch voltages are then obtainable from the b branch equations.

- The cut-set analysis is the dual of the loop analysis. The n tree-branch voltages $e_1, e_2, \ldots, e_n$ are used as network variables, and n linearly independent equations in terms of branch currents are written by applying KCL for all the fundamental cut sets associated with the tree. In linear time-invariant networks the n equations can be put explicitly in terms of the n tree-branch voltages. In general, the resulting matrix equation is

  $$\mathbf{Y}_q(D)\mathbf{e} = \mathbf{i}_s$$

  Once **e** is determined, the b branch voltages can be found immediately from

  $$\mathbf{v} = \mathbf{Q}^T\mathbf{e} \qquad \text{(KVL)}$$

  The b branch currents are then obtainable from the b branch equations.

- Given any oriented graph $\mathcal{G}$ and any of its trees, the resulting fundamental loop matrix **B** and the fundamental cut-set matrix **Q** are such that

  $$\mathbf{BQ}^T = \mathbf{0} \qquad \text{and} \qquad \mathbf{QB}^T = \mathbf{0}$$

  Furthermore,

  $$\mathbf{B} = [\underbrace{\mathbf{1}_l}_{l} \mid \underbrace{\mathbf{F}}_{n}]\}l \quad \text{and} \quad \mathbf{Q} = [\underbrace{-\mathbf{F}^T}_{l} \mid \underbrace{\mathbf{1}_n}_{n}]\}n$$

- The analogies between the four methods of analysis deserve to be emphasized:

$\mathbf{Y}_n(j\omega) = \mathbf{A}\mathbf{Y}_b(j\omega)\mathbf{A}^T$ for node analysis
$\mathbf{Z}_m(j\omega) = \mathbf{M}\mathbf{Z}_b(j\omega)\mathbf{M}^T$ for mesh analysis
$\mathbf{Y}_q(j\omega) = \mathbf{Q}\mathbf{Y}_b(j\omega)\mathbf{Q}^T$ for cut-set analysis
$\mathbf{Z}_l(j\omega) = \mathbf{B}\mathbf{Z}_b(j\omega)\mathbf{B}^T$ for loop analysis

- Each one of the "connection" matrices **A**, **M**, **Q**, and **B** is of full rank.

Problems

Trees, cut sets, and loops

1. For the oriented graph shown in Fig. P11.1 and for the tree indicated,

a. Indicate all the fundamental loops and the fundamental cut sets.
b. Write all the fundamental loop and cut-set equations.
c. Can you find a tree such that all its fundamental loops are meshes?

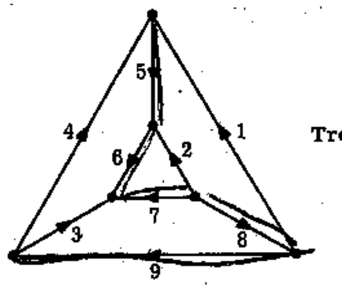


Fig. P11.1

Loop analysis

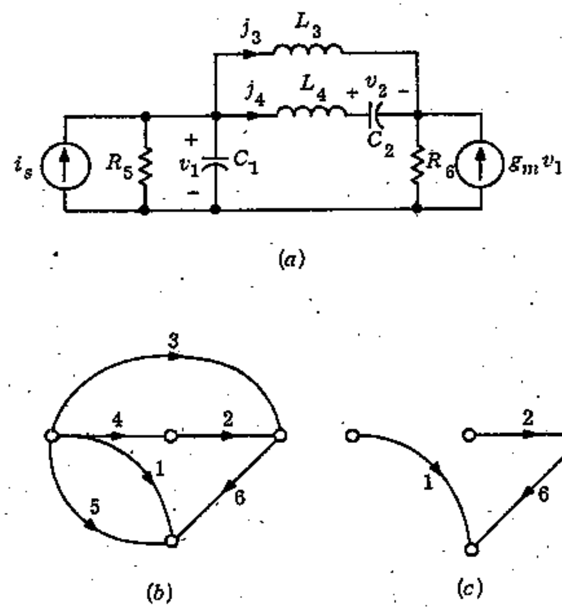
2. Your roommate analyzed a number of passive linear time-invariant *RLC* circuits. He found the loop impedance matrices given below. Which ones do you accept as correct? Give your reasons for rejecting any.

a. $\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3+j & -2j \\ -2j & 5+7j \end{bmatrix}$ $\begin{bmatrix} 3 & -j \\ -j & 2 \end{bmatrix}$ $\begin{bmatrix} 5 & 7j \\ 6j & 8+3j \end{bmatrix}$.

Loop analysis

3. The linear time-invariant network of Fig. P11.3*a*, having a (topological) graph shown in Fig. P11.3*b*, is in the sinusoidal steady state. From the (topological) graph a tree is picked as shown in Fig. P11.3*c*.



$g_m = 2$ mhos
$C_1 = 2$ F
$C_2 = 1$ F
$L_3 = 4$ H
$L_4 = 3$ H
$R_5 = 1\ \Omega$
$R_6 = 2\ \Omega$
$i_s = 3 \sin 2t$ amp

Fig. P11.3

a. Write the fundamental loop matrix **B**.
b. Calculate the loop impedance matrix $\mathbf{Z}_l$.
c. Write the loop equations in terms of voltage and current phasors; that is, $\mathbf{Z}_l\mathbf{I} = \mathbf{E}_s$.

Loop analysis

4. Assume that the linear time-invariant network of Fig. P11.3 is in the sinusoidal steady state. Write the fundamental loop equations for the tree indicated by the shortcut method. (First assume that the dependent current source is independent, and introduce its dependence in the last step.)

Loop analysis

5. The linear time-invariant network shown in Fig. P11.5 is in the sinusoidal steady state. For the reference directions indicated on the inductors, the inductance matrix is

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Write the fundamental loop equation for a tree of your choice.

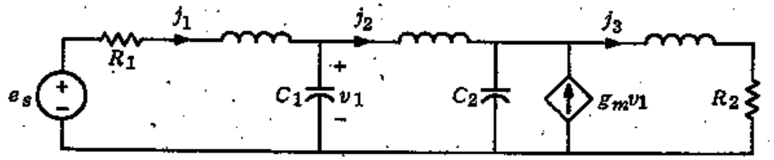


Fig. P11.5

Cut-set analysis

6. Consider the linear time-invariant network shown in Fig. P11.3*a*. Suppose it is in the sinusoidal steady state. Consider the tree shown in Fig. P11.3*c*.

a. Write the fundamental cut-set matrix **Q**.

b. Calculate the cut-set admittance matrix $\mathbf{Y}_q$.

c. Write the cut-set equations in terms of the cut-set voltage and source current phasors; that is, $\mathbf{Y}_q\mathbf{E} = \mathbf{I}_s$.

Cut-set analysis

7. The linear time-invariant network shown in Fig. P11.7 is in the sinusoidal steady state. It originates from delay-line designs; each inductor is coupled to its neighbor and to his neighbor(s) once removed, and the bridging capacitors compensate the coupling of the neighbors once removed. The coupling between inductors is specified by the reciprocal inductance matrix

$$\mathbf{\Gamma} = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & 0 \\ \Gamma_1 & \Gamma_0 & \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_1 & \Gamma_0 & \Gamma_1 \\ 0 & \Gamma_2 & \Gamma_1 & \Gamma_0 \end{bmatrix}$$

Pick a tree such that the corresponding cut-set equations are easy to write. Write these cut-set equations.

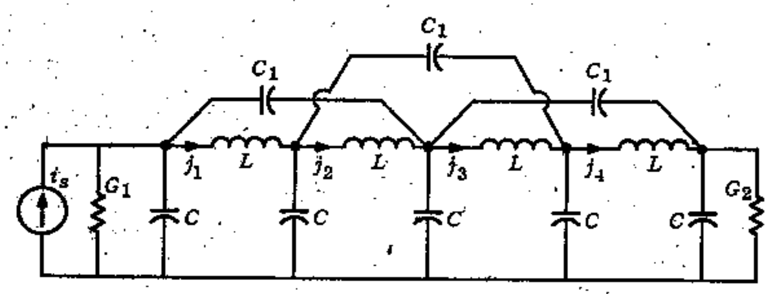


Fig. P11.7

Cut set and loop matrix

8. For a given connected network and for a fixed tree, the fundamental loop matrix is given by

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

a. Write, by inspection, the fundamental cut-set matrix which corresponds to the same tree.

b. Draw the oriented graph of the network.