EE 701
ROBOT VISION

Shape from Shading (SfS)

Problem definition

Linear Reflectance (special case)

General solution to SfS

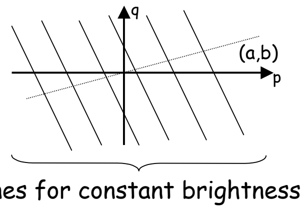
Relaxation methods for SfS

Shape from Shading

- Although photometric stereo allows direct shape recovery from two images, one can perceive a lot even from a single image
- Without additional information, it is not possible to obtain *(p,q)* from a single image irradiance equation : *E(x,y)=R(p,q)*
- Extra information is either some "singular" points with known orientation or the "smoothness" of surfaces, which lack discontinuities in depth
- With smoothness constraint, neighboring patches can not assume arbitrary assumptions

SfS for Linear Reflectance Map (1/3)

- Assume the reflectance map has the form
  R(p,q) = f(ap+bq)
  - a,b are known constants,
  - p,q unknown surface normals,
  - f is a strictly monotonic function with an inverse

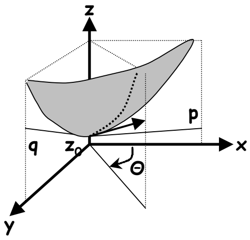


- Using image irradiance equation
  $E(x,y)=R(p,q) \rightarrow f^{-1}(E(x,y))=ap+bq$

- The slope of any surface at an arbitrary angle to the x-axis can be obtained using *directional derivative* :

$$m(\Theta) = \nabla z(x,y).n_{\Theta} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} . \begin{bmatrix} \cos\Theta \\ \sin\Theta \end{bmatrix}$$

$$= p\cos\Theta + q\sin\Theta$$



SfS for Linear Reflectance Map (2/3)

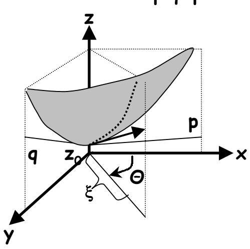
- Choose a particular direction perpendicular to constant brightness lines : $m(\Theta_0) = p\cos\Theta_0 + q\sin\Theta_0 = \frac{ap+bq}{\sqrt{a^2+b^2}} = \frac{f^{-1}(E(x,y))}{\sqrt{a^2+b^2}}$
- In this particular direction (which gives the maximum brightness change in the image), one can determine the slope of the surface
- Using the slope of the surface, one can determine z(x,y) along a direction on xy-plane which is parallel to Θ_0 direction on pq-plane

$$\delta z = m(\Theta_0)\delta\xi \quad : \quad \delta\xi \textit{ is a step along direction } \Theta_0$$

$$\frac{\delta z}{\delta \xi} = \frac{f^{-1}(E(x,y))}{\sqrt{a^2+b^2}} \quad \textit{where}$$

$$x(\xi) = x_0 + \xi\cos(\Theta_0) \textit{ and } y(\xi) = y_0 + \xi\sin(\Theta_0)$$

$$z(\xi) = z_0 + \frac{1}{\sqrt{a^2+b^2}} \int_0^{\xi} f^{-1}(E(x,y))d\xi$$



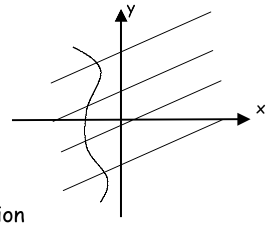
SfS for Linear Reflectance Map (3/3)

$$z(\xi) = z_0 + \frac{1}{\sqrt{a^2 + b^2}} \int_0^{\xi} f^{-1}(E(x,y))\,d\xi$$

- Unknown z_0 should be determined at one point to find an absolute depth; otherwise depth will be relative
- In order to recover the shape of whole surface, we need several strips (still in the same direction) and several starting points
- However, in general R(p,q) is not linear, hence strips are not lines

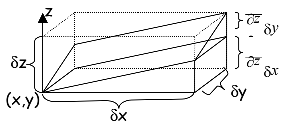
<u>SfS algorithm for a linear reflectance map</u> :

1. Start the solution at a known point (x_0, y_0, z_0)
2. Take a small step along Θ_0 direction
3. Calculate new (x,y,z)
4. Reset starting point as (x,y,z)
5. Repeat until there is a discontinuity in the image function



SfS for General Reflectance Map (1/3)

- Taking a small step along (δx, δy), the change in depth is equal to δz= pδx + qδy
- Provided that p and q are known at a given point, the changes in p and q are given by :



$$\delta p = \frac{\partial^2 z}{\partial x^2}\delta x + \frac{\partial^2 z}{\partial x \partial y}\delta y \,,\, \delta q = \frac{\partial^2 z}{\partial x \partial y}\delta x + \frac{\partial^2 z}{\partial y^2}\delta y \quad \Rightarrow \quad \begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = H \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

- The Hessian matrix, H, is required to find the change in p & q
- Image irradiance equation can be used to find the unknowns :

$$\frac{\partial E(x,y)}{\partial x} = \frac{\partial R(p,q)}{\partial x} = \frac{\partial R}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial R}{\partial q}\frac{\partial q}{\partial x}$$

$$\frac{\partial E(x,y)}{\partial y} = \frac{\partial R(p,q)}{\partial y} = \frac{\partial R}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial R}{\partial q}\frac{\partial q}{\partial y} \quad \Rightarrow \quad \begin{pmatrix} E_x \\ E_y \end{pmatrix} = H \begin{pmatrix} R_p \\ R_q \end{pmatrix}$$

SfS for General Reflectance Map (2/3)

$$\begin{pmatrix}\delta p\\ \delta q\end{pmatrix} = H\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \qquad \begin{pmatrix}E_x\\ E_y\end{pmatrix} = H\begin{pmatrix}R_p\\ R_q\end{pmatrix}$$

- It is not possible to find the elements of H, but one way to proceed is to eliminate H from these equations.
- Since we can not find a solution in an arbitrary direction on the surface, lets assume we can solve these equations for a specific direction, along the gradient of R(p,q) :

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}R_p\\ R_q\end{pmatrix}\delta\xi \quad \Rightarrow \quad \begin{pmatrix}\delta p\\ \delta q\end{pmatrix} = \begin{pmatrix}E_x\\ E_y\end{pmatrix}\delta\xi$$

- If the direction of change in the image xy-plane is parallel to the gradient of R(p,q), then change in (p,q) can be computed
- One can summarize all these relations with 5 ordinary DE :

$$\frac{\partial x}{\partial \xi} = R_p \quad \frac{\partial y}{\partial \xi} = R_q \quad \frac{\partial z}{\partial \xi} = pR_p + qR_q \quad \frac{\partial p}{\partial \xi} = E_x \quad \frac{\partial q}{\partial \xi} = E_y$$

SfS for General Reflectance Map (3/3)

- In order to obtain the whole surface, one must patch together all the (characteristic) strips that are seperately obtained by solving 5 ODE
- However, we require a point (for each set of ODE) with all initial values given in order to start a solution
- Such set of points are determined around a neighborhood of isolated (singular) maximum at $R(p_0, q_0)=E(x_0, y_0)$ or occluding boundaries

<u>SfS algorithm for a general case</u> (Characteristic Strip Expansion) :

1. Start the solution at a known point $(x_0, y_0, z_0, p_0, q_0)$
2. Take a small step from (x_0, y_0) along direction parallel to gradient of $R(p_0, q_0)$
3. Compute changes in p and q from changes in E(x,y)
4. Calculate new values for R, R_p and R_q .
5. Compute new value for z
6. Reset starting point as (x,y,z,p,q)
7. Repeat until there is a discontinuity in the image function

SfS using Relaxation Methods (1/3)

- The method of *characteristic strip expansion* suffers from :
  - sensitivity to measurement noise in practice
  - problems between neighboring strips due to numerical integration
  - hard to utilize info about surface orientation on occluding boundaries
  - not possible to obtain parallel implementations
- An iterative scheme similar to finite-difference methods may be more desirable (easy to incorporate boundary conditions, parallel implementation)
- Since there is only one (image irradiance) equation against two unknowns (p,q), the common imposed constrained is smoothness of p and q
- A smooth surface is characterized by slowly varying gradients p and q
- Hence, one can specify the smoothness constraint as minimizing :

$$e_s = \iint \left(\left(p_x^2 + p_y^2 \right) + \left(q_x^2 + q_y^2 \right) \right) dxdy$$

SfS using Relaxation Methods (2/3)

- In order to account for noise which causes departure from ideal image irradiance equation, the problem is posed as a minimization of total error e given by

$$e = e_s + \lambda e_i \quad where\ e_i = \iint \left(E(x,y) - R(p,q) \right)^2 dxdy$$

- There are two approaches to minimize e
  - Use calculus of variations (Euler equations) to convert analytic integral equation to 2nd order differential equations and solve these differential equations using numerical methods
  - Directly minimize a discrete version of this integral equation

$$e = \sum_i \sum_j \left(s_{ij} + \lambda t_{ij} \right) \quad where \quad \begin{aligned} t_{ij} &= (E_{ij} - R_s(p_{ij}, q_{ij}))^2 \\ s_{ij} &= \frac{1}{4}\left(\left(p_{i+1j} - p_{ij} \right)^2 + \left(p_{ij+1} - p_{ij} \right)^2 + \left(q_{i+1j} - q_{ij} \right)^2 + \left(q_{ij+1} - q_{ij} \right)^2 \right) \end{aligned}$$

SfS using Relaxation Methods (3/3)

$$e = \sum_i \sum_j (s_{ij} + \lambda t_{ij}) \quad where \quad \begin{aligned} t_{ij} &= (E_{ij} - R_s(p_{ij}, q_{ij}))^2 \\ s_{ij} &= \frac{1}{4}\left((p_{i+1j} - p_{ij})^2 + (p_{ij+1} - p_{ij})^2 + (q_{i+1j} - q_{ij})^2 + (q_{ij+1} - q_{ij})^2\right) \end{aligned}$$

- For directly minimizing the discrete version, differentiate wrt p_{ij} and q_{ij}

$$\frac{\partial e}{\partial p_{ij}} = 2\left(p_{ij} - \underbrace{\bar{p}_{ij}}_{\text{local aver.}}\right) - 2\lambda(E_{ij} - R_s(p_{ij}, q_{ij}))\frac{\partial R_s}{\partial p}$$

$$\frac{\partial e}{\partial q_{ij}} = 2\left(q_{ij} - \underbrace{\bar{q}_{ij}}_{\text{local aver.}}\right) - 2\lambda(E_{ij} - R_s(p_{ij}, q_{ij}))\frac{\partial R_s}{\partial q}$$

- After equating these two equations to zero, we obtain the iterative solution :

$$p_{ij}^{n+1} = \bar{p}_{ij}^n + \lambda\left[E_{ij} - R(\bar{p}_{ij}^n, \bar{q}_{ij}^n)\right]\frac{\partial R}{\partial p} \qquad q_{ij}^{n+1} = \bar{q}_{ij}^n + \lambda\left[E_{ij} - R(\bar{p}_{ij}^n, \bar{q}_{ij}^n)\right]\frac{\partial R}{\partial q}$$

Photometric Stereo using Relaxation Methods

- Similar ideas can be applied to photometric stereo in order account measurement errors. If there are *n* images, we formulate problem as :

$$e = \iint \left((p_x^2 + p_y^2) + (q_x^2 + q_y^2)\right) dxdy + \sum_{k=1}^{n} \lambda_k \iint (E_k(x,y) - R_k(p,q))^2\, dxdy$$

where E_i is the brightness measured at ith image and R_i is the corresponding reflectance map

- Corresponding discrete equations suggest an iterative scheme :

$$p_{ij}^{n+1} = \bar{p}_{ij}^n + \sum_{k=1}^{n} \lambda_k \left[E_{ijk} - R_k(\bar{p}_{ij}^n, \bar{q}_{ij}^n)\right]\frac{\partial R_k}{\partial p}$$

$$q_{ij}^{n+1} = \bar{q}_{ij}^n + \sum_{k=1}^{n} \lambda_k \left[E_{ijk} - R_k(\bar{p}_{ij}^n, \bar{q}_{ij}^n)\right]\frac{\partial R_k}{\partial q}$$

Recovering Depth from A Needle Diagram

- The algorithms produce surface shape information in the form of a *needle diagram* in which surface orientation is given for pels
- In most applications, a *depth map*, which is the height above a reference plane, is a desirable objective :

$$z(x,y) = z(x_0, y_0) + \int_{(x_0,y_0)}^{(x,y)} (p\,dx + q\,dy)$$

- Since p and q are obtained from noisy data, the above integral may yield erroneous; e.g. integral along a closed path may give nonzero results
- Use a least squares method to find the surface z(x,y) that best fits to the imperfect estimate of surface gradient

$$\iint \left((z_x - p)^2 + (z_y - q)^2 \right) dxdy \quad z_{x,y} : partial\ derivatives\ of\ z(x,y)$$

Final words on Shape from Shading …

- Although SfS is an elegant idea, there are some practical difficulties which limits its usefulness
- Reflectance properties of surfaces are not always known accurately
- It is difficult to control the illumination of the scene
- Constant albedo assumption also limits performance