On Parallelized Serially Concatenated Codes

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Abstract—Serial concatenated codes show very good performance at low signal to noise ratios. However, large decoding delays due to long input frame lengths constitute a major disadvantage for these type of codes. In this study we introduce a new class of concatenated convolutional codes which are very suitable for parallel decoding, thus have much less decoding delays and also show comparable performance to that of classical serial concatenated convolutional codes. The analytical upper bound expressions for performance of the proposed structures are derived using the uniform interleaver concept.

I. INTRODUCTION

CONCATENATED codes are proposed by Forney [1] in 1966. The usage of a posteriori probabilities (APPs) as opposed to hard decision estimates is essential for enhancing decoder performance. Following Forney's work, new APP algorithms were developed. The BCJR algorithm [2] which was ignored for a long time is one of the most widely known APP algorithms. With the introduction of turbo codes (parallel concatenated convolutional codes, i.e., PCCCs) [3] in 1993 a huge interest on iterative decoding was aroused. The use of soft-in soft-out decoding algorithms was a key in the astonishing performance of turbo codes. This showed the way to the invention of serially concatenated convolutional codes SCCCs in [4], where authors showed that they were better than PCCCs in some respects. In the last decade, similar codes such as, low density parity check codes [5], block product codes [6], and single parity check (SPC) product codes [7] have been extensively studied.

Convolutional codes have some appealing properties over linear block codes. Convolutional codes' trellis structure does not vary in time as in block codes. Moreover, the number of states in the trellis structure of an (n, k) block code is upper bounded by and usually on the order of $2^{(n-k)}$ where k is the information sequence length and n is the codeword length in bits [8]. However, the number of states in a convolutional code can be set as desired. The time invariant trellis structure of convolutional codes makes them more convenient for implementation [4]. In addition, numerous practical techniques such as trellis coded modulation (TCM) and puncturing can be easily utilized with convolutional codes, as opposed to linear block codes.

Concatenated convolutional codes CCCs are potential candidates for practical applications in many communication systems. Low latency is necessary if high data rates are

required. Since CCCs are iteratively decoded, the decoding delay may not be affordable for high data rate communication systems. One possible solution is to implement CCC decoders in a parallel fashion. In [9], [10], [11] techniques for parallel realization of turbo decoders were suggested. The common idea of these studies is to divide the received information block into sub-blocks and decode the sub-blocks in a parallel manner. However some difficulties arise in these applications. In [10] and [11] trellis blocks are divided into overlapped sub-blocks and these sub-blocks are decoded using the same soft-in soft-out (*T-Algorithm* or *MAP*) decoders which utilizes only partial observations. Thus, the method in [10] and [11] is sub-optimal unless a reasonable overlapping depth is used. A similar method was introduced in [9] where forward and backward variables computed in the previous iteration are used to provide boundary distributions for each block sub-block MAP decoders. This method requires additional memory in hardware to store the boundary distributions.

Turbo product codes based on convolutional codes are studied in detail in [12] where parallelization is applied at the encoder side so that parallel processing at the receiver is directly available without adding any extra algorithms. The receiver side directly employs MAP decoders in parallel. No overlapping, no boundary distributions or any other extra considerations are needed at the receiver side. We extend the studies in [12] with this paper and propose general structures for serially and parallel concatenated codes. SCCCs and PCCCs are in fact special cases of the proposed structures. We show that PDSCCs can be used instead of the SCCCs considerably reducing decoding delays.

The outline of the paper is as follows. In Section II definitions of weight spectrum functions are specified and analytical upper bound expressions are given for concatenated codes using the uniform interleaver concept. In Section III the proposed code configurations are explained. The advantage of the proposed codes is mentioned in Section IV. Simulation and analytical analysis results are given in Sections V and VI. Finally, concluding remarks are given in Section VII.

II. BACKGROUND

Analytical bound expressions for the performance of serial and parallel concatenated codes are defined in [4]-[13] where the authors employ uniform interleaving which is defined in [13] as a probabilistic device that maps a given input information sequence of length K and Hamming weight w into all distinct $\binom{K}{w}$ permutations with equal probability $1/\binom{K}{w}$. The uniform interleaver permits the estimation of

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the average interleaver gain, independent of the particular interleaver used in a concatenated code system.

A. Serially Concatenated Codes

The encoder for serially concatenated codes (SCCs) is depicted in Fig. 1. The constituent codes can be chosen from block and convolutional codes. If both constituent codes are chosen from convolutional codes, the well known serially concatenated convolutional codes (SCCCs) are obtained.

Although two constituent encoders are employed in Fig. 1, the idea can be extended to any number of constituent encoders by incorporating more encoders and interleavers.

Fig. 1. Serial concatenated code. CC_1 and CC_2 are constituent codes CC s 1 and 2, respectively. The interleaver size is N .

B. Analytical Analysis of SCCCs

1) *Conditional Weight Enumerating Function (CWEF)*: The conditional weight distribution function gives us information about the weight distribution of codeword sequences or parity sequences generated from a Hamming weight w input sequences:

$$A(w, X) = \sum_i A_{w,i} X^i$$

where i is the Hamming weight of the codewords generated by input sequences of Hamming weight w .

2) *Input Output Weight Enumerating Function (IOWEF)*: IOWEF contains information about the Hamming weight of the input sequences along with the produced codewords:

$$A(W, X) = \sum_{w,i} A_{w,i} W^w X^i,$$

where $A_{w,i}$ is the number of codewords of Hamming weight i generated by information sequences of Hamming weight w . W and X are dummy variables.

3) *IOWEF of SCCCs*: Referring to Fig. 1, IOWEF of serial concatenated code $A^{CC_s}(W, X)$ is expressed as a product of the two CWEFs of the constituent codes, which is normalized by the number of the possible permutations (i.e., uniform interleaver)

$$A^{CC_s}(W, X) = \sum_{i=0}^N \frac{A^{CC_1}(W, i) \times A^{CC_2}(i, X)}{\binom{K}{i}}.$$

Once IOWEF of a concatenated code is available, upper bounds of the bit error probability can be calculated using (2) and (3).

4) *Analytical Bound*: Once spectrum functions of the concatenated codes are available, performance upper bound to the bit error probability for maximum likelihood soft decoding of the code is found in the form

$$P_b(e) \leq \sum_{w,i} \frac{w}{2k} A_{w,i} \text{erfc}\left(\sqrt{\frac{i R_c E_b}{N_0}}\right). \quad (1)$$

This is further simplified as [13]

$$P_b \cong \frac{1}{2} \sum_i D_i \text{erfc}\left(\sqrt{\frac{i R_c E_b}{N_0}}\right), \quad (2)$$

where R_c is the code rate, $\frac{E_b}{N_0}$ is the bit energy to noise ratio of the AWGN channel, i is the Hamming weight of the codewords generated by Hamming weight w input sequences. For SCCCs D_i 's are obtained from the IOWEF coefficients according to

$$D_i = \sum_w \frac{w}{k} A_{w,i}. \quad (3)$$

III. PARALLEL DECODABLE CONCATENATED CODES (PDCCs)

To reduce the decoding delays of turbo-like codes, researches developed some techniques which involve parallel processing in the decoding operation [9], [10], [11]. In these studies, parallelization is considered at the receiver side. In our study we employ the parallelization at the transmitter side and use it directly at the receiver side without requiring any extra algorithms. The parallel encoder structure for SCCCs is depicted in Fig. 2 where N and M are the number of outer and inner constituent encoders for SCCCs. Serial to parallel (S/P) and parallel to serial (P/S) converters are employed.

A. Parallel Decodable Serially Concatenated Codes (PDSCCs)

The general structure of the PDSCCs is depicted in Fig. 2 where $C_{11} \dots C_{1N}$ and $C_{21} \dots C_{2M}$ are the outer and

Fig. 2. PD-SCC model. $CC_n^o, n = 1 \dots N$ and $CC_m^i, m = 1 \dots M$ are constituent encoders.

inner constituent code (CC) encoders respectively. N and M are the number of constituent encoders in the outer and inner clusters, respectively. An interleaver is placed between the inner and outer encoder clusters. If all the constituent codes are chosen from the same linear block code and a rectangular interleaver is employed between code clusters, the well known block product codes are obtained. Block product codes are usually expressed using a matrix notation for ease of illustration. The matrix equivalent representation for the block product code is depicted in Fig. 3.

Fig. 3. Regular product code encoding procedure, where a block code is used to encode rows and columns.

1) *Parallel Decodable Serially Concatenated Convolutional Codes (PDSCCCs)*: If convolutional code encoders are employed in *PDSCCs*, *PDSCCCs* are obtained. When $M = N = 1$ *PDSCCCs* turn out to be *SCCCs*, i.e., *SCCCs* are a special case of *PDSCCCs*. Assuming that the rate of all the constituent encoders equals $1/2$, for an input sequence of length $(N \times M)/2$ where $N \geq 2$ and $M = 2k$, k a positive integer, the overall encoding procedure can be demonstrated using a matrix notation. This special case will be named as matrix *PDSCCCs* (i.e., *M-PDSCCCs*). If M value is chosen as $2N$ then the input matrix becomes a square matrix. The encoding procedure is depicted in Fig. 4. We didn't show the addition of trellis termination bits to the encoded data in the Fig. 4, however the effects of trellis termination bits are taken into account in simulations. The length of the input subsequences of the outer and inner *CC* encoders equals $M/2$ and N respectively. The interleaver size is $N \times M$. Although a rectangular interleaver is employed between constituent encoder clusters for ease of illustration any type of interleaver can be used. We will use a square input matrix ($M = 2N$) in our simulations. The encoding operation using the matrix

Fig. 4. *M-PDSCCC* encoding procedure.

structure proceeds as follows. The data to be sent is put into a matrix. Each row of the matrix is encoded using a code encoder. Once all the rows are encoded, the matrix is sent to an interleaver. The interleaved matrix is coded column-wise. Data matrix dimension is $k \times k$, and encoded data matrix dimension is $n \times n$, i.e., our code is an $(n \times n, k \times k)$ code.

2) *PDSCCC Decoder*: The decoder of a *PDSCCC* is seen in Fig. 5. It is clear from the decoder structure that decoders $D_{11} \dots D_{1N}$ or decoders $D_{21} \dots D_{2M}$ can run in parallel.

Fig. 5. *PD-SCCC* Decoder. r_x is the received signal value. INT, D-INT are the interleaver and de-interleaver. L_k and L_l are the bit probabilities. L_{ke} and L'_{ke} are the extrinsic bit probabilities.

IV. PRACTICAL IMPLEMENTATION ADVANTAGES

Definition: The complexity operator $C(L)$ for the convolutional code C is $C(L) = k \times L$ where k is the number of parameters (number of alphas, betas, gammas) to be computed by the decoder in a single stage of the code trellis, i.e., for an input frame of length L the total computation amount is approximately $k \times L$. k depends on the convolutional code generator matrix.

Definition: The latency operator is $T(L) = t \times L$ where t is the decoding latency for a single stage of the code trellis. L is the length of the input information sequence.

The implementation advantage of *M-PDSCCC* will be discussed herein with the parameters used in this study. Trellis termination will be neglected in calculation and will not alter the results significantly. In *SCCC*, for a given transmit data vector of length L , two log-MAP decoders are needed. The first decoder has a complexity of $C(2L)$ and a time delay of $T(2L)$. The second decoder has a shorter input, thus it has a complexity of $C(L)$ and a time delay of $O(L)$. In total, the complexity is of $C(3L)$ and the time delay is of $T(3L)$. An input sequence of length L corresponds to a square matrix of size $\sqrt{L} \times \sqrt{L}$. In *M-PDSCCC* columns (inner code cluster) are decoded first. The use of separate log-MAP decoders for each row and column makes parallel processing operation possible. Each column decoder (inner decoder) has complexity of $C(\sqrt{L})$ and time delay of $T(\sqrt{L})$. Since these decoders are run in parallel, the total column decoding complexity is of $C(2L)$ but the time delay is of $T(\sqrt{L})$. Similarly, row decoding (outer decoder) has a total complexity of $C(L)$ and time delay of $T(\sqrt{L})$. Although both complexities are the same, *M-PDSCCCs* decrease the decoding delay by factor $T(\sqrt{L})$.

V. PDSCCC SIMULATION

We used the recursive systematic convolutional code *RSC* with the generator $(1, 5/7)_{\text{octal}}$ for all constituent code encoders in *M-PDSCCC*. An *S-random interleaver* ($S=18$) is employed between clusters. Input information sequence frame length is chosen as 1024 bits. We formed a size 32×32 information matrix. Thus, N equals 32 and M equals 68

(i.e trellis termination bits are added after row encoding). The encoded data in matrix form is multiplexed to a single stream and binary phase shift key (BPSK) modulated. The BPSK modulated signal is passed through an additive white Gaussian noise (AWGN) channel with double-sided power spectral density $\frac{N_0}{2}$. We used the *log*-MAP soft decoding algorithm [3] to iteratively decode the *M*-PDSCCs. We used 12 iterations for decoding. We also simulated the serially concatenated convolutional code which is a special case of *M*-PDSCC with $M = N = 1$. The signal-to-noise ratio values given in all the Figs. are normalized with the proper code rates for all scenarios. Non-recursive systematic convolutional codes were also tried and it was seen that their performance is not as good as the *M*-PDSCC system where RSC encoders are employed.

A. Trellis Termination Effects

We investigated the effects of trellis termination for three different scenarios. Trellis termination bits are added by the outer RSC encoders (row encoders) (R-T) with *rate* ≈ 0.235 , both outer and inner constituent RSC encoders (row and column encoders) added trellis termination bits to the codeword (TT) with *rate* ≈ 0.221 , neither outer and inner constituent RSC encoders added trellis termination bits to the codewords (No-TT) with *rate* ≈ 0.25 . Although addition of trellis termination bits decreases the code rate, they are critical for good performance of the *M*-PDSCC as seen in Fig. 6. Addition of trellis termination bits in turbo or serially concatenated code shows negligible improvement on the code performance [14]. However, without trellis termination the performance of the *M*-PDSCC degrades drastically. The addition of trellis termination bits slightly decreases the code rate, and this leads to a small loss in bandwidth efficiency.

The performance graphs are seen in Fig. 6. *M*-PDSCC has better performance at very low $\frac{E_b}{N_0}$ levels when only inner codes are trellis terminated. Its performance is worse at higher $\frac{E_b}{N_0}$ levels when it is compared to the inner and outer codes trellis terminated case. Though quite close up to BER 10^{-7} , SCCC seems to have an error curve of higher slope compared to TT at higher $\frac{E_b}{N_0}$ values. The analytical bound is evaluated for the TT case using the uniform interleaver approach. However as seen from the Fig. 6 the bound obtained by uniform interleaver concept gives a rough idea about the code performance.

B. Symmetric PDSCC

We also simulated the scenario where an equal number of parallel branches are used for the outer and inner clusters. The performance graph is depicted in Fig. 7. It is obvious from the Fig. 7 that the performance of the PDSCC is slightly better than that of the SCCC at high $\frac{E_b}{N_0}$ values.

VI. ANALYTICAL ANALYSIS OF PDSCC

The uniform interleaving assumption can be used to evaluate the average performance of the PDSCCs. Since the information bits of parallel subsequences of inner the and

Fig. 6. *M*-PDSCC and SCCC performance graph. *M*-PDSCC with no trellis termination (*M*-PDSCC No TT). *M*-PDSCC when rows (outer codes) are trellis terminated (*M*-PDSCC R-T). *M*-PDSCC when rows and columns (outer and inner codes) are trellis terminated (*M*-PDSCC TT). Frame length=1024, Iteration number=12.

Fig. 7. PDSCCs performance graph. The number of outer encoder and inner constituent encoders are equal and it is 16. Trellis termination bits are added by both outer and inner encoders. Frame length=1024. Iteration number=12.

outer clusters are independent, *IOWEFs* of the inner and outer clusters can be evaluated as

$$A^{C1}(W, X) = \prod_{i=1}^N A_{1i}(W, X)$$

$$A^{C2}(W, X) = \prod_{i=1}^M A_{2i}(W, X),$$

where $A_{1i}(W, X)$ and $A_{2i}(W, X)$ are the *IOWEFs* of the CCs in the inner and outer clusters. We considered different scenarios and evaluated analytical bounds for the performance of the *M*-PDSCCs. The analytical bounds for square

input matrices are shown in Fig. 8 for different interleaver lengths. Trellis termination bits are used by both inner and outer clusters. It is clear from Fig. 8 that as the interleaver size increases better performance is obtained.

Fig. 8. *M*-PDSCCs' analytical bounds for different interleaver sizes. Square input matrices are used.

Analytical bounds are also evaluated for the case when the trellis termination is not applied to the constituent codes. The bounds are shown in Fig. 9. When compared to Fig. 8, the degradation in performance is obvious.

Fig. 9. *M*-PDSCCs' analytical bounds when trellis termination bits are not used.

VII. CONCLUSION

In this article we propose new classes of serial concatenated convolutional codes. Using the uniform interleaver assumption we showed that the proposed codes perform as well as classical serially convolutional codes. Simulation results also support the analytical outcomes. The decoding delay is reduced by a factor of the number of parallel

branches. This provides a significant advantage in practical applications.

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